ON THE CONJUGACY CLASSES OF $p^2 : GL_2(p)$ — "p ODD PRIME"

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The object of this paper is to develop a general method for constructing the conjugacy classes of $p^2 : GL_2(p)$, where p^2 is an elementary abelian p-group of order p^2.

INTRODUCTION

A particular procedure has been followed in [1] to construct the conjugacy classes of the split extension $p^2 : GL_2(p)$. The object of this paper is to develop a general method for constructing the conjugacy classes of $p^2 : GL_2(p)$, where p^2 is an elementary abelian p-group of order p^2. This procedure can be used to construct the conjugacy classes of the split extension $p^n : k$ where k is any finite group. A brief description of the character table of $p^2 : GL_2(p)$ is also given, the character table of $p^2 : GL_2(p)$ plays a big role in the construction of the character table of the maximal subgroup $p^{1+2} : GL_2(p)$ of the projective symplectic group $PSP_4(p)$ p-prime [4], where p^{1+2} is the extra special group of order p^3, this is because $(p^{1+2} : GL_2(p)) / Z(p^{1+2}) \simeq p^2 : GL_2(p)$ where $Z(p^{1+2})$ is the center of p^{1+2} in $p^{1+2} : GL_2(p)$, the extra special group $p^{1+2} = < a, b \mid a^p = b^p = (a\,b)^p = [a, b]^p = 1 >$, where $[a, b] = a^{-1}\, b^{-1}\, a\, b$.

1. THE CONJUGACY CLASSES OF $GL_2(p)$

The conjugacy classes of $GL_2(p)$ have been taken from Steinberg paper [5], and they are presented below. Let ρ and σ be a primitive element of $GF(p)^*$ and $GF(p^2)^*$ respectively such that $\rho = \sigma^{p+1}$, where $GF(p)^* = GF(p) \setminus \{0\}$.

Family	Element	Number of Classes	Number of Elements in each Class
A_1	$\begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix}$	$p-1$	1
A_2	$\begin{pmatrix} \rho^a & \\ 1 & \rho^a \end{pmatrix}$	$p-1$	p^2-1
A_3	$\begin{pmatrix} \rho^a & \\ & \rho^b \end{pmatrix}$ $a \neq b$	$\frac{1}{2}(p-1)(p-2)$	$p(p+1)$
B	$\begin{pmatrix} \sigma^a & \\ & \sigma^b \end{pmatrix}$ $a \neq \text{mult}(p+1)$ $b \not\equiv ap \bmod (p^2-1)$	$\frac{1}{2}p(p-1)$	$p(p-1)$

2. THE CONJUGACY CLASSES OF $p^2 : GL_2(p)$

Denote $p^2 : GL_2(p)$ by $H : K$, to find the conjugacy classes of the split extension $H : K$, we need to find the conjugacy classes of a general element (h,k). Two elements (h_1, k_1) and (h_2, k_2) cannot be conjugate if $(1, k_1)$ is not conjugate to $(1, k_2)$. We can assume that $k_1 = k_2$. Then in order to see whether (h_1, k_1) and (h_2, k_1) are conjugate, we need only conjugate by elements (x, y) such that :

$$(x, y)(h_1, k_1)(x, y)^{-1} = (h_2, k_1).$$

This means that (h_1, k_1) is conjugate to (h_2, k_1) if $(x, y)(h_1, k_1)(x, y)^{-1} = (h_2, k_1)$, for some (x, y), and also this means that (h_1, k_1) is conjugate to (h_2, k_1) if and only if (h_2, k_1) lies in the orbit of (h_1, k_1) under the set of all elements (x, y) such that $(x, y)(h, k_1)(x, y)^{-1} = (h', k_1)$, where $h, h' \in H$ (i. e. stabilizer of the coset $\{(h, k_1) \mid h \in H\}$). Clearly $\{(h, 1)\}$ lies in the stabilizer of $\{(h, k_1)\}$. Since

$$(h, 1)(h', k_1)(h^{-1}, 1) = (h, 1)(h' h^{-1}, k_1) = (h h' h^{-1}, k_1),$$

where $h h' h^{-1}$ might not be h' (if H is not abelian), H is contained in stabilizer of $\{(h, k_1) \mid h \in H\}$.

Also $(h, x)(h', k_1)(h, x)^{-1} = (*, x k_1 x^{-1}) = (*, k_1)$ if and only if $(1, x) \in C_K(k_1)$, and so the stabilizer of the coset $\{(h, k_1)\}$ is $H : C_K(k_1)$, where $C_K(k_1)$ is the centralizer of k_1 in K.

The elementary abelian p-group H can be considered as a 2-dimensional vector space $v_2(p)$ over $GF(p)$. Let $k \in K$ be a representative of the conjugacy class $\widehat{k}$. The classes of $H : K$ which lie below k are of the form hk for some h's $\in H$. The action of K on H,

$$h \xrightarrow{k} h^k = k^{-1} h k$$

can be identified with

$$\underline{u} \xrightarrow{k} \underline{u} k$$

where $\underline{u}$ is the 2-tuple which corresponds to h with respect to the basis $A = \{(1, 0), (0, 1)\}$ of $V_2(p)$, and the element hk can be represented by 3×3 matrix

$$\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & \\ 0 & k \end{array}\right].$$

Because if $k_1, k_2 \in K = GL_2(p)$ and $\underline{u}_1, \underline{u}_2$ are the two 2-tuples which correspond to $h_1, h_2 \in H$, respectively, we have

$$\left[\begin{array}{c|c} 1 & \underline{u}_1 \\ \hline 0 & \\ 0 & k_1 \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{u}_2 \\ \hline 0 & \\ 0 & k_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{u}_1 k_2 + \underline{u}_2 \\ \hline 0 & \\ 0 & k_1 k_2 \end{array}\right]$$

which corresponds to $(h_1, k_1)(h_2, k_2) = (h_1^{k_2} + h_2, k_1 k_2)$.

Now we give a general description for the construction of the conjugacy classes of $H : K$.

Choose an element $(h^*, k) \in H : K$, this element can be identified with $\left[\begin{array}{c|c} 1 & \underline{u}^* \\ \hline 0 & \multirow{2}{*}{k} \\ 0 & \end{array}\right]$, where $\underline{u}^*$ is the 2-tuple corresponding to h^* with respect to the basis A, then we have

$$\left[\begin{array}{c|c} 1 & \underline{u}_1 \\ \hline 0 & \\ 0 & I \end{array}\right]\left[\begin{array}{c|c} 1 & \underline{u}^* \\ \hline 0 & \\ 0 & k \end{array}\right]\left[\begin{array}{c|c} 1 & -\underline{u}_1 \\ \hline 0 & \\ 0 & I \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{u}_1 k + \underline{u}^* \\ \hline 0 & \\ 0 & k \end{array}\right]\left[\begin{array}{c|c} 1 & -\underline{u}_1 \\ \hline 0 & \\ 0 & I \end{array}\right] =$$

$$= \left[\begin{array}{c|c} 1 & \underline{u}_1 k + \underline{u}^* - \underline{u}_1 \\ \hline 0 & \\ 0 & k \end{array}\right]$$

This multiplication can be abbreviated to

$$(\underline{u}_1, I)\,(\underline{u}^*, k)\,(-\underline{u}_1, I) = (\underline{u}_1 k + \underline{u}^* - \underline{u}_1, k).$$

We first determine the length of the block of imprimitivity containing $(\underline{u}^*, \underline{k})$ by considering expressions of the form

$((ru_{11} + u_1^* - u_{11} + tu_{21}, su_{11} + u_2^* - u_{21} + vu_{21}), k)$ where $\underline{u}_1 = (u_{11}, u_{21})k = \begin{pmatrix} r & s \\ t & v \end{pmatrix}$ and $\underline{u}^* = (u_1^*, u_2^*)$. Suppose that $r = 1$ and $t = 0$, this means that we get $\left((0, *), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right)$ which is the same orbit. Now if $\underline{u}^* = (u_1^*, u_2^*) \neq \underline{0}$ and if we conjugate $\left((u_1^*, u_{21}\rho^a + u_2^* - u_{21}), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right)$ by $\begin{pmatrix} l & \\ & m \end{pmatrix}$ we get an orbit of form $\left((l^{-1} u_1^*, *), \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right)$ of length $p(p-1)$, this means that we have two conjugacy classes of $H : K$ lie below $\begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}$; their representatives are $\left(\underline{0}, \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right)$ and $\left(\underline{u}^*, \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right)$, $\underline{u}^* \neq \underline{0}$ and the order of these classes are p, $p(p-1)$ respectively. The other conjugacy classes of K were treated in a similar manner. The complete results are given in the following table :

Class Representative	$\left(\underline{0}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)$	$\left(\underline{u}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right) \underline{u} \neq 0$	$\left(\underline{0}, \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right) a \neq p-1$	$\left(\underline{u}, \begin{pmatrix} 1 & \\ & \rho^a \end{pmatrix}\right) a \neq p-1, u \neq 0$	$\left(\left(\underline{0}, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\right)\right)$	$\left(\underline{u}, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\right) \underline{u} \neq 0$	$\left(\underline{0}, \begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix}\right) a \neq p-1$
Number of classes	1	1	$p-2$	$p-2$	1	1	$p-2$
Orbit length	1	p^2-1	p	$p(p-1)$	p	$p(p-1)$	p^2
Centralizer	$p^2(p^2-1)(p^2-p)$	$p^2(p^2-p)$	$p(p^2-1)(p^2-p)$	$(p+1)(p^2-p)$	$p(p^2-1)(p^2-p)$	$p(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$

Class Representative	$\left(\underline{0}, \begin{pmatrix} \rho^a & \\ & \rho^a \end{pmatrix}\right) a \neq p-1$	$\left(\underline{0}, \begin{pmatrix} \rho^a & \\ & \rho^b \end{pmatrix}\right) a \neq b$	$\left(\underline{0}, \begin{pmatrix} \sigma^a & \\ & \sigma^b \end{pmatrix}\right) \begin{matrix} a \neq \text{mult}(p+1) \\ b \not\equiv ap \bmod (p^2-1) \end{matrix}$
Number of classes	$p-2$	$\frac{(p-2)(p-3)}{2}$	$\frac{1}{2}p(p-1)$
Orbit length	p^2	p^2	p^2
Centralizer	$(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$	$(p^2-1)(p^2-p)$

The total number of the conjugacy classes of $p^2 : GL_2(p)$ is p^2+p-1. The character table of $p^2 : GL_2(p)$ can be constructed as follows: We extend the whole character table of $GL_2(p)$ to $p^2 : GL_2(p)$. The character table of $GL_2(p)$ has been taken from [5] and presented below. Next we induce the

1-representations of $GL_2(p)$ to $p^2 : GL_2(p)$. The extension gives $p^2 - 1$ irreducible characters of $p^2 : GL_2(p)$ and the induction gives $p - 1$ irreducible characters. The tensor product of one of these $p-1$ irreducible characters with an irreducible character of $p^2 : GL_2(p)$ of degree $p - 1$ completes the character table of $p^2 : GL_2(p)$.

Note : The extension, induction and tensor product of characters can be easily handled using Clifford Programme [2].

CHARACTERS OF $GL_2(p)$

In this table, χ_p^r for example, will denote a character of degree p. The superscript being used to distinguish between two characters of the same degree.

Element	χ_1^n	$\chi_p^{(n)}$	$\chi_{p+1}^{(m,n)}$	$\chi_{p-1}^{(n)}$
	$n=1, 2, \dots, p-1$ $\varepsilon^{p-1}=1$	$n=1, 2, \dots, p-1$ $\varepsilon^{p-1}=1$	$m, n=1, 2, \dots, p-1$ $m \neq n$; $(m,n)\equiv(n,m)$ $\varepsilon^{p-1}=1$	$n=1, 2, \dots, p^2-1$ $n \neq$ mult $(p+1)$ $\varepsilon^{p^2-1}=1$
A_1	ε^{2na}	$p\ \varepsilon^{2na}$	$(p+1)\ \varepsilon^{(m+n)a}$	$(p-1)\ \varepsilon^{na(p+1)}$
A_2	ε^{2na}	0	$\varepsilon^{(m+n)a}$	$-\varepsilon^{na(p+1)}$
A_3	$\varepsilon^{n(a+b)}$	$\varepsilon^{n(a+b)}$	$\varepsilon^{ma+nb} + \varepsilon^{na+mb}$	0
B_1	ε^{na}	$-\varepsilon^{na}$	0	$-(\varepsilon^{na} + \varepsilon^{np})$

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ÖZET

Bu çalışmada, p^2 mertebesi p^2 olan bir elemanter abelyen p-grubu göstermek üzere, $p^2 : GL_2(p)$ nin eşlenik eleman sınıflarını inşa etmek için genel bir yöntem verilmektedir.