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ON THE ALGEBRAIC INDEPENDENCE OF CERTAIN TRANSCENDENTAL NUMBERS

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(Dedicated to the memory of 70 th birthday of Professor Orhan Ş. İçen)

In this paper it is proved by using of Durand's Lemma that some transcendental numbers are algebraic independent.

Let

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{a_n} x^{e_n} \tag{1}$$

be a power series with $a_n \in \mathbb{Z}$, $a_n > 1$ and increasing integers e_n satisfying the following conditions:

$$\liminf_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = \sigma > 1, \tag{2}$$

$$\limsup_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = + \infty , \tag{3}$$

$$\lim_{n \to \infty} \frac{\log a_n}{e_n} = + \infty . \tag{4}$$

It follows from (2) that the radius of convergence of (1) is infinity and that the number

$$u = \limsup_{n \to \infty} \frac{\log \{l.\ c.\ m.\ (a_0, a_1, \ldots, a_n)\}}{\log a_n}$$

is finite with $1 \leq u \leq \dfrac{\sigma}{\sigma - 1}$.

It is proved by the author that for a non-zero algebraic number α of degree m smaller than $\sigma/(2u)$ $f(\alpha)$ is an U-number of degree $\leq m$ (See [3, Theorem 2, p. 144]).

In this paper we prove the following

THEOREM. Let $f(x)$ be a power series as in (1) such that (2), (3) and (4) hold. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be non-zero algebraic numbers with pairwise different absolute values. Then the numbers $f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_m)$ are algebraic independent.

For the proof of the theorem we use the following lemma of A. Durand [2, Theorem 2, p. 260]

LEMMA 1 (Durand). Let $\sigma_1, \ldots, \sigma_m$ be complex numbers and assume that sequences $(\sigma_{1,n})_{n=1}^{\infty}, \ldots, (\sigma_{m,n})_{n=1}^{\infty}$ of algebraic numbers exist with the following properties :

(i) $0 < | \sigma_{j+1} - \sigma_{j+1,n} | \leq \dfrac{1}{n} | \sigma_j - \sigma_{j,n} |$ (for $j = 1, \ldots, m-1$)

(ii) $0 < | \sigma_1 - \sigma_{1,n} | \leq \prod\limits_{j=1}^{m} \wedge (\sigma_{j,n})^{-\frac{n\delta}{\delta_j}}$,

where $\delta = [\mathbf{Q}(\sigma_{1,n}; \ldots; \sigma_{m,n}) : \mathbf{Q}]$ and $\delta_j = [\mathbf{Q}(\sigma_{j,n}) : \mathbf{Q}]$.

Then the numbers $\sigma_1, \ldots, \sigma_m$ are algebraic independent.

Proof of Theorem. Let $K = \mathbf{Q}(\alpha_1, \ldots, \alpha_m)$, $d = [K : \mathbf{Q}]$ and

$$0 < | \alpha_m | < | \alpha_{m-1} | < \ldots < | \alpha_1 |.$$

It holds $1 \leq u \leq \dfrac{\sigma}{\sigma - 1}$ and $a_n \leq A_n \leq a_n^{u+\varepsilon_1}$ for $n \geq n_0$, $\varepsilon_1 > 0$ sufficiently small, where $A_n = l.\,c.\,m.\,(a_0, \ldots, a_n)$. Let

$$\sigma_{j,n} = \sum_{\nu=0}^{n} \frac{1}{a_\nu} \alpha_j^{e_\nu}$$

then we get

$$| f(\alpha_j) - \sigma_{j,n} | = | r_{j,n} | = \left| \frac{\alpha_j^{e_{n+1}}}{a_{n+1}} + \frac{\alpha_j^{e_{n+2}}}{a_{n+2}} + \ldots \right| \leq$$

$$\leq \frac{| \alpha_j |^{e_{n+1}}}{a_{n+1}} \left[1 + \left| \frac{a_{n+1}}{a_{n+2}} \right| . | \alpha_j |^{e_{n+2} - e_{n+1}} + \ldots \right].$$

It follows from (2) that $\lim\limits_{n \to \infty} \dfrac{a_n}{a_{n+1}} = 0$. Therefore we obtain for $n \geq n_1(\alpha_j) \geq n_0$

$$| r_n | \leq \frac{2 \,.\, | \alpha_j |^{e_{n+1}}}{a_{n+1}}. \tag{5}$$

Moreover we have from (5) for $n \geq n_2(\alpha_j) \geq n_1(\alpha_j)$

$$|r_n| \leq \frac{1}{3} \cdot \frac{|\alpha_j|^{e_n}}{a_n}. \tag{6}$$

We have further from (6)

$$\left|\sum_{\nu=n}^{\infty} \frac{1}{a_\nu} \alpha_j^{e_\nu}\right| \leq \left|\frac{1}{a_n} \alpha_j^{e_n}\right| + |r_n| \leq \frac{4}{3} \cdot \frac{|\alpha_j|^{e_n}}{a_n}$$

and

$$\left|\sum_{\nu=n}^{\infty} \frac{1}{a_\nu} \alpha_j^{e_\nu}\right| \geq \frac{|\alpha_j|^{e_n}}{a_n} - |r_n| \geq \frac{2}{3} \cdot \frac{|\alpha_j|^{e_n}}{a_n}.$$

So we obtain for $n \geq n_2(\alpha_j)$

$$\frac{2}{3} \cdot \frac{|\alpha_j|^{e_n}}{a_n} \leq \left|\sum_{\nu=n}^{\infty} \frac{1}{a_\nu} \alpha_j^{e_\nu}\right| \leq \frac{4}{3} \cdot \frac{|\alpha_j|^{e_n}}{a_n}. \tag{7}$$

We obtain for $N > N_2 = \max(n_2(\alpha_1), \ldots, n_2(\alpha_m))$

$$\begin{aligned} 0 < |f(\alpha_{j+1}) - \sigma_{j+1, N}| &< \frac{4}{3} \cdot \frac{|\alpha_{j+1}|^{e_{N+1}}}{a_{N+1}} \leq \\ &\leq \frac{2}{3n} \cdot \frac{|\alpha_j|^{e_{N+1}}}{a_{N+1}} \leq \frac{1}{n} |f(\alpha_j) - \sigma_{j,n}| \end{aligned} \tag{8}$$

for $1 \leq j \leq m$, if $2n \leq \left|\frac{\alpha_j}{\alpha_{j+1}}\right|^{e_{N+1}}$. Because of $\min\limits_{1 \leq j < m} \left|\frac{\alpha_j}{\alpha_{j+1}}\right| > 1$ it exists for every $n \in \mathbf{N}$ a natural number $N_3 = N_3(n, \alpha_1, \ldots, \alpha_m)$ such that for all $N \geq N_3$ it holds

$$2n \leq \left(\min_{1 \leq j < m} \left|\frac{\alpha_j}{\alpha_{j+1}}\right|\right)^{e_{N+1}}.$$

For arbitrary $n \in \mathbf{N}$ we get (8) for $1 \leq j < m$ and for $N \geq \max(N_2, N_3)$ which is the first condition of Durand's lemma.

For the proof (ii) we need the following lemma of Cijsouw and Tijdemann [[1], Lemma 1, p. 302].

LEMMA 2. Let α be an algebraic number of degree m and height H. Suppose d is a positive integer such that $d\alpha$ is an algebraic integer. Then

$$H \leq (2d \cdot \max(1, \overline{|\alpha|}))^m.$$

For sufficiently large n and for sufficiently small $\varepsilon_2 > 0$ we get from (4) and Lemma 2 that

$$H(\sigma_{j,n}) \leq a_n^{ud+\varepsilon_2}. \tag{9}$$

We have therefore

$$\wedge(\sigma_{j,N}) = 2^{\partial(\sigma_j, N)} \cdot L(\sigma_{j,n})$$
$$\leq 2^d \cdot (1 + \partial(\sigma_{j,N})) \cdot H(\sigma_{j,N}) \leq 2^d \cdot (1 + d) \cdot a_N^{ud+\varepsilon_2}$$

and

$$\wedge(\sigma_{j,N})^{\overline{\frac{\delta}{\partial(\sigma_j, N)}}} \leq \wedge(\sigma_{j,N})^d \leq 2^{d^2} \cdot (1 + d)^d \cdot a_N^{(ud+\varepsilon_2)d} .$$

Because of (3) for every $n \in \mathbb{N}$ there exists an integer $N_4 = N_4(n, \alpha_1, \ldots, \alpha_m)$ such that for infinitly many $N \geq N_4$ it holds

$$0 < |f(\alpha_1) - \sigma_{1,N}| < \frac{4}{3} \cdot \frac{|\alpha_1|^{e_{N+1}}}{a_{N+1}} < \prod_{j=1}^{m} \wedge(\sigma_{j,N})^{-\overline{\frac{n^\delta}{\partial(\sigma_j, N)}}}. \qquad (10)$$

To every $n \in \mathbb{N}$ we correspond the least integer N for which $N > \max(N_2, N_3, N_4)$ and (10) hold. We call this integer $N(n)$. We put in Durand's Lemma

$$\sigma_{j,n} := \sigma_{j,N(n)} \text{ for } j = 1, 2, \ldots, m \; ; \; n = 1, 2, \ldots,$$

so we obtain the conditions (i) and (ii) from (8) and (10). Hence the theorem is proved.

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ÖZET

Bu çalışmada Durand Lemması kullanılarak bazı transandant sayıların cebirsel bağımsız oldukları gösterilmektedir.