İstanbul Üniv. Fen Fak. Mat. Der. **51** (1992), 23-28

ON PRECROSSED MODULES AND GROUP GRAPHS

M. I. M. AL ALI

In [2] J. Shrimpton has defined for a graph (Γ) the symmetry group graph SYM (Γ), which contains the traditional automorphism group Aut (Γ) as set of edges. In this paper we study the relation between precrossed modules and group graphs, also we introduce the notion of sub-precrossed module.

INTRODUCTION

Throughout this paper we will deal with reflexive directed graph, called simply digraph. This consists of a set (Γ) and functions $s, t : \Gamma \longrightarrow \Gamma$, called respectively the source and target maps, such that $st=t$, $ts=s$. It follows that $s^2=s$, $t^2=t$ and s, t coincide on $\text{Im } s = \text{Im } t$. The elements of Γ are called the edges of the digraph, and the elements of Im s are called vertices. If x, y are vertices of Γ, it is common to write $\Gamma(x, y)$ for $s^{-1}(x) \cap t^{-1}(y)$, and to write $u : x \longrightarrow y$ for $u \in \Gamma(x, y)$. An edge element such that $su=tu$ is called a loop, other edges are called links. It is common to draw a diagram of a digraph in the form

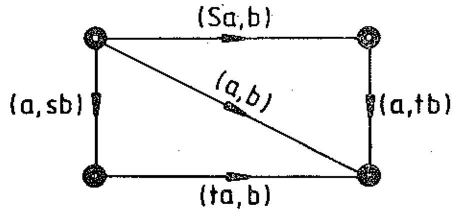
where the dot denotes a vertex.

A morphism $f : (\Gamma, s, t) \longrightarrow (\Gamma', s', t')$ of digraphs is a function $f : \Gamma \longrightarrow \Gamma'$ commuting with s, t, i.e. $sf=fs$, $tf=ft$. So we have a category Digr of digraphs and their morphisms.

Note that this definition allows for a morphism of digraphs to map an edge to a vertex. It is possible to set up another category Digr of irreflexive digraphs, in which this possibility of mapping an edge to a vertex is not allowed. However, the category Digr has some properties which are preferable to those of J Digr.

There are a number of constructions in graph theory whose properties are more easily comprehended from the view point of the category Digr. For example, the category Digr has products, where the product $\Gamma\times\Delta$ of digraphs (Γ_E , s , t) , (Δ_E , s , t) has $(\Gamma\times\Delta)_E = \Gamma_E\times\Delta_E$ and source and target maps $s\times s'$, $t\times t'$.

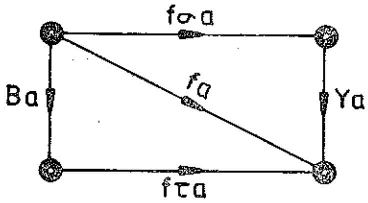
This implies that if a , b are edges of Γ , Δ respectively, then in $\Gamma\times\Delta$ we have a diagram :



Of course the universal property of $\Gamma\times\Delta$ is that there are two morphisms $p_1 : \Gamma\times\Delta \longrightarrow \Gamma$, $p_2 : \Gamma\times\Delta \longrightarrow \Delta$, and a morphism $f: X \longrightarrow \Gamma\times\Delta$ is entirely determined by its two components $p_1 f: X \longrightarrow \Gamma$, $p_2 f: X \longrightarrow \Delta$. More generally, one can say that Digr admits all limits and all colimits (i.e. is complete and cocomplete), but this more general statement will not be used here.

If Γ , Δ are graphs, then Digr(Γ , Δ) is the set of digraph morphisms $\Gamma \longrightarrow \Delta$. This set is equivalent to the set of vertices of a digraph DIGR(Γ , Δ), which we now describe.

The edges of DIGR(Γ , Δ) are triples (f, β , γ) such that β , γ are morphisms $\Gamma \longrightarrow \Delta$ and $f: \Gamma \longrightarrow \Delta$ is a function such that $\sigma f = \beta\sigma$, $\tau f = \gamma\tau$ so for each $a \in \Gamma$ we obtain a diagram



DEFINITIONS AND PRELIMINARIES

1. Δ cat^1-group is defined by Loday (using here the notion of [1]) to be a group G with two endomorphisms s, t of G such that $st = t$, $ts = s$ and $[\mathrm{Ker}(s) , \mathrm{Ker}(t)] = 1$ the group of commutators of Ker(s) and Ker(t). If the condition $[\mathrm{Ker}(s) , \mathrm{Ker}(t)] = 1$ is dropped, we get what is known as pre-cat^1-group, or group graph.

Given any group-graph (G, s, t), there is therefore a canonical method of constructing a cat^1-group from it: we can form $\overline{G} = G/[\mathrm{Ker}(s), \mathrm{Ker}(t)]$ with the induced endomorphisms $\overline{s}, \overline{t} : \overline{G} \longrightarrow \overline{G}$.

2. A precrossed module is consisting of two groups M, P where P acts on M and a homomorphism $\mu : M \longrightarrow P$ such that $\mu(m^p) = p^{-1}(\mu m) p$, $p \in P$ and $m \in M$.

Theorem 1. Any precrossed module is associated with a group graph.

Proof. Consider the following mapping $P \alpha M \underset{t'}{\overset{s'}{\rightrightarrows}} P \overset{\theta}{\longrightarrow} P \alpha M$ where $P \alpha M$ is the semidirect product of P by M and $(p, m)(p', m') = (pp', m^{p'} m')$ where $p, p' \in P$ and $m, m' \in M$. The mappings s', t' and θ are defined as follows:

$$s'(p, m) = p, \quad t'(p, m) = p(\mu m) \text{ and } \theta(p) = (p, 1).$$

It is clear that s' is a homomorphism.

$t'(pp', m^{p'} m') = pp' \; \mu(m^{p'} m') = pp' p'^{-1} (\mu m) p' (\mu m') = t'(p, m) t'(p', m)$, so, t' is a homomorphism and hence (G, s, t) is a group graph where $s, t : P \alpha M \longrightarrow P \alpha M$ are homomorphisms defined by $s = \theta s'$ and $t = \theta t'$. It is very clear that $st = t$ and $ts = s$.

Theorem 2. Every group graph is associated with precrossed module.

Proof. Let (G, s, t) be a group graph, then $t^* : \mathrm{Ker}\, s \longrightarrow \mathrm{Im}\, s$, where t^* is the restriction of t on $\mathrm{Ker}\, s$, and $\mathrm{Im}\, s$ acts on $\mathrm{Ker}\, s$ by conjugation, gives a precrossed module structure; for, let $b \in \mathrm{Im}\, s$ and $a \in \mathrm{Ker}\, s$, this implies that $b = s(b^*)$ for some $b^* \in G$, so $t^*(a^b) = t^* \in (a^{s(b^*)}) = t^* s((b^*)^{-1}) \; t(a) \, s(b^*) =$ $= t^* s(b^*)^{-1} \; t(a) \; s(b^*) = s^*(b^*)^{-1} \; t(a) \; s(b^*) = b^{-1} \; t(a) \; b$. It is clear that $\mathrm{Im}\; s \,\alpha\, \mathrm{Ker} = G$.

The symmetry digraph $\mathrm{SYM}(\Gamma)$ of Γ consists of the invertible elements of $\mathrm{DIGR}(\Gamma, \Gamma)$. In other words the triples (f, β, γ) consist of permutations of Γ such that β and γ are morphisms and $\sigma f = \beta \sigma$, $\tau f = \gamma \tau$ (Note that we use the term "permutation" for any one to one correspondence of a set to itself, whether the set is finite or infinite. Thus the automorphism group $\mathrm{Aut}(\Gamma)$ forms the set of vertices of Γ (Note that an automorphism is an invertible morphism).

The monoid multiplication on $\mathrm{DIGR}(\Gamma, \Gamma)$ is given by pointwise composition of functions: $(f, \beta, \gamma)(f', \beta', \gamma') = (ff', \beta\beta', \gamma\gamma')$, and this multiplication gives $\mathrm{SYM}(\Gamma)$ a group structure. It can easily be checked that the two structures on SYM(Γ) are compatible, in the sense that source and target functions are group homomorphisms.

Theorem 3. Let k_n be a 1-complete graph on n vertices (i.e. there is only one directed edge from x to y and one directed edge from y to x for any vertices x, y in k_n) then $\mathrm{SYM}(k_n) \cong S_n \,\alpha\, S_n$ where S_n is the symmetric group of degree n.

Proof. It is clear that $\mathrm{Aut}(k_n) \cong S_n$. The edges of the group $\mathrm{SYM}(k_n)$ are (f, β, γ), where $\beta, \gamma : k_n \longrightarrow k_n$ are automorphisms. Let $u : x \longrightarrow y$ be an edge in k_n then $fu : \beta x \longrightarrow \gamma y$, so fu is determined by β, γ, x and y, this means that f is determined by β, γ, so $\mathrm{SYM}(k_n)$ is 1-complete.

We have $\sigma, \tau : \mathrm{SYM}(k_n) \longrightarrow \mathrm{SYM}(k_n)$ and $\sigma(f, \beta, \gamma) = (\beta, \beta, \beta)$ and $\tau(f, \beta, \gamma) = (\gamma, \gamma, \gamma)$, also Ker σ consists of (f, β, γ) such that $\beta = 1$.

Hence Ker $\sigma \cong S_n$, so the associated precrossed module is isomorphic to $S_n \xrightarrow{\tau^*} S_n$, where τ^* is the restriction of τ on Ker σ. So,

$$\mathrm{SYM}(k_n) \cong S_n \,\alpha\, S_n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} S_n \,\alpha\, S_n.$$

Now we will discuss the corresponding notion associated to subgroup graphs.

Let (G, s, t) be a group graph and H be any subgroup of G, such that $S(H) \subseteq H$, $t(H) \subseteq H$, then (H, s^*, t^*) is a group-graph, called subgroup-graph of (G, s, t).

Lemma. Let Q be a subgroup of P, N be a subgroup of M which is Q equivalent (i.e. if $n \in N, q \in Q$, then $n^q \in N$), then $N \longrightarrow Q$ is a precrossed module if $M \longrightarrow P$ is a precrossed module, and $Q \,\alpha\, N$ is a subgroup-graph of $P \,\alpha\, M$.

Proof. The proof is obvious and similar to that of theorem 1. Note that the action of Q on N is the restriction of action of P on M.

Let Δ_n be a discrete graph on n vertices and k_n be a 1-complete graph on n vertices then we have the following theorem :

Theorem 4. Let $\{1\} \xrightarrow{1} S_n$ and $S_n \xrightarrow{1} S_n$ be two precrossed modules, then the associated group graphs are $\mathrm{SYM}(\Delta_n)$ and $\mathrm{SYM}(k_n)$ respectively (S_n acts on itself by conjugation, and the same for the action of S_n on $\{1\}$).

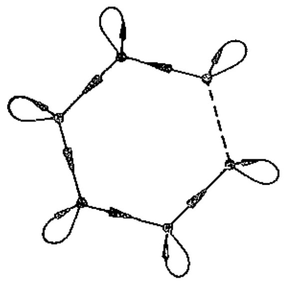
Proof. It is clear that the associated group graphs are isomorphic to $S_n \,\alpha\, \{1\} \rightrightarrows S_n \,\alpha\, \{1\}$, where $s(p, 1) = (p, 1)$, $t(p, 1) = (p(tm), 1)$ and

$S_n \alpha S_n \overset{s}{\underset{t}{\rightrightarrows}} S_n \alpha S_n$ respectively where $s(p, m)=(p, 1)$ and $t(p, m)=(p(tm), 1)$, $p, m \in S_n$, but $S_n \alpha \{1\} \cong \text{Im } \sigma \alpha \text{ Ker } \sigma$, where σ is the source map : $\text{SYM}(\Delta_n) \longrightarrow \text{SYM}(\Delta_n)$ (note that $\text{Aut}(\Delta_n) \cong S_n$), also $S_n \alpha S_n \cong \text{Im } \sigma \alpha \text{ Ker } \sigma$ where σ is the source map :

$$\text{SYM}(k_n) \longrightarrow \text{SYM}(k_n).$$

So, $k_n \alpha S_n \cong \text{Im } \sigma \alpha \text{ Ker } \sigma \cong \text{SYM}(k_n)$, and $S_n \alpha \{1\} \cong \text{Im } \sigma \alpha \text{ Ker } \sigma \cong \text{SYM}(\Delta_n)$.

Theorem 5. If Γ is an n circuit digraph, i.e.

$\Gamma =$ 

then $\text{SYM}(\Gamma) \cong C_n \alpha \{1\}$, C_n is the cyclic group of order n.

Proof. It is clear that $\text{Aut}(\Gamma) \cong C_n$. Let $\sigma, \tau : \text{SYM}(\Gamma) \longrightarrow \text{SYM}(\Gamma)$ be the source and target maps of $\text{SYM}(\Gamma)$, then $\text{Im } \sigma \cong C_n$ and $\text{Ker } \sigma \cong \{1\}$. So, $\{1\} \xrightarrow{\tau^*} C_n$ is the associated precrossed module. Hence $\text{SYM}(\Gamma) \cong C_n \alpha \{1\}$.

Now we end this paper with the following open problems :

Problem 1. What is the action of a group graph on a group graph?

Problem 2. What is the precrossed module associated with $\text{SYM}(k_n \cup k_n \cup \dots \cup k_n)$, where $\Gamma = k_n \cup k_n \cup \dots \cup k_n$ is a graph consisting of n-copies of k_n?

Problem 3. What is the semidirect product of $\text{SYM}(\Gamma)$ by a group?

Problem 4. Suppose Γ is a graph such that $\text{SYM}(\Gamma)$ is discrete graph. Is the precrossed module associated to $\text{SYM}(\Gamma \cup \Gamma)$ discrete? Is it

$$\{1\} \xrightarrow{1} \text{Aut}(\Gamma) \alpha z_2$$

where $\text{Aut}(\Gamma) \alpha z_2$ is the direct product of $\text{Aut}(\Gamma)$ with the cyclic group of order 2?

REFERENCES

[1] LODAY, J.L. : *Spaces with finitely many non trivial homotopy groups*, J. Pure Appl. Alg., 24 (1982), 179-202.

[2] SHRIMPTON, J. : **Graphs, symmetry and categorical methods**, Univ. of Wales, Ph. D. thesis, 1989.

DEPARTMENT OF MATHEMATICS
MU'TAH UNIVERSITY
P.O. BOX 7
MU'TAH - ALKARAK
JORDAN

ÖZET

[2] de J. Shrimpton, bir Γ grafı için geleneksel Aut (Γ) otomorfiler grubunu kenarlar cümlesi olarak içeren SYM (Γ) simetri grubu grafını tanımlamıştır. Bu çalışmada "precrossed" modüller ile grup grafları arasındaki ilişki incelenmekte ve ayrıca, "sub-precrossed" modül kavramı ithal edilmektedir.