# SEMI-SYMMETRIC CONNEXION ON GENERALIZED CO-SYMPLECTIC MANIFOLDS

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Recently R. S. Mishra [2] defined some new manifolds. In this paper we have studied semi-symmetric connexion on Generalized Co-symplectic manifolds.

## 1. INTRODUCTION

Let there exist an odd-dimensional differentiable manifold $M_n$, of differentiability class $C^\infty$ on which there are defined a tensor field $F$ of type (1,1), a vector field $U$ and a 1-form $u$, satisfying for arbitrary vector fields $X, Y, Z, \ldots$

$$\begin{aligned} &\text{a) } \bar{\bar{X}} + X = u(X)U, \quad \text{b) } \bar{X} \overset{def}{=} F(X), \quad \text{c) } \bar{U} = 0, \\ &\text{d) } u(\bar{X}) = 0, \quad \text{e) } u(U) = 1, \quad \text{f) } n \text{ is odd} = 2\,\mathrm{m} + 1. \end{aligned} \tag{1.1}$$

Then $M_n$ is called an almost contact manifold.

An almost contact manifold $M_n$, on which a metric tensor $g$ satisfying

$$\text{a) } 'F(X, Y) = g(\bar{X}, Y) = -g(X, \bar{Y}), \qquad \text{b) } g(X, U) = u(X) \tag{1.2}$$

has been introduced is called an almost contact metric manifold [1].

An almost contact metric manifold satisfying

$$(D_X\, 'F)\,(Y, Z) = u(Y)\,(D_X\, u)\,(\bar{Z}) - u(Z)\,(D_X\, u)\,(\bar{Y}) \tag{1.3}$$

is called a generalized co-symplectic manifold [2].

If an almost contact metric manifold $U$ satisfies

$$\begin{aligned} &\text{a) } (D_X\, u)\,(\bar{Y}) = -\,(D_{\bar{X}} - u)\,(Y) = (D_Y u)\,(\bar{X}), \\ &\text{b) } (D_X\, u)\,(Y) = (D_{\bar{X}}\, u)\,(\bar{Y}) = -\,(D_Y u)\,(X), \end{aligned} \tag{1.4}$$

and

$$\text{c) } D_U\, F = 0,$$

then the manifold is said to be generalized co-symplectic manifold of first class or (nqs) manifold whose equation is [2]

$$(D_X {'F}) (Y, Z) = u(Y) (D_Z u) (\bar{X}) + u(Z) (D_{\bar{X}} u) (Y). \tag{1.5}$$

If on an almost contact metric manifold $U$ satisfies

$$\begin{aligned} &\text{a) } (D_X u) (\bar{Y}) = (D_{\bar{X}} u) (Y) = - (D_Y u) (\bar{X}) \Longleftrightarrow \\ &\text{b) } (D_X u) (Y) = - (D_{\bar{X}} u) (\bar{Y}) = - (D_Y u) (X), \end{aligned} \tag{1.6}$$

and

$$\text{c) } D_U F = 0,$$

then $U$ is said to be of the second class and the generalized co-symplectic manifold of second class whose equation is [2]

$$(D_X {'F}) (Y, Z) = u(Y) (D_{\bar{X}} u) (Z) + u(Z) (D_Y u) (\bar{X}). \tag{1.7}$$

The Nijenhuis tensor $N$ of $F$ is a tensor field of type (1,2) given by [1]

$$\begin{aligned} &\text{a) } N(X, Y) = (D_{\bar{X}} F) (Y) - (D_{\bar{Y}} F) (X) - \overline{(D_X F) (Y)} + \overline{(D_Y F) (X)} \\ &\text{b) } {'N}(X, Y, Z) = (D_{\bar{X}} {'F}) (Y, Z) - (D_{\bar{Y}} {'F}) (X, Z) + \\ &\qquad + (D_X {'F}) (Y, \bar{Z}) - (D_Y {'F}) (X, \bar{Z}). \end{aligned} \tag{1.8}$$

## 2. SEMI-SYMMETRIC CONNEXION B

Let $D$ be a Riemannian connexion on an almost contact manifold $M_n$. We consider another connexion $B$ on $M_n$ defined by [3]

$$B_X Y = D_X Y - u(X) Y. \tag{2.1}$$

The torsion tensor of $B$ is given by

$$S(X, Y) = u(Y) X - u(X) Y. \tag{2.2}$$

Hence the connexion $B$ is semi-symmetric connexion. We have

$$\begin{aligned} X {'F}(Y, Z) &= (B_X {'F}) (Y, Z) + {'F}(B_X Y, Z) + {'F}(Y, B_X Z) \\ &= (D_X {'F}) (Y, Z) + {'F}(D_X Y, Z) + {'F}(Y, D_X Z). \end{aligned}$$

Then [3]

$$(B_X {'F}) (Y, Z) = (D_X {'F}) (Y, Z) + 2u(X) {'F}(Y, Z). \tag{2.3}$$

The Nijenhuis tensor $N$ in terms of connexion $B$ is given by [3]

$$\begin{aligned} &\text{a) } N(X, Y) = (B_{\bar{X}} F) (Y) - (B_{\bar{Y}} F) (X) - \overline{(B_X F) (Y)} + \overline{(B_Y F) (X)} \\ &\text{b) } {'N}(X, Y, Z) = (B_{\bar{X}} {'F}) (Y, Z) - (B_{\bar{Y}} {'F}) (X, Z) + \\ &\qquad + (B_X {'F}) (Y, \bar{Z}) - (B_Y {'F}) (X, \bar{Z}). \end{aligned} \tag{2.4}$$

Theorem (2.1). If $F$ is killing, then on generalized co-symplectic manifold with semi-symmetric connexion $B$ we have

$$(B_X{}'F)(Y, Z) + (B_Y{}'F)(X, Z) = 2u(X)\,{}'F(Y, Z) + 2u(Y)\,{}'F(X, Z). \quad (2.5)$$

**Proof.** From (2.3) we have

$$(B_X{}'F)(Y, Z) + (B_Y{}'F)(X, Z) = (D_X{}'F)(Y, Z) + 2u(X)\,{}'F(Y, Z) + \\ + (D_Y{}'F)(X, Z) + 2u(Y)\,{}'F(X, Z).$$

Since ${}'F$ is killing then putting $(D_X{}'F)(Y, Z) + (D_Y{}'F)(X, Z) = 0$ in the above equation we get (2.5).

**Theorem (2.2).** If $U$ is killing then

$$(B_X{}'F)(Y, \bar{Z}) + (B_Y{}'F)(\bar{Z}, X) + (B_{\bar{Z}}{}'F)(X, Y) = \\ = {}'N(X, Y, Z) = 2u(Z)(D_{\bar{Y}}u)(\bar{X}). \quad (2.6)$$

**Proof.** From (2.4) (b) we have

$${}'N(X, Y, Z) - (B_X{}'F)(Y, \bar{Z}) - (B_Y{}'F)(\bar{Z}, X) - (B_{\bar{Z}}{}'F)(X, Y) = \\ = (B_{\bar{X}}{}'F)(Y, Z) - (B_{\bar{Y}}{}'F)(X, Z) - (B_{\bar{Z}}{}'F)(X, Y).$$

By (2.3) and (1.3) in the above equation we get

$${}'N(X, Y, Z) - (B_X{}'F)(Y, \bar{Z}) - (B_Y{}'F)(\bar{Z}, X) - (B_{\bar{Z}}{}'F)(X, Y) = \\ = -u(X)\{(D_{\bar{Y}}u)(\bar{Z}) + (D_{\bar{Z}}u)(\bar{Y})\} + \\ + u(Y)\{(D_{\bar{X}}u)(\bar{Z}) + (D_{\bar{Z}}u)(\bar{X})\} + \\ + u(Z)\{(D_{\bar{Y}}u)(\bar{X}) - (D_{\bar{X}}u)(\bar{Y})\}.$$

Since $U$ is killing then putting $(D_X u)(Y) + (D_Y u)(X) = 0$ in this equation we get at once (2.6).

**Theorem (2.3).** On a generalized co-symplectic manifold with semi-symmetric connexion $B$ we have

$$\text{a) } (B_X{}'F)(\bar{Y}, \bar{Z}) = 2u(X)\,{}'F(\bar{Y}, \bar{Z}) \\ \text{b) } (B_{\bar{X}}{}'F)(Y, \bar{Z}) + (B_{\bar{Y}}{}'F)(Z, \bar{X}) + (B_{\bar{Z}}{}'F)(X, \bar{Y}) = \\ = u(Y)(D_{\bar{X}}u)(\bar{Z}) + u(Z)(D_{\bar{Y}}u)(\bar{X}) + u(X)(D_{\bar{Z}}u)(\bar{Y}). \quad (2.7)$$

**Proof.** Barring $Y$ and $Z$ in (2.3) we find

$$(B_X{}'F)(\bar{Y}, \bar{Z}) = (D_X{}'F)(\bar{Y}, \bar{Z}) + 2u(X)\,{}'F(\bar{Y}, \bar{Z}).$$

From (1.3), using $(D_X{}'F)(\bar{Y}, \bar{Z}) = 0$, then

$$(B_X{}'F)(\bar{Y}, \bar{Z}) = 2u(X)\,{}'F(\bar{Y}, \bar{Z}).$$

By virtue of (2.3) we have

$$(B_{\overline{X}}\,'F)\,(Y, Z) = (D_{\overline{X}}\,'F)\,(Y, Z),$$

then

$$(B_{\overline{X}}\,'F)\,(Y, \overline{Z}) + (B_{\overline{Y}}\,'F)\,(Z, \overline{X}) + (B_{\overline{Z}}\,'F)\,(X, \overline{Y}) =$$
$$= (D_{\overline{X}}\,'F)\,(Y, \overline{Z}) + (D_{\overline{Y}}\,'F)\,(Z, \overline{X}) + (D_{\overline{Z}}\,'F)\,(X, \overline{Y}).$$

Using (1.3) in the above equation we obtain (2.7) b).

## 3. N. Q. S. MANIFOLD

**Theorem (3.1).** N.q.s. manifold is generalized co-symplectic of second class iff

$$(D_Z\, u)\,(\overline{X}) = (D_{\overline{X}}\, u)\,(Z). \tag{3.1}$$

**Proof.** We have

$$(D_X\,'F)\,(Y, Z) = u(Y)\,(D_Z\, u)\,(\overline{X}) + u(Z)\,(D_{\overline{X}}\, u)\,(Y).$$

Using (3.1) in this equation we get

$$(D_X\,'F)\,(Y, Z) = u(Y)\,(D_{\overline{X}}\, u)\,(Z) + u(Z)\,(D_Y\, u)\,(\overline{X})$$

which is the generalized co-symplectic manifold of second class. The converse is also true.

**Theorem (3.2).** On the nqs manifold the following hold:

$$\left.\begin{aligned} &\text{a) } 'N(X, Y, Z) = 2u(Z)\,(D_Y\, u)\,(X) \\ &\text{b) } 'N(X, \overline{Y}, Z) + 'N(\overline{X}, Y, Z) = 0. \end{aligned}\right\} \tag{3.2}$$

**Proof.** From (1.8) b) and (1.5) we have

$$'N(X, Y, Z) = u(Y)\,(D_Z\, u)\,(\overline{\overline{X}}) + u(Z)\,(D_{\overline{\overline{X}}}\, u)\,(Y) - u(X)\,(D_Z\, u)\,(\overline{\overline{Y}}) -$$
$$- u(Z)\,(D_{\overline{\overline{Y}}}\, u)\,(X) + u(Y)\,(D_{\overline{Z}}\, u)\,(\overline{X}) - u(X)\,(D_{\overline{Z}}\, u)\,(\overline{Y}).$$

Using (1.4) in the above equation we find (3.2) a). By virtue of (3.2) a),

$$'N(X, \overline{Y}, Z) + 'N(\overline{X}, Y, Z) = 2u(Z)\,\{(D_{\overline{Y}}\, u)\,(X) + (D_Y\, u)\,(\overline{X})\}.$$

Using (1.4) in this equation we get at once (3.2) b).

**Theorem (3.3).** On nqs manifold with the semi-symmetric connexion $B$ we have

$$\left.\begin{aligned} &(B_X\,'F)\,(Y, \overline{Z}) + (B_Y\,'F)\,(\overline{Z}, X) + (B_{\overline{Z}}\,'F)\,(X, Y) = \\ &\qquad = 'N(X, Y, Z) - 2u(Z)\,(D_Y\, u)\,(X). \end{aligned}\right\} \tag{3.3}$$

**Proof.** From (2.4) b) we have

$$'N(X, Y, Z) - (B_X\,'F)\,(Y, \bar{Z}) - (B_Y\,'F)\,(\bar{Z}, X) - (B_{\bar{Z}}\,'F)\,(X, Y) =$$
$$= (B_{\bar{X}}\,'F)\,(Y, Z) - (B_{\bar{Y}}\,'F)\,(X, Z) - (B_{\bar{Z}}\,'F)\,(X, Y).$$

By (2.3) and (1.5) this equation assumes the form

$$'N(X, Y, Z) - (B_X\,'F)\,(Y, \bar{Z}) - (B_Y\,'F)\,(\bar{Z}, X) - (B_{\bar{Z}}\,'F)\,(X, Y) =$$
$$= u(Y)\,(D_Z u)\,(\bar{\bar{X}}) + u(Z)\,(D_{\bar{\bar{X}}} u)\,(Y) - u(X)\,(D_Z u)\,(\bar{\bar{Y}}) -$$
$$- u(Z)\,(D_{\bar{\bar{Y}}} u)\,(X) - u(X)\,(D_Y u)\,(\bar{\bar{Z}}) - u(Y)\,(D_{\bar{\bar{Z}}} u)\,(X).$$

Using (1.4) in the above equation we obtain (3.3).

**Theorem (3.4).** On nqs manifold with the semi-symmetric connexion $B$, the following hold:

$$\begin{aligned} &\text{a) } (B_X u)\,(\bar{Y}) = (B_Y u)\,(\bar{X}),\\ &\text{b) } (B_{\bar{X}}\,'F)\,(Y, Z) - (B_{\bar{Y}}\,'F)\,(Z, X) + (B_{\bar{Z}}\,'F)\,(X, Y) =\\ &\qquad = -\,2u(Y)\,(D_Z u)\,(X). \end{aligned} \tag{3.4}$$

**Proof.** In the almost contact manifold we have the relation [3]

$$(B_X u)\,(\bar{Y}) = (D_X u)\,(\bar{Y}).$$

Then

$$(B_X u)\,(\bar{Y}) - (B_Y u)\,(\bar{X}) = (D_X u)\,(\bar{Y}) - (D_Y u)\,(\bar{X}).$$

Using (1.4) a) we get $(B_X u)\,(\bar{Y}) = (B_Y u)\,(\bar{X})$. From (2.3) and (1.5) we have

$$(B_{\bar{X}}\,'F)\,(Y, Z) - (B_{\bar{Y}}\,'F)\,(Z, X) + (B_{\bar{Z}}\,'F)\,(X, Y) =$$
$$= u(Y)\,\{(D_Z u)\,(\bar{\bar{X}}) + (D_{\bar{\bar{Z}}} u)\,(X)\} +$$
$$+ u(X)\,\{(D_Y u)\,(\bar{\bar{Z}}) - (D_{\bar{\bar{Y}}} u)\,(Z)\} +$$
$$+ u(Z)\,\{(D_{\bar{\bar{X}}} u)\,(Y) - (D_X u)\,(\bar{\bar{Y}})\}.$$

Using (1.4) a) in this equation we get (3.4) b).

## 4. SEMI-SYMMETRIC CONNEXION ON GENERALIZED COSYMPLECTIC MANIFOLD OF SECOND CLASS

**Theorem (4.1).** On a generalized co-symplectic manifold of second class with the semi-symmetric connexion $B$, we have

$$\begin{aligned} &\text{a) } (B_U\,'F)\,(Y, Z) = 2\,'F(Y, Z),\\ &\text{b) } (B_X u)\,(\bar{Y}) + (B_Y u)\,(\bar{X}) = 0,\\ &\text{c) } (B_{\bar{X}}\,'F)\,(Y, Z) + (B_{\bar{Y}}\,'F)\,(Z, X) + (B_{\bar{Z}}\,'F)\,(X, Y) =\\ &\quad = -\,2\,\{u(X)\,(D_Z u)\,(Y) + u(Y)\,(D_X u)\,(Z) + u(Z)\,(D_Y u)\,(X)\}. \end{aligned} \tag{4.1}$$

**Proof.** From (2.3) we get

$$(B_U{}'F)(Y, Z) = (D_U{}'F)(Y, Z) + 2{}'F(Y, Z).$$

Using (1.7) in the above equation we get (4.1) a). In the almost contact manifold we have the relation [3] $(B_X u)(\overline{Y}) = (D_X u)(\overline{Y})$, then by (1.6) we get

$$(B_X u)(\overline{Y}) + (B_Y u)(\overline{X}) = 0.$$

By virtue of (2.3), (1.7) and (1.6) we at once have (4.1) c).

**Theorem** (4.2). On generalized co-symplectic manifold of second class with the semi-symmetric connexion $B$ the Nijenhuis tensor is given by

$$\begin{aligned} {}'N(X, Y, Z) = {} & 2u(X)\{(D_Y u)(Z) + {}'F(Y, \overline{Z})\} - \\ & - 2u(Y)\{(D_X u)(Z) + {}'F(X, \overline{Z})\} - 2u(Z)(D_X u)(\overline{Y}). \end{aligned} \tag{4.2}$$

**Proof.** By virtue of (2.4) b), (2.3) and (1.7) we have

$$\begin{aligned} {}'N(X, Y, Z) = {} & u(Y)\{(D_{\overline{\overline{X}}} u)(Z) + (D_{\overline{X}} u)(\overline{Z}) - 2{}'F(X, \overline{Z})\} + u(Z)(D_Y u)(\overline{\overline{X}}) \\ & - u(X)\{(D_{\overline{\overline{Y}}} u)(Z) + (D_{\overline{Y}} u)(\overline{Z}) - 2{}'F(Y, \overline{Z})\} - u(Z)(D_X u)(\overline{\overline{Y}}). \end{aligned}$$

Using (1.6) in the above equation we get

$$\begin{aligned} {}'N(X, Y, Z) = {} & 2u(X)\{(D_Y u)(Z) + {}'F(Y, \overline{Z})\} - \\ & - 2u(Y)\{(D_X u)(Z) + {}'F(X, \overline{Z})\} - 2u(Z)(D_X u)(\overline{Y}). \end{aligned}$$

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## ÖZET

Evvelce R.S. Mishra [2] bazı yeni manifoldlar tanımlamıştır. Bu çalışmada genelleştirilmiş "co-symplectic" manifoldlar üzerindeki yarı-simetrik bağıntı incelenmektedir.