İstanbul Üniv. Fen Fak. Mat. Der. 50 (1991), 179-184

ON THE ANALYSIS OF THE COSETS OF T = E : GL_n (p), p PRIME

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The main purpose of this paper is to study some properties of the cosets of the semidirect product $E : GL_n(p)$, where E is an elementary abelian normal subgroup of order p^n and Γ/E is isomorphic to the general linear group $GL_n(p)$. These properties are of great importance for the construction of the conjugacy classes of T.

1. The matrix form of T

The elementary abelian group E can be regarded as an n-dimensional vector space over $GF(p)$. Let $A \in GL_n(p)$ be a representative of the conjugacy class $cl(p)$ of $GL_n(p)$.

The action of $GL_n(p)$ on E

$$A(e) = e^A = A^{-1}\, e\, A \quad \text{for } A \in GL_n(p) \text{ and } e \in E$$

can be identified with

$$\underline{v} \xrightarrow{A} \underline{v}\, A$$

where $\underline{v}$ is an n-tuple which corresponds to e w.r.t. the standard basis $B = \{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$. And the element $Ae \in \Gamma$ can represented by the $(n+1)(n+1)$ matrix

$$\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & \\ 0 & \\ \vdots & A \\ 0 & \end{array}\right]$$

because if A_1, A_2 are two elements of $GL_n(p)$ and $\underline{v}_1$, $\underline{v}_2$ are the two n-tuples which corresponds to $e_1, e_2 \in E$ respectively, we have

$$\left[\begin{array}{c|c} 1 & \underline{v}_1 \\ \hline 0 & A_1 \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{v}_2 \\ \hline 0 & A_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{v}_1 A_2 + \underline{v}_2 \\ \hline 0 & A_1 A_2 \end{array}\right]$$

which corresponds to

$$(A_1, e_1)(A_2, e_2) = (A_1 A_2, e^{A_2} + e_2). \tag{1.1}$$

2. Analysis of the cosets of Γ

The map $\Phi : \Gamma \longrightarrow GL_n(p)$ defined by $\Phi \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] = A \in GL_n(p)$ is a homomorphism, this is clear from (1.1).

Lemma. Let $\mathscr{F}$ denote the stabilizer in Γ of $\Phi^{-1}(A)$ and ε denote the stabilizer in E of h in $\Phi^{-1}(A)$, then $|\Theta(h)| = \dfrac{p^n}{|\varepsilon|}$ where $\Theta(h)$ is the orbit of $\Phi^{-1}(A)$ corresponding to h under the action of E on $\Phi^{-1}(A)$.

Proof. It is clear that E is a normal subgroup of $\mathscr{F}$, let $T = \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & B \end{array}\right] \in \mathscr{F}$ and $h = \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & A \end{array}\right]$ then

$$\Phi\left(\left[\begin{array}{c|c} 1 & -\underline{v}B^{-1} \\ \hline 0 & B^{-1} \end{array}\right]\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & A \end{array}\right]\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & B \end{array}\right]\right) =$$

$$= \Phi\left(\left[\begin{array}{c|c} 1 & -\underline{v}B^{-1}AB + \underline{u}B + \underline{v} \\ \hline 0 & B^{-1}AB \end{array}\right]\right) = \Phi\left(\left[\begin{array}{c|c} 1 & -\underline{v}A^B + \underline{u}B + \underline{v} \\ \hline 0 & A^B \end{array}\right]\right) =$$

$$= \Phi\left(\left[\begin{array}{c|c} 1 & -\underline{v}A + \underline{u}B + \underline{v} \\ \hline 0 & A \end{array}\right]\right) = A$$

this means that $\mathscr{F}/\varepsilon$ is isomorphic to the centralizer of A in $GL_n(p)$ i. e. $\mathscr{F} = E.\,C_{GL_n(p)}(A)$ where $E.\,C_{GL_n(p)}(A)$ is the nonsplit extension of E by the centralizer of A in $GL_n(p)$. Also the orbits of E on $\Phi^{-1}(A)$ all have the same length, for let $\underline{w} \in \varepsilon$ then

$$-\underline{w}\left(\underline{u}^*\left[\begin{array}{c|c} 1 & -\underline{u} \\ \hline 0 & A \end{array}\right]\right)(\underline{w}) = \left[\begin{array}{c|c} 1 & (-\underline{w} + \underline{u}^* + \underline{w})A + \underline{u} \\ \hline 0 & A \end{array}\right] =$$

$$= \left[\begin{array}{c|c} 1 & (-\underline{w} + \underline{u}^* + \underline{w})A + (-\underline{w} + \underline{w})A + \underline{u} \\ \hline 0 & A \end{array}\right] =$$

$$= \left(-\underline{w}\left[\begin{array}{c|c} 1 & \underline{u}^* \\ \hline 0 & A \end{array}\right]\underline{w}\right)\left(-\underline{w}\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & A \end{array}\right]\underline{w}\right) = \underline{u}^*\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & A \end{array}\right]$$

so ε is the stabilizer in E of $\underline{u}^*\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & A \end{array}\right] \in \Phi^{-1}(A)$ and hence $|\theta(h)| = \dfrac{p^n}{|\varepsilon|}$.

Remark 1. Assume that $|\varepsilon| = p^r$ where r divides n. Let $\theta_1, 0_2, \ldots, \theta_{p^r}$ be the orbits of E on $\Phi^{-1}(A)$ and $\Sigma_1, \Sigma_2, \ldots, \Sigma_q$ be the orbits of $\mathscr{F}$ on $\Phi^{-1}(A)$. Then each Σ_i is an orbit of E or is a union of some orbits of E. Also the orbits $\theta_i \subseteq \Sigma_i$ are blocks of primitivity of $\mathscr{F}$, see [3] .

If $\sigma \in \Sigma_i$ then $|C_\Gamma(\sigma)| = \dfrac{|\mathscr{F}|}{|\Sigma_i|}$ and since $\Gamma \diagdown E \approx \mathscr{F}$, the extension of E by Γ, then $|C_\Gamma(\sigma)| = \dfrac{p^n}{d_i p^{n-r}} \left| \dfrac{C(A)}{GL_n(p)} \right| = \dfrac{p^r}{d_i} \left| \dfrac{C(A)}{GL_n(p)} \right|$ where d_i is the number of distinct orbits of E contained in Σ_i.

Lemma. Let $A \in GL_n(p)$ then

(i) A determines a homomorphism $A : E \longrightarrow E$ defined by

$$A(\underline{e}) = \underline{e}^{\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]} = \underline{e}\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e}) =$$

$$= \underline{e}\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}\left[\begin{array}{c|c} 1 & -\underline{v}A^{-1} \\ \hline 0 & A^{-1} \end{array}\right]$$

where $\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]$ is a coset representative in Γ.

(ii) If $\beta \in \mathscr{F}$, then the action of β on E commutes with this homomorphism.

Proof. (i) The above map is well defined i. e. it is independent of the choice of the coset representative $\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]$ for let $\underline{e}^* \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{e}^*A + \underline{v} \\ \hline 0 & A \end{array}\right]$ be another representative of the same coset, then

$$A(\underline{e}) = \underline{e}^{\left[\begin{array}{c|c} 1 & \underline{e}^*A + \underline{v} \\ \hline 0 & A \end{array}\right]} =$$

$$= \underline{e}\left[\begin{array}{c|c} 1 & \underline{e}^*A + \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e}) - \underline{e}\left[\begin{array}{c|c} 1 & \underline{e}^*A + \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}\left[\begin{array}{c|c} i & -\underline{v}A^{-1} \\ \hline 0 & A^{-1} \end{array}\right](-\underline{e}^*) =$$

$$= \underline{e}\ \underline{e}^*\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}\left[\begin{array}{c|c} 1 & -\underline{e}^*A^{-1} \\ \hline 0 & A \end{array}\right][-\underline{e}^*] =$$

$$= \underline{e}\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}\left[\begin{array}{c|c} 1 & \underline{e}^*A^{-1} - \underline{v}A^{-1} - \underline{e}^*A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] =$$

$$= \underline{e}\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}\left[\begin{array}{c|c} 1 & -\underline{v}A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] = \underline{e}^{\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]}.$$

To prove that $A : E \longrightarrow E$ is a homomorphism, let $\underline{e}_1$, $\underline{e}_2$ be two elements in E, then

$$A(\underline{e}_1 \underline{e}_2) = (\underline{e}_1 \underline{e}_2)^{\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]} =$$

$$= (\underline{e}_1 \underline{e}_2)\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e}_1 \underline{e}_2) =$$

$$= \underline{e}_1 \underline{e}_2 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}_1 \underline{e}_2 \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] =$$

$$= \underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}_1 \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] =$$

$$= \underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}_1 \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] \underline{e}_2 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}_2 \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] =$$

$$= \underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e}_1) \cdot \underline{e}_2 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e}_2) =$$

$$= \underline{e}_1^{\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]} \underline{e}_2^{\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]} = A(\underline{e}_1) A(\underline{e}_2).$$

(ii) Let $\beta = \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \in \mathscr{F}$, then

$$\left(\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right]\left(\underline{e}\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right](\underline{e})\right)\left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right] =\right.$$

$$= \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right]\left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right] =$$

$$= \left(\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right]\right) \cdot$$

$$\left(\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right]\left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right]\right) =$$

$$= \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right] \underline{e}^* \cdot$$

$$\cdot \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{u} B^{-1} \\ \hline 0 & B^{-1} \end{array}\right]\left[\begin{array}{c|c} 1 & -\underline{v} A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] [-\underline{e}^*] =$$

$$= \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & -\underline{u}\,B^{-1} \\ \hline 0 & B^{-1} \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right].$$

$$\underline{e} \left[\begin{array}{c|c} 1 & -\underline{u}\,B^{-1} \\ \hline 0 & B^{-1} \end{array}\right] \left[\begin{array}{c|c} 1 & -\underline{u}\,A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] =$$

$$= \underline{e}^{\beta} \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] (\underline{e}^{\beta})$$

where $\underline{e}^{\beta} = \beta\,\underline{e}\,\beta^{-1}$ .... (1), and

$$\underline{e}^* \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] = \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \left[\begin{array}{c|c} 1 & -\underline{u}\,B^{-1} \\ \hline 0 & B^{-1} \end{array}\right].$$

Remark 2. From (1) we can see that $\mathscr{F}$ normalizes ε^* and I where $\varepsilon^* = \left\{ \underline{e} \in E : \left[\begin{array}{c|c} 1 & \underline{v} + \underline{e}\,A \\ \hline 0 & A \end{array}\right] \right\} = \underline{e}$ which is the stabilizer in E of $\left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right]$ and $I = \{A(\underline{e}) : \underline{e} \in E\}$. Also

$$\underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] (\underline{e}_1) \cdot \underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] =$$

$$= \underline{e}_1 \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \underline{e}_1 \left[\begin{array}{c|c} 1 & -\underline{v}\,A^{-1} \\ \hline 0 & A^{-1} \end{array}\right] \underline{e} \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] =$$

$$= \underline{e}_1\,\underline{e}_2\,\underline{e} \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] = \underline{e}' \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right].$$

So the orbits of I under multiplication are the blocks of the cosets.

Remark 3. Let $r_1 \leq r_2 \leq \ldots \leq r_q$, where r_i is the number of blocks in Σ_i. If $r_l = 1$ then Γ has a class with centralizer $\varepsilon^* . C_{GL_n(p)}(A)$, where $\varepsilon^* . C_{GL_n(p)}(A)$ is the extension of $C_{GL_n(p)}(A)$ by ε^*. The action of $\mathscr{F}$ on the blocks is isomorphic to the group action $\varepsilon^* : C_{GL_n(p)}(A)$.

Proof. Let $\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \in \varepsilon^*. C_{GL_n(p)}(A)$ then

$$\left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] \left(\underline{e} \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right] \right) = \left[\begin{array}{c|c} 1 & \underline{u} \\ \hline 0 & B \end{array}\right] (\underline{e}) \left[\begin{array}{c|c} 1 & \underline{v} \\ \hline 0 & A \end{array}\right],$$

the block-orbits correspond to the action of $\varepsilon^* . C_{GL_n(p)}(A)$ on E/I of order p^r with $\{0\}$ corresponding to Σ_i.

Hence each class of Γ in the coset of A corresponds to an orbit of $C_{GL_n(p)}(A)$ on the group E/I.

N. B. The conjugacy classes of Γ where studied in detail in [2], and the above results can abbriviate a lot of computations carried out in [1] and in [2].

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ÖZET

Bu çalışmada, E mertebesi P^n olan bir elemanter abelyen normal alt grup ve T/E genel lineer grup $GL_n(p)$ ye izomorf olmak üzere, $E:GL_n(p)$ yarı direkt çarpımının kalan sınıflarının bazı özelikleri incelenmektedir.