CONE MAXIMAL POINTS IN LINEAR TOPOLOGICAL PRODUCTS*)

M. TURINICI

A maximality result due to Sterna-Karwat is shown to have a natural extension to the product setting. Some related questions occasionated by these developments are also discussed.

1. PRELIMINARIES

Let E be a Hausdorff (real) topological vector space. By a *cone* in E we shall mean any part C of E with

$$C + C \subseteq C \; ; \; \lambda C \subseteq C, \; 0 < \lambda \in R \; ; \; 0 \in C. \tag{1.1}$$

Given such an object, denote

$$\text{lin}(C) = C \cap (-C), \quad \text{pt}(C) = C \cap (\text{lin}(C))^c$$

(Here, for any subset X of E, X^c stands for the absolute complement of X). Of course, $\text{lin}(C)$ is the larger linear subspace included in C. And, $\text{pt}(C)$ is a cone without origin; that is, (1.1) takes place -with $\text{pt}(C)$ in place of C- but without its last part.

For a nonempty subset Y of E, denote by $\max(Y, C)$ the (eventual empty) subset of all $z \in Y$ with the maximal (mod C) property

$$w \in Y, \; z \leq w \,(\text{mod } C) \Longrightarrow z \leq w \,(\text{mod lin}(C)),$$

where, by $\leq$ (mod C) we understand the quasi-ordering over E induced by C, in the usual way

$$x \leq y \text{ if and only if } y - x \in C.$$

We shall be interested in the sequel to determine (structural) conditions upon C so that the following property -referred to as C is a *comp-max cone*- be fulfilled

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$\max(H, C)$ is nonempty, for each (nonempty) compact part H of E (1.2)

(Here, as usually, the term "compact" is taken as in Kelley [9, ch. 5, §1]; that is, each net in H admits an accumulation point in H). To give a practical motivation of this, we note (cf. Penot [12]) that an element of $\max(H, C)$ may be deemed as a *Parote efficient point* of a certain multi-criterion optimization problem. But, our interest has also a theoretical motivation; this emerges from the fact that (1.2) may be interpreted as a (linear) topological version of the Zorn Maximality Principle. In this perspective, as a basic answer to the problem we dealt with, we must consider the one obtained in 1954 by Ward [18]:

Theorem 1. Suppose that

K_1) C is a closed cone.

Then, C has the comp-max property, in the sense

$\max(H, C)$ is nonempty and cofinal (mod C) in H, for each (nonempty) compact part H of E. (1.2′)

Now, for the above precised reasons, it is natural to ask of whether or not is Theorem 1 extendable in the sense of (1.2). Note that any such extension must be purely technical; because, as results from the paper by Borwein [1], Theorem 1 is actually equivalent to the Axiom of Choice. The answer is positive. For example, to state an extension of this type, call the cone C in E, *correct* when $\mathrm{cl}(C)+\mathrm{pt}(C) \subseteq C$ (Here, "cl" stands for the closure operator). Now, it may be shown that, under

K_2) C is a correct cone

in place of K_1), the above conclusion of Theorem 1 is retainable. This is essentially the 1989 Luc's result [11, ch. 2, §3] (where a more general compactness assumption about H is being used). But this is only a step towards a larger extonsion. To be more specific, let us call the cone C in E, *admissible*, when

L = closed subspace of $\mathrm{lin}(\mathrm{cl}(C))$ and $\mathrm{cl}(L \cap C)$ = linear subspace imply $L \cap C$ = linear subspace. (1.3)

This property was introduced in Sterna-Karwat [15]. It is important to note that

K_3) C is an admissible cone

includes K_2) (This fact will be clarified in Section 3). Of course, the implication in (1.3) must be checked only if $\mathrm{cl}(L \cap C)$ is not reduced to the null subspace; since, otherwise, it becomes trivial. For example, this happens when C fulfills a condition like in Corley [5]; namely

K_4) $\mathrm{cl}(C)$ is pointed ($\mathrm{lin}\,\mathrm{cl}(C) = \{o\}$).

An interesting situation described by such a condition is the following: Denote

$$[Y] = (Y + C) \cap (Y - C)\ ,\ Y \subseteq E,$$

and term the subset Y of E, *full*, in case $Y = [Y]$. Call the underlying cone C, *normal* when the origin of E has a fundamental system $\mathcal{U}$ of full neighborhoods. Now, it turns out that

K_4') C is a normal cone

is a particular case of K_4). To verify this, we firstly note that all members in $\mathcal{U}$ may be taken as balanced (hence symmetric) parts of E. Secondly, by a well known formula (to be found, e.g., in Schaefer [13, ch. 1, §1])

$$\mathrm{cl}(C) = \cap\{C + U;\ U \in \mathcal{U}\} = \cap\{C - U;\ U \in \mathcal{U}\}.$$

This finally gives (by the Hausdorff property)

$$\mathrm{lin}\ \mathrm{cl}(C) = \cap\{(C + U) \cap (-C + U);\ U \in \mathcal{U}\} = \cap\,\mathcal{U} = \{o\}$$

and the claim follows. Passing to the general case, call the cone C in E, *non-flat* when for each closed subspace L of lin cl(C) with cl$(L \cap C)$ = nondegenerate linear subspace, it is the case that $L \cap C$ has a nonnempty interior in cl$(L \cap C)$ (endowed with the relative (linear) topology). Then

K_5) C is non-flat

is also a particular case of K_3). In fact, let the closed subspace L of lin cl(C) be taken as above. Then, by the classical Eidelheit's separation theorem (see, e.g., Cristescu [6, ch. 1, §2)] it is not hard to show that

$$L \cap C = \mathrm{cl}(L \cap C) = \text{linear subspace}, \tag{1.4}$$

and the claim follows. Now, K_5) is fulfilled when

K_5') $C' = \{x \in C;\ x \neq o\}$ is open (in E).

To verify this, let the closed subspace L of lin cl(C) be such that cl$(L \cap C)$ is a nondegenerate linear subspace. We therefore have

$$L \cap C \neq \{o\} \text{ (or, equivalently, } L \cap C' \neq \phi).$$

By the double inclusion

$$L \cap C' = (L \cap C) \cap C' \subseteq \mathrm{cl}(L \cap C) \cap C' \subseteq L \cap C',$$

we deduce $L \cap C'$ is open in cl$(L \cap C)$ (endowed with the relative (linear) topology); hence, the interior of $L \cap C$ in cl$(L \cap C)$ is nomenpty, as claimed. The similar statement of the above quoted author (Remark 2.2, iii), in [15]) corresponds to C' being, in addition, convex in K_5'). This is, of course, redundant by the argument above; and moreover, it makes K_5) be vacuously satisfied. Indeed, C must be pointed under such an assumption about C'; for otherwise, if x is a nonzero element of lin(C) then, $o = (1/2)\ (x + (-x))$ must belong to C', contradiction. So, combining with (1.4),

$$L \cap C = \text{lin}(L \cap C) = L \cap \text{lin}(C) = \{o\}.$$

But, this is absurd by the admitted hypothesis about $\text{cl}(L \cap C)$; hence the claim. Finally, it is werth noting that, by Theorem C (ch. 1, §2) and Lemma A (ch. 2, §11) in Helmes [8], condition K_5) will be also fulfilled when

K_6) lin cl(C) is finite dimensional.

In particular, this condition will trivially hold for any cone C in E provided $\dim(E) = $ finite; and this shows K_3) includes effectively K_1).

Further aspects concerning this notion will be delineated in Section 3. Here, we merely note that the following statement obtained in the 1986 paper by Sterna-Karwat we already quoted is -technically speaking- a strict extension of the above one :

Theorem 2. Suppose K_3) is admitted. Then, C is a comp-max cone, in the sense

$$\begin{array}{l}\max(H, C) \text{ is nonempty and cofinal (mod cl}(C)) \text{ in } H,\\ \text{for each (nonempty) compact part } H \text{ of } E.\end{array} \tag{1.2*}$$

The proof goes by transfinite induction. It consists essentially in the following. Take

$$L_0 = E,\ C_0 = C,\ H_0 = H,\ y_0 = \text{arbitrary in } H_0.$$

Generally, for the ordinal λ, put

$$\begin{array}{l}L_\lambda = \text{lin}(\text{cl}(C_{\lambda-1})),\ C_\lambda = L_\lambda \cap C_{\lambda-1},\ y_\lambda = \text{arbitrary in}\\ \max(H_{\lambda-1}, \text{cl}(C_{\lambda-1})),\ H_\lambda = H_{\lambda-1}(y_\lambda + L_\lambda)\end{array} \tag{1.5}$$

when λ is of the first kind ($\lambda - 1$ exists), and

$$\begin{array}{l}L_\lambda = \cap\{L_\xi ; \xi < \lambda\},\ C_\lambda = \cap\{C_\xi ; \xi < \lambda\},\\ H_\lambda = \cap\{H_\xi ; \xi < \lambda\},\ y_\lambda = \text{arbitrary in } \max(H_\lambda, \text{cl}\, C_\lambda))\end{array} \tag{1.5'}$$

when λ is of the second kind ($\lambda - 1$ does not exist). Now, it may be shown that

$$\max(H, C) \supseteq \max(H_\xi, C_\xi), \text{ for all } \xi.$$

So, to complete the proof, it suffices that

$$\max(H_\xi, C_\xi) \neq \phi, \text{ for some } \xi.$$

This clearly holds when

$$C_\xi = \text{lin}(C_\xi), \text{ for some } \xi.$$

Hence, the only point is what happens in the opposite case. Then

$$L_\xi \text{ is constructible for all } \xi, \tag{1.6}$$

and

$$L_\xi \supset L_\eta, \text{ whenever } \xi < \eta. \tag{1.7}$$

(Here, $\supset$ denotes the strict inclusion). This, in the author's view, yields a contradiction if one takes into account the Axiom of Replacement (see, e.g., Cohen [4, ch. 2, §3]) in the presence of the other axioms of the Zermelo-Fraenkel set theory. Unfortunately, the author's argument is unacceptable from a logical viewpoint. Indeed, it will follow by (1.6)+(1.7) that the transfinite sequence $\mathscr{L}$ of all such L_ξ appears as an effective/actual model of the class W of all ordinals, whose contradictory character is well known (see, e.g., Sierpinski [14, ch. 14, §2]). And consequently, any argument founded on this premise has no logical value. But, in this case, the last part of the above proof hangs in the air; and then, the question arises of whether or not is this removable. To give an appropriate answer, let us remember that any transfinite induction process is to be considered over a certain segment $W(\mu)=\{\xi\in W\,;\,\xi<\mu\}$ of ordinals, where the limiting ordinal μ is *apriori* given by our data. Having this in mind, let k denote the cardinal number of the family

$$\mathscr{S}(C)=\{L\,;\,L=\text{closed subspace of lin cl}(C)\}.$$

Also, let m be any (Hartogs type) aleph number for which relation $m\le k$ does not hold (The existence of such an aleph is a consequence of a result in Sierpinski [14, ch. 16, §1] proved without the aid of the Axiom of Choice). Denote by μ the initial ordinal associated to m; that is, $\min\{\xi\in W\,;\,\text{card}\,W(\xi)=m\}$. Now, assume the transifinite sequence $\mathscr{L}$ of all such L_ξ is constructible over $W(\mu)$. Then, $\mathscr{L}$ is order isomorphic with $W(\mu)$; and consequently, it has the cardinality m. This, however, is in contradiction with

$$m=\text{card}(\mathscr{L})\le\text{card}(\mathscr{S}(C))=k.$$

Hence, the sequence in question must stop for a certain ordinal $\beta<\mu$; and the proof of Theorem 2 is complete. We must however say that the argument is technically complicated. It is our aim in the following to get a simplified form of this; details will be given in Section 4. The main tool of these investigations is the well known Bourbaki fixed point principle [2]; some constructive aspects of it were delineated in Section 2. We also show, in Section 3, that the statement of Theorem 2 may be put in a product setting. And, finally, Section 5 is devoted to the converse question to Theorem 2.

2. THE BOURBAKI FIXED POINT PRINCIPLE

Let A be a nonempty set and $\le$, an *ordering* over A. We let $f:A\to A$ be a *progressive* mapping; that is

H$_1$) $x\le f(x)$, for all x in A.

Concerning the question of what can be said about the set Fix(f) of all fixed points for this mapping, the following facts will be in effect for us. Let us call the ambient set A, *semi-complete*, when

H_2) $\sup(X)$ exists, for each part X of A.

It is now clear that, with such a hypothesis about A, the set $\mathrm{Fix}(f)$ is not empty; in fact, $\sup(A)$ is an element of it. But, for the developments below, this will not suffice. Our objective is to determine, for any point $a \in A$, the "shortest" iterative process starting from a, having as endpoint an element of $\mathrm{Fix}(f)$. In this direction, the following result due to Bourbaki [2] must be noted.

Proposition 1. Let f and A be as in H_1) plus H_2). Then, for each $a \in A$, there may be determined a well ordered part $B = B(a)$ of A with the properties

a) $a \in B$, $f(B) \subseteq B$, and $\sup(X) \in B$, whenever $X \subseteq B$.

b) $u, v \in B \Longrightarrow u \geq v$ or $f(u) \leq v$ (i.e., $x \to f(x)$ is the immediate successor mapping of B).

c) $\sup(B)$ is the only fixed point of f in B (that is, $x \in B$ and $x \neq \sup(B)$ imply $x < f(x)$).

Actually, B may be defined as the intersection of all nonempty parts Y of A fulfilling a) (with Y in place of B); see the quoted paper for details. Now, in view of b), the iterative process we are looking for is that defined by the well ordered set B. To explain this, we need some preliminary facts. Let W stand for the class of all ordinals; it has a contradictory character, by the well known Burali-Forti paradoxe (see, e.g., Sierpinski [14, ch. 14, §2]). However, when one restricts the considerations to a Grothendieck *universe* $\mathscr{G}$ (introduced as in Hasse and Michler [7, ch. 1, §2]) this contradictory character is removed for the class $W(\mathscr{G})$ of all *admissible* (modulo $\mathscr{G}$) ordinals (that is, ordinals generated by well-ordered (non-contradictory) sets in $\mathscr{G}$). In the following, we drop the subscript $(\mathscr{G})$ for simplicity. So, by an ordinal (in W) we shall actually understand a $\mathscr{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathscr{G}$. This will be referred to as an *admissible ordinal* (to indicate the fact that a generic universe $\mathscr{G}$ is considered in its construction). Clearly,

$$\xi = \text{admissible ordinal}, \ \eta \leq \xi \Longrightarrow \eta = \text{admissible ordinal}.$$

Hence, in the formula

$$W(\lambda) = \{\xi \in W; \xi < \lambda\}, \ \lambda \in W,$$

the set W in the brackets may be taken as the "absolute" set of all ordinals.

Let in the following the generic Grothendieck universe $\mathscr{G}$ be so large that A is a member/part of it. For each $a \in A$, let the transfinite iterates of f at this point be introduced as

$$
\begin{aligned}
f^{o}(a) &= a \text{ (the considered point)} \\
f^{\lambda}(a) &= f(f^{\lambda-1}(a)), \text{ if } \lambda \text{ is a first kind ordinal} \\
&= \sup\{f^{\xi}(a)\,;\,\xi<\lambda\}, \text{ otherwise}
\end{aligned}
\tag{2.1}
$$

(Here, by "ordinal" we actually mean "admissible ordinal". But, in the following, we shall not make any distinction between these; because, in view of the accepted hypothesis, only admissible ordinals are considered). The definition above is meaningful, in view of the semi-completeness condition H_2). Moreover, by H_1),

$$f^{\xi}(a) \leq f^{\eta}(a), \text{ whenever } \xi \leq \eta\,;$$

that is, the transfinite sequence $(f^{\xi}(a))$ increases. Now, in principle, it would be possible that such a sequence be *nonstationary*; i.e.,

$$f^{\xi}(a) < f^{\eta}(a), \text{ provided } \xi < \eta\,. \tag{2.2}$$

Therefore, what the above result says is that the transfinite sequence $(f^{\xi}(a))$ becomes stationary beyond a certain ordinal number $\beta = \beta(a)$ (which also depends on f and A), in the sense

$$f^{\beta}(a) = f^{\xi}(a), \text{ for all } \xi \geq \beta.$$

Precisely, let γ denote the order type of $(B, \leq)$. Hence B is order isomorphic with $W(\gamma)$; and this, in conjunction with c), shows γ is necessarily a first kind ordinal (that is, $\beta=\gamma-1$ exists) and proves the assertion above in view of the remark made in b). Two important facts about this ordinal must be noted:

i) The ordinal in question is admissible (in a sense we already precised) with respect to the ambient Grothendieck universe $\mathscr{G}$ including/containing A. In fact, under this assumption (about A), the family $\mathscr{P}(A)$ is necessarily a member/part of the same universe. As a consequence, any subfamily of $\mathscr{P}(A)$ -in particular, the one appearing in the definition of B- is again endowed with such a property. But, in this case, B is a member/part of the same universe; and then, by definition, β is an admissible ordinal.

ii) The process of determining this ordinal is not depending on the Axiom of Choice. Nevertheless, it is true that, with the aid of this axiom, a more direct proof of the statement above is available. Precisely, denote $k = \text{card}\,(A)$, and let m be another cardinal with

$$k < m = \text{aleph number}$$

(Note that each of these relations requires the Axiom of Choice; because, the former is not in general valid without the trichotomy law, and the latter may fail, in general without the aleph hypothesis; cf. Sierpinski [[14], ch. 16, §1]). Denote also by μ the initial ordinal associated to m. It follows by the same procedure as in Section 1 that the transfinite sequence $(f^{\xi}(a))$ cannot satisfy a relation like (2.2) over $W(\mu)$; hence, it becomes stationary beyond a certain ordinal $\beta < \mu$ and the claim is proved.

Now, as a consequence of these facts, the mapping from A into itseif

$$f^{\infty}(a) = \sup_{(\xi)} \{f^{\xi}(a)\}, \ a \in A$$

is well defined. The basic properties of it are collected in

Proposition 2. The following are valid :

m) $f^{\xi}(a) \leq f^{\infty}(a)$, for all ξ and all $a \in A$; and consequently, f^{∞} is progressive (over A),

n) $f^{\xi}(f^{\infty}(a)) = f^{\infty}(f^{\xi}(a)) = f^{\infty}(a)$, for all ξ and all $a \in A$; hence, in particular, $f^{\infty}(a)$ is an element of $\mathrm{Fix}(f)$, for each $a \in A$.

p) if f is increasing over A, then so is f^{∞}.

Proof. The first part is obvious. For the second part, it suffices to note that the property in question is valid for $\xi = 1$; and, from this, the general case follows easily by transfinite induction. The third part is an immediate consequence of the fact that, under the precised requirements, all transfinite iterates (f^{ξ}) of the considered function are increasing over A. Hence the result. q.e.d.

Now, as f^{∞} is progressive too, the transfinite sequence of all its iterates $((f^{\infty})^{\xi}(a))$ is again stationary beyond a certain ordinal $\gamma = \gamma(a)$, for each $a \in A$. Hence, the mapping (from A to itself)

$$(f^{\infty})^{\infty}(a) = \sup_{(\xi)} \{(f^{\infty})^{\xi}(a)\}, \ a \in A,$$

is well defined, etc. This procedure may continuate indefinitely and seems to generate interesting problems. But, for the developments below, these will not be needed.

3. MAIN RESULTS

We now return to the framework of Section 1. Let E be a Hausdorff (real) topological vector space. Denote by $\mathscr{C}(E)$ the class of all cones in E; it is easily shown to be semi-complete with respect to the converse inclusion ($\supseteq$) over $\mathscr{P}(E)$. Let the self-mapping T of $\mathscr{C}(E)$ be introduced as

$$T(X) = X \cap \operatorname{lin} \operatorname{cl}(X), \ X \in \mathscr{C}(E)$$

(Here, as precised, "cl" is the closure operator associated to the ambient (linear) topology over E). This map is trivially shown to be progressive (again with respect to the converse inclusion). Hence, for any cone C in E, the transfinite sequence of iterates $(T^{\xi}(C))$ is constructible, in accordance with the convention

$$T^0(C) = C \text{ (the considered cone)}$$
$$T^\lambda(C) = T(T^{\lambda-1}(C)), \text{ if } \lambda \text{ is a first kind ordinal} \quad (3.1)$$
$$= \cap \{T^\xi(C);\ \xi < \lambda\}, \text{ otherwise}$$

(Here, the term "ordinal" means "admissible ordinal" with respect to a sufficiently large Grothendieck universe $\mathcal{G}$ containing/including E. But in the following the word "admissible" will be deleted). Now, it is clear that

$$T^\xi(C) \supseteq T^\eta(C), \text{ whenever } \xi \leq \eta.$$

So, the question arises of whether or not is a property like

$$T^\xi(C) \supset T^\eta(C), \text{ provided } \xi < \eta, \quad (3.2)$$

avoidable, for any cone C in E. Moreover -supposing this would be true- it is natural to ask of which supplementary properties has the associated self-map T^∞ of $\mathcal{C}(E)$, introduced as

$$T^\infty(X) = \bigcap_{(\xi)} \{T^\xi(X)\},\ X \in \mathcal{C}(E).$$

An appropriate answer to these is precised in

Proposition 3. Let the notations above be accepted. Then

A) For any cone C in E there may be determined an ordinal $\beta = \beta(C)$ (which also depends on E and T) such that the transfinite sequence $(T^\xi(C))$ becomes stationary beyond β, in the sense

$$T^\beta(C) = T^\xi(C), \text{ for all } \xi \geq \beta.$$

B) The self-mapping T^∞ of $\mathcal{C}(E)$ (introduced as above) is well defined and increasing (with respect to the usual inclusion over $\mathcal{C}(E)$).

Proof. The first part follows by Proposition 1 (in the preceding section) and the remarks following it. The second part is immediate, via Proposition 2, because T is increasing with respect to the usual inclusion over $\mathcal{C}(E)$. q.e.d.

Now, it must be remarked that part A) of the statement above is due to Sterna-Karwat [17]. The author's proof consists in the following:

"Observe that $(T^\xi(C))$ is a chain in $(\mathcal{C}(E), \supseteq)$. Since, by the Hausdorff Maximality Principle (see, e.g., Kelley [9, ch. 0, § 12]) there exists a maximal chain in every ordered set, we must have an ordinal β with the stated property. Hence the result."

That such an argument cannot be accepted from a logical viewpoint follows, essentially, by the same way as the one precised in Section 1. In fact, the author's remark involving the chain property of $(T^\xi(C))$ must be necessarily coupled with condition (3.2) being accepted; for, otherwise, there is nothing to prove. Now, two cases are open before us:

a) No reference is made about a Grothendieck universe $\mathscr{G}$ which should contain/include E. Then, "ordinal" means "absolute ordinal". And, in such a situation, the transfinite sequence in question becomes an effective model of the contradictory class W of all ordinals. Therefore, we cannot assign any logical value (true/false) to the premise of the argument above.

b) An implicit assumption is made about a Grothendieck Universe $\mathscr{G}$ which should contain/include E. Then, the transfinite sequence $(T^{\xi}(C))$ is no longer contradictory; but, from this one cannot see to what extent is available the Hausdorff Maximality Principle to get -in a direct way- the conclusion of the above argument.

Under these preliminaries, we may now return to the notion of admissible cone introduced in Section 1. A useful characterization of it may be given under the lines below.

Proposition 4. The following are equivalent:

i) C is admissible (in the sense of (1.3)),

ii) $D =$ subcone of C and $\mathrm{cl}(D) =$ linear subspace imply
$C \cap \mathrm{cl}(D) =$ linear subspace,

iii) $T^{\infty}(C) = \mathrm{lin}(C)$.

Proof. iii) $\Longrightarrow$ i). Let L be a closed subspace of $\mathrm{lin}\,\mathrm{cl}(C)$ with $\mathrm{cl}(L \cap C) =$ linear subspace. It is not hard to see, via transfinite induction, that

$$L \cap C = L \cap T^{\xi}(C), \text{ for all } \xi. \tag{3.3}$$

Actually, this relation holds for any subspace L of E with $\mathrm{cl}(L \cap C) =$subspace (of E). The deep part of the induction argument is the verification for $\xi = 1$. This, in turn is immediate, in view of

$$L \cap C = L \cap C \cap \mathrm{lin}(\mathrm{cl}(L \cap C)) \subseteq L \cap C \cap \mathrm{lin}(\mathrm{cl}(L) \cap \mathrm{cl}(C)).$$

As a direct consequence,

$$L \cap C = L \cap T^{\infty}(C) = \text{linear subspace;}$$

and so, the assertion is proved.

i) $\Longrightarrow$ ii). Let D be a subcone of C with $\mathrm{cl}(D) =$ linear subspace (Here, the term "subcone" means: a subset of a cone which is itself a cone). Put $L = \mathrm{cl}(D)$. We have (by some elementary arguments)

$$\mathrm{cl}(L \cap C) = L \ (= \text{closed linear subspace of } \mathrm{lin}\,\mathrm{cl}(C)).$$

Hence, by i), $L \cap C = \mathrm{cl}(D) \cap C$ is a linear subspace.

ii) $\Longrightarrow$ iii). Denote for simplicity $D = T^{\infty}(C)$. That $\mathrm{cl}(D)$ is a linear subspace of $\mathrm{lin}\,\mathrm{cl}(C)$ is simply to verify, via $D = T(D)$. Hence, by ii), $C \cap \mathrm{cl}(D)$ is a linear subspace. On the other hand, we have by (3.3), with $L = \mathrm{cl}(D)$, that

$$C \cap \mathrm{cl}(D) = T^{\xi}(C) \cap \mathrm{cl}(D), \text{ for all } \xi.$$

This immediately gives (by the adopted notation)

$$C \cap \mathrm{cl}(D) = D \cap \mathrm{cl}(D) = D\,;$$

or, in other words, D is a linear subspace (of $\mathrm{lin}(C)$). Now, it is easy to verify, via transfinite induction, that

$$\mathrm{lin}(C) = \mathrm{lin}(T^{\xi}(C)), \text{ for all } \xi \tag{3.4}$$

(As before, the deep part of the induction argument is the one concerning the case $\xi = 1$; and this, by the definition of T, is immediate). As a consequence,

$$\mathrm{lin}(C) = \mathrm{lin}(D) = D,$$

and the assertion is proved. q.e.d.

It is to be noted that the equivalence i)$\Longrightarrow$iii) was obtained in Sterna-Karwat [17], through a similar technique. Also, we formally conclude from the above argument that the admissibility condition upon C may be also expressed, respectively, by

i*) the property (1.3), with L, an arbitrary subspace of E (instead of being closed in $\mathrm{lin}\,\mathrm{cl}(C)$);

ii*) the property ii), with the subcone D of C being, in addition, linearly compatible with C (i.e., $\mathrm{lin}(D) = \mathrm{hn}(C)$);

iii*) $T^{\infty}(C)$ is a linear subspace of E.

Immediate applications of these developments are:

P) Let $\mathcal{M}$ be a family of admissible cones, and put $C = \cap\mathcal{M}$. We have, by part B) of Proposition 3,

$$T^{\infty}(C) \subseteq \cap\{T^{\infty}(K)\,;\, K \in \mathcal{M}\} = \cap\,\{\mathrm{lin}(K)\,;\, K \in \mathcal{M}\} = \mathrm{lin}(C).$$

On the other hand, it follows by the relations (3.4) that $T^{\infty}(C) \supseteq \mathrm{lin}(C)$. Hence, $T^{\infty}(C) = \mathrm{lin}(C)$; and this proves (cf. Sterna-Karwat [17]):

> The class of all admissible cones in E is semi-complete with respect to the converse inclusion. (3.5)

Q) We introduced in Section 1 the notion of correctness for a cone C in E, by means of

$$\mathrm{cl}(C) + \mathrm{pt}(C) \subseteq C \text{ (or, equivalently, } \mathrm{cl}(C) + \mathrm{pt}(C) \subseteq \mathrm{pt}(C))\,;$$

see Luc [11, ch. 1, §1] for details. It is our intention to study the relationships between this notion and the one of admissibility. So, let C be a correct cone in E, and denote for simplicity $D = T^{\infty}(C)$. By hypothesis, it clearly follows that $\mathrm{cl}(D) + \mathrm{pt}(D) \subseteq C$. On the other hand, from the remarks made in Proposition 4, cl(D) is a linear space; and consequently, cl(D) + pt(D) = cl(D). Hence, combining these facts, cl$(D) \subseteq C$. And, from this,

$$\mathrm{cl}(D) \subseteq C \cap \mathrm{cl}(D) = D,$$

if we again take into account the developments in that statement. This immediately gives

$$D = \mathrm{cl}(D) = \text{linear subspace}.$$

Summing up, we have the implication

$$C \text{ is correct} \Longrightarrow C \text{ is admissible.} \tag{3.6}$$

This cannot be reversed; because, e.g., a non-closed linear space of E is admissible but not correct.

A dimensional consequence of these facts may be stated along the following lines. Let $\{E_i\,;\, i \in I\}$ be a family of Hausdorff (real) topological vector spaces, and $E = \Pi\{E_i\,;\, i \in I\}$ their *topological product* (introduced as in Cristescu [6, ch. 1, §2]). For each $j \in I$, let γ_j denote the projection of E into E_j, defined as

$$\gamma_j(x) = x_j, \text{ when } x = (x_i)_{i \in I} \text{ is in } E.$$

Put $\Gamma = (\gamma_i)_{i \in I}$. We shall term the cone C of E, *Γ-decomposable*, in case

$$C = \Pi\{C_i\,;\, i \in I\}, \text{ where } C_i = \gamma_i(C),\ i \in I.$$

The class of all such cones in E will be denoted by $\mathscr{D}(E)$. It is easily shown to be semi-complete with respect to the converse inclusion ($\supseteq$) over $\mathscr{C}(E)$.

The following fact is useful for us:

Proposition 5. Let $C = \Pi\{C_i\,;\, i \in I\}$ be a T-decomposable cone in E. Then, C is admissible in E if and only if C_i is admissible in E_i, for each $i \in I$. In other words: the property of being admissible is closed with respect to cartesian products.

Proof. Suppose that, for $i \in I$, the cone C_i in E_i is admissible. Then, evidently,

$$C_i^* = \gamma_i^{-1}(C_i) \text{ is admissible in } E, \text{ for each } i.$$

And this, in conjunction with (3.5), proves the sufficieny. Conversely, suppose the Γ-decomposable cone $C = \Pi\{C_i\,;\, i \in I\}$ is admissible in E. Denote by T the mapping from $\mathscr{C}(E)$ to itself introduced as in a previous convention (where "cl" denotes the closure operator induced by the product topology). It follows by some well known facts involving topological products (see, e.g., Bourbaki [3, ch. 1, §4]) that the restriction of T to $\mathscr{D}(E)$ is a self-mapping of $\mathscr{D}(E)$. Precisely,

$$T(X) = \Pi\{T_i(X_i)\,;\, i \in I\},\ X = \Pi\{X_i\,;\, i \in I\} \in \mathscr{D}(E),$$

where, for each $i \in I$, the self-mapping T_i of $\mathscr{C}(E_i)$ is introduced again as in the discussed convention (with "cl" denoting this time the closure operator associated to the linear topology over E_i). And, from this, one immediately gets, by transfinite induction, that

$$T^{\xi}(C) = \Pi\{T_i^{\xi}(C_i)\ ;\ i \in I\}, \text{ for all } \xi. \tag{3.7}$$

Now, by part B) of Proposition 3, the self-map T_i^{∞} of $\mathscr{C}(E_i)$ is well defined, for each $i \in I$; likewise, the self-map T^{∞} of $\mathscr{C}(E)$ is again well defined. This, in conjunction with (3.7), yields

$$T^{\infty}(C) = \Pi\{T_i^{\infty}(C_i)\ ;\ i \in I\}.$$

But, in view of the accepted hypothesis, $T^{\infty}(C)$ is a subspace of E. Hence, $T_i^{\infty}(C_i) = \gamma_i(T^{\infty}(C))$ is a subspace of E_i, for each $i \in I$; and this, in view of a remark following Proposition 4, gives the desired conclusion. q.e.d.

We now introduce a basic convention. Let us call a cone C in E, *Γ-admissible*, in case

$$C_i = \gamma_i(C) \text{ is admissible in } E_i, \text{ for each } i \in I. \tag{3.8}$$

By the above statement we have that, for a Γ-decomposable cone in E, the notions of admissible and Γ-admissible are identical. Hence, taking into account Theorem 2, we formally deduce

Theorem 3. Let the Γ-decomposable cone C in E be Γ-admissible. Then, it necessarily has the comp-max property, in the sense of (1.2*).

Of course, this methodological implication may be reversed; i.e., Theorem 2 is a particular case of Theorem 3 (obtained whenever the index set I reduces to a single point).

A natural question is to see what happens when the underlying cone is no longer Γ-decomposable. For a partial answer, denote

$$\Gamma(C) = \Pi\{\gamma_i(C)\ ;\ i \in I\},\ C \in \mathscr{C}(E).$$

Clearly, $\Gamma(C)$ is a Γ-decomposable cone in E which, in addition, includes C (a cone in E).

Theorem 4. Let the cone C in E be Γ-admissible and linearly compatible with $\Gamma(C)$. Then, it has the comp-max property, in the sense precised by (1.2*).

Proof. Denote for simplicity $D = \Gamma(C)$. We have $\mathrm{lin}(D) = \mathrm{lin}(C) \subseteq C \subseteq D$.

So, if we apply the (increasing) map T^{∞} to these, one immediately gets $T^{\infty}(C) = \mathrm{lin}(C)$; or, in other words, C is admissible (by Proposition 4). But, in this case, Theorem 2 applies. Hence the conclusion. q.e.d.

So far, Theorem 4 above is a particular case of the main result in Section 1. But, the converse is also valid, as it can be directly seen.

The above statement -as well as the ones in the introductory part- are non-trivial only if the ambient Hausdorff linear topology over E is, roughly speaking, "not very strong". To explain this, let E be a (real) vector space. We introduce over E the Hausdorff linear topology having as zero-neighborhoods basis the class of all convex balanced sets for which the origin of E is an algebraic interior point. This object -called the *convex core topology*- is actually the strongest locally convex topology over E, as it can be directly seen.

Theorem 5. Let the ambient Hausdorff linear topology over E be stronger than the convex core topology. Then, any cone C in E has the comp-max property.

Proof. Let H be a (nonempty) compact part of E (with respect to the ambient (linear) topology). Then, necessarily, H is compact in the convex core topology. And this, combined with a known result (see, e.g., Kelley and Namioka [10, ch. 2, §6, ex. I]) gives

$H \subseteq L$, for some finite dimensional subspace L of E. But, in this case,

$$\max(H, C) = \max(H, C \cap L),$$

for any cone C in E. Adding to these informations the remarks in Section 1 following K_6), conclusion is clear. q.e.d.

This result partially answers a question raised by Corley [5]; namely

"Is it true that any cone C in E has the comp-max property with respect to any Hausdorff (linear) topology over the ambient space E?"

Precisely, Theorem 5 says that the class of all such topologies cannot be empty. On the other hand, as we shall see in Section 5, not any Hausdorff linear topology over E has such a property, when E is infinite dimensional. An interesting open problem is that of determining to what extent is the convex core topology the minimal one to solve the question above. We conjecture that the answer is affirmative.

4. A DIRECT PROOF OF THEOREM 2

As already precised in Section 1, the argument in Sterna-Karwat [15] may be appropriately completed so that an acceptable proof of Theorem 2 be reached. However, a close analysis shows it may be improved. It is our aim in the following to get such a simplified proof. Let E be a Hausdorff (real) topological vector space. The following "diagonal" version of Theorem 1 will be in effect for us.

Proposition 6. Suppose H is a (nonempty) compact part of E. And, $\alpha>0$ denoting an ordinal number, let $(C_\xi\,;\xi<\alpha)$ be a family of closed cones in E, which is descending with respect to the usual inclusion in $\mathscr{C}(E)$; namely

$$C_\xi \supseteq C_\eta\,,\ \text{whenever } \xi\leq\eta<\alpha. \tag{4.1}$$

Then, $K=\cap\{\max(H, C_\xi)\,;\xi<\alpha\}$ is nonempty and cofinal (mod C_0) in H.

Proof. Let $x\in H$ be arbitrary fixed. By Theorem 1, we find a $y_0\in\max(H,C_0)$ with $x\leq y_0$ (mod C_0). For this y_0 there exists, again by Theorem 1, a $y_1\in\max(H, C_1)$ with $y_0\leq y_1$ (mod C_1); note that, as $C_0\supseteq C_1$, one gets

$$y_0\leq y_1\,(\text{mod } C_0)\ (\text{hence } x\leq y_1\,(\text{mod } C_0)).$$

Generally, suppose that, for the ordinal number $\lambda<\alpha$, we constructed a net $(y_\xi)_{\xi<\lambda}$ in H, with the properties

$$y_\xi\leq y_\eta\,(\text{mod } C_\xi),\ \text{when } \xi<\eta<\lambda \tag{4.2}$$

$$y_\xi\in\max(H, C_\xi),\ \text{for each } \xi<\lambda. \tag{4.3}$$

If λ is a first kind ordinal, put $\lambda-1=\mu$. We thus have

$$y_\xi\leq y_\mu\,(\text{mod } C_\xi),\ \text{when } \xi\leq\mu.$$

Now, again by Theorem 1, choose a $y_\lambda\in\max(H, C_\xi)$ with $y_\mu\leq y_\lambda$ (mod C_λ). By (4.1) + (4.2), it is clear that

$$y_\xi\leq y_\lambda\,(\text{mod } C_\xi),\ \text{for each } \xi<\lambda\,; \tag{4.2'}$$

that is, (4.2) holds with $\eta=\lambda$ (We formally remark that (4.3) also holds with $\xi=\lambda$, by the choice of y_λ). If λ is a socond kind (limit) ordinal, the net $(y_\xi)_{\xi<\lambda}$ has, by the compactness of H, an accumulation point (in H), say t. In view of

$$\{y_\xi\,;\zeta\leq\xi<\lambda\}\subseteq y_\zeta+C_\zeta\,,\ \zeta<\lambda,$$

plus the closedness of C_ζ, one gets

$$y_\zeta\leq t\,(\text{mod } C_\xi),\ \text{when } \zeta<\lambda.$$

Now, again by Theorem 1, we may determine a $y_\lambda\in\max(H, C_\lambda)$ with $t\leq y_\lambda$ (mod C_λ). Hence (4.2′) is valid; and, from this, (4.2) holds with $\eta=\lambda$ (That (4.3) also holds for $\zeta=\lambda$ is trivial). Summing up, the net (y_ξ) is constructible over $W(\alpha)$ so that (4.2) + (4.3) be fulfilled (with $\lambda=\alpha$). But, in this case, the procedure we just employed for the ordinal λ may be used as well to produce a point $y\in H$ with

$$y_\zeta\leq y\,(\text{mod } C_\zeta),\ \text{for all } \zeta<\alpha.$$

The obtained point is an element of K (the set defined in the statement); and, moreover, $x\leq y$ (mod C_0). Hence the conclusion. q.e.d.

We now introduce a useful convention. Let C, D be a couple of cones in E with $C \supseteq D$. For a (nonempty) part Y of E, denote by $\max(Y; C, D)$ the (eventual empty) subset of all $z \in Y$ with the (maximal type) property

$$w \in Y,\ z \leq w \,(\mathrm{mod}\ C) \Longrightarrow z \leq w \,(\mathrm{mod}\ D).$$

Note at this time the trivial implication

$$D = \mathrm{lin}(C) \Longrightarrow \max(Y; C, D) = \max(Y, C). \tag{4.4}$$

Under the model of Section 1, let us say the pair (C, D) has the *comp-max property*, when

$$\max(H; C, D) \text{ is nonempty, for each (nonempty) compact part } H \text{ of } E. \tag{4.5}$$

Now, as an application of the developments above, one has

Theorem 6. Let C be a cone in E. Then, the pair $(C, T^{\infty}(C))$ has the comp-max property, in the sense

$$\begin{array}{l}\max(H; C, T^{\infty}(C)) \text{ is nonempty and cofinal (mod cl}(C)) \\ \text{in } H, \text{ for each (nonempty) compact part } H \text{ of } E.\end{array} \tag{4.5'}$$

Proof. Let the self-mapping T of $\mathscr{C}(E)$ be introduced as in Section 3. Denote for simplicity

$$C_{\xi} = T^{\xi}(C), \text{ for all } \xi\ ;\ D = T^{\infty}(C).$$

It follows by Proposition 3 that, an ordinal $\beta = \beta(C)$ may be found so that the (descending) transfinite sequence (C_{ξ}) becomes stationary beyond β; that is,

$$C_{\beta} = C_{\xi}, \text{ for all } \xi \geq \beta \quad (\text{hence } C_{\beta} = D).$$

The transfinite sequence $(\mathrm{cl}(C_{\xi})\ ;\ \xi < \beta + 1)$ is again descending (with respect to the usual inclusion in $\mathscr{C}(E)$). Let H be a (nonempty) compact part of E. By Proposition 6,

$$H_C = \cap\{\max(H, \mathrm{cl}(C_{\xi}))\ ;\ \xi < \beta + 1\}$$

is nonempty and cofinal (mod cl(C)) in H. We now claim that $H_C \subseteq \max(H; C, D)$ (and this will complete the argument). Let x be arbitrary fixed in H_C, and let $y \in H$ be such that $x \leq y$ (mod C); or, in other words, $x \leq y$ (mod C_0). We thus have $x \leq y$ (mod cl(C_0)); this, plus $x \in \max(H, \mathrm{cl}(C_0))$, gives $x \leq y$ (mod lin cl(C_0)) wherefrom (again by the information above)

$$x \leq y \ (\mathrm{mod}\ C_0 \cap \mathrm{lin\ cl}(C_0) = C_1).$$

Generally, assume that, for the fixed ordinal $\lambda < \beta + 1$, one has the information like

$$x \leq y \ (\mathrm{mod}\ C_{\xi}), \text{ for all } \xi < \lambda. \tag{4.6}$$

If λ is a first kind ordinal, then the argument above (with $C_{\lambda-1}$ in place of C_0) gives $x \leq y \,(\mathrm{mod}\; C_\lambda)$; i.e., (4.6) is true for $\xi = \lambda$. If λ is a second kind ordinal then, (4.6) plus the construction of C_λ yield the same conclusion. Hence, (4.6) is necessarily true over $W(\beta+1)$. In particular, this must happen for $\xi=\beta$; that is, $x \leq y \,(\mathrm{mod}\; D)$. Combining with our initial information yields $x \in \max(H; C, D)$. As x is arbitrary in H_C, the claim follows. q.e.d.

Now, Theorem 1 follows immediately from this statement, by virtue of (4.4). As a matter of fact, the converse implication also holds; hence these two results are equivalent to each other.

It is not without importance to specify that, in all these arguments, the (non-integer) scalar multiplication were not effectively used. Hence, these results will remain valid in case of E being a Hausdorff topological (additive) *abelian group* and C, a *semigroup* in E; that is, a part of E fulfilling (1.1) with Z in place of R. Actually, a close analysis shows that a further extension of these statements is obtainable in the context of Hausdorff topological spaces endowed with quasiorderings. We shall treat these questions elsewhere.

5. SOME CONVERSE RESULTS

Let again E be a Hausdorff (real) topological vector space. By the developments in Secton 1, we have that the class of admissible cones in E is included in the class of comp-max cones in E. Concerning this fact, it is natural to ask of what can be said about the converse inclusion (implication). Loosely speaking, a complete answer is to be given in the case of E being *metrizable* ; that is,

an invariant to translations metric $(x, y) \Longrightarrow d(x, y)$ exists so that its associated (linear) topology over E is equivalent to the initial one (5.1)

(Equivalently, this may be also expressed as the ambient space having a *countable* (neighborhoods) *basis*; see, e.g., Schaefer [13, ch. 1, §6] for details). The main results in this direction are the ones due to Sterna-Karwat [15,16]. It is our aim in the following to make a few remarks about these, imposed by some technical reasons to be explained. Some aspects involving the general (non-metrizable) case will be also discussed.

We start by remembering a notation made in Section 1. Namely, for each cone C in E, we denoted by $\mathrm{pt}(C)$ the set-difference between C and $\mathrm{lin}(C)$. As precised there, $\mathrm{pt}(C)$ is a cone without origin. Of course, this notion is operant only when C is not a linear space.

The following auxiliary fact will be needed.

Proposition 7. Let the cone C in E be singular, in the sense

$$C \text{ is not a linear subspace, but } \mathrm{cl}(C) \text{ is.} \tag{5.2}$$

Then, for each x in $\mathrm{pt}(C)$ and each neighborhood U of zero there exists y in $\mathrm{pt}(C)$ with $x+y\in U$.

Proof. Let $x\in \mathrm{pt}(C)$ be arbitrary fixed. Note that we anyway have $-x\in \mathrm{cl}(C)$. So, there exists a net $(y_j)_{j\in J}$ in C with $y_j\to(-x)$. Denote

$$z_j=2y_j+x,\ j\in J. \tag{5.3}$$

Clearly, $z_j\to(-x)$ (hence $z_j+x\to 0$) and

$$-z_j \text{ is not belonging to } C \text{ (i.e., } z_j\in \mathrm{pt}(C)),\ j\in J;$$

since, otherwise (for certain ranks j in J)

$$-x=2y_j+(-z_j)\in C, \text{ contradiction.}$$

This ends the argument. q.e.d.

As an application of this, we deduce

Proposition 8. Suppose the cone C in E is nonadmissible and lin cl(C) has a countable basis. Then, C is necessarily not comp-max.

Proof. By the admitted hypothesis, we may find a closed subspace L of lin cl(C) with $L\cap C$ being singular (in the sense of (5.2)). Now, in view of a remark in Section 3, it will suffice proving that

$$\max(H, L\cap C)=\phi, \text{ for some (nonempty) compact part } H \text{ of } L.$$

So, without any loss, one may replace the couple (E, C) by $(L, L\cap C)$; or, in other words, we assume E itself has a countable basis and C is a singular cone in E. Let $\{U_0, U_1, \ldots\}$ be a countable descending (module $\subseteq$) basis of zero-neighborhoods. Fix y in $\mathrm{pt}(C)$. By Proposition 7 we find a $y_0\in \mathrm{pt}(C)$ with $y+y_0\in U_0$. Further, as $y+y_0$ is in $\mathrm{pt}(C)$, we determine, again by Proposition 7, a $y_1\in \mathrm{pt}(C)$ with $y+y_0+y_1\in U_1$, etc. This procedure may be continued indefinitely, via ordinary induction, to get a sequence $\{y_0, y_1, \ldots\}$ in $\mathrm{pt}(C)$ with the above properties. Denote

$$z_0=-y,\quad z_1=y_0,\quad z_2=y_0+y_1,\ldots .$$

The (nonempty) subset $H=\{z_0, z_1, \ldots\}$ is evidently compact and $\max(H, C)=\phi$, because

$$z_i\le z_j \text{ (mod } C) \text{ is impossible for } i>j.$$

Hence the conclusion. q.e.d.

We are now in position to state

Theorem 7. Let the space E have a countable basis. Then, for a cone C in E, the property of being admissible is equivalent with the property of being comp-max.

As already said, this result was obtained by Sterna-Karwat [15]. The only point in which the argument differs from the original one is that concerning Proposition 7; or, to be more specific, the one related to the use of definition (5.3) for the net (z_j) used there. Although minor from a technical perspective, it is methodologically useful, because makes the argument above independent of (non-integer) scalar multiplications. That is, Theorem 7 remains valid in case of E being a Hausdorff topological abelian (additive) group and C, a semigroup in E.

Now, a natural question raised by this statement is to determine what happens beyond the metrizable context. As precised in Sterna-Karwat [16], the answer is, in general, negative. Precisely, let the ambient Hausdorff (linear) topology over E be the convex core topology (introduced as in Section 3). By Theorem 5, any cone C in E has the comp-max property. This, in particular, will be valid for that cone C in E consisting of the null vector in E and all nonzero vectors in E whose last coordinate with respect to a certain Hamel basis (e_ξ) is strictly positive. Of course, in such a construction E is infinite dimensional. But, in this case, by Theorem D (ch. 1, §2) in Holmes [8], we have $\mathrm{cl}(C) = E$; in other words, C is non-admissible, and the assertion is proved. Nevertheless, the answer in question is not essentially negative. To clarify this, let $\{E_i\,;\, i \in I\}$ be a family of Hausdorff (real) topological vector spaces and $E = \Pi\{E_i\,;\, i \in I\}$, its topological (hnear) product.

Proposition 9. Suppose the T-decomposable cone $C = \Pi\{C_i\,;\, i \in I\}$ in E is non-admissible, in the sense

$$\begin{array}{l}\text{for at least one } j \in I,\ C_j \text{ is non-admissible}\\ \text{and lin cl}(C_j) \text{ has a countable basis.}\end{array} \tag{5.4}$$

Then, C is not a comp-max cone.

Proof. By Proposition 8, we have promised, for that rank j, a (nonempty) compact part H_j of E_j with $\max(H_j, C_j) = \phi$. On the other hand, for any rank i different from j, let H_i be a (nonempty) compact subset of E_i. Denote $H = \Pi\{H_i\,;\, i \in I\}$; it is nonempty and compact, by the well known Thyconoff's theorem (see, e.g., Kelley [9, ch. 5, §3]). In addition, $\max(H, C)$ is empty, as it can be directly seen. Therefore, C is not a comp-max cone. q.e.d.

Now, clearly,

$$\text{lin cl}(C) = \Pi\{\text{lin cl}(C_i)\,;\, i \in I\}$$

does not admit a countable basis when the index set I is not denumerable; see Kelley-Namioka [10, ch. 2, §6] for details. Hence, conclusion of Proposition 8 may be valid even if lin cl(C) is non-metrizable.

Finally, as a completion of Theorem 7, we have

Theorem 8. Suppose all factors $\{E_i ; i \in I\}$ have a countable basis. Then, for a Γ-decomposable cone C in E, the property of being admissible is equivalent with the property of being comp-max.

In other words, the equivalence stated in Theorem 7 is extendable beyond the metrizable context. Note that the structural requirement of the statement above is a natural one; because, by Theorem 9 (ch. 2, §6) in Kelley and Namioka [10], any Hausdorff (real) topological vector space can be imbedded in a product of metrizable (real) topological vector spaces. As a matter of fact, this context is also the most appropriate one for the main results in Section 3. Some related aspects of these were discussed in Luc [11, ch. 2, §3].

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SEMINARUL MATEMATIC "AL. MYLLER"
UNIVERSITY OF IAŞI
6600 IAŞI
ROMANIA

ÖZET

Bu çalışmada, Sterna-Karwat'a ait bir maksimallik sonucu genelleştirilmekte ve bu arada ortaya çıkan bazı sorular tartışılmaktadır.