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THE AR-PROBLEM IN LINEAR METRIC SPACES

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1. Introduction. The aim of this paper is to supply the reader with some open problems in the area of functional analysis and topology. These problems were posed by the founders of functional analysis: Banach, Schauder and other in early 1930's but they are still open until now. It is of interest to say that although the solutions of these problems have not yet been found, the searching for answers to these problems have received marvellous successes: a lot of important results were discovered. For instance, a new branch of mathematics, called infinite dimensional topology, was born on the way of searching for a solution to Banach problem. We hope that this paper will provide young researchers a source of open problems for their research study which are still very active nowadays.

For convenience for the reader we state some criteria for attacking these problems.

Notation and Conventions. In this paper all maps are assumed to be continuous. By a linear metric space we mean a topological linear space X which is metrizable. We write $\| x - y \| = \mathrm{p}(x, y)$, where p is an invariant metric, see [Re]. We may assume that $\| . \|$ is monotonous, that is $\| \lambda x \| \leq \| x \|$ for every $x \in X$ and $\lambda \in \mathbf{R}$ with $|\lambda| \leq 1$.

$\| \quad \|$ is called an F-norm.

The zero element of X is denoted by θ. A locally convex space is a linear metric space which possesses a basis of neighbourhoods of θ consisting of convex sets.

Let E be a subset of a linear space X. By conv E we denote the convex hull of E and span E denotes the linear subspace of X spanned by E.

Let E be a subset of a metric space X and $x \in X$. We denote

$$\| x - E \| = \inf \{ \| x - y \| : y \in E \}.$$

We recall that for $p \in [0, 1)$ the linear metric space L_p is defined by

$$L_p = \left\{ f : [0, 1] \to R, \int_0^1 |f(t)|^p \, dt < \infty \right\} \text{ for } 0 < p < 1 \text{ and}$$

$$L_0 = \left\{ f : [0, 1] \to R, \int_0^1 \frac{|f(t)|}{1 + |f(t)|} \, dt < \infty \right\}.$$

For undefined notation, see [Bo], [BP].

2. The AR-problem. We say that a metric space X is an ANR if and only if for any metric space Y which contains X topologically as a closed subset there exist a neighbourhood U of X in Y and a map (called a retraction) $r : Y \to X$ such that $r(x) = x$ for every $x \in X$.

We say that X is an AR if in the above definition we can take $U = Y$. The AR-problem in linear metric spaces is stated as follows:

2.1. Problem. Is every convex set in a linear metric space an AR ? See [G], [W], Problems LS I, LS 6.

For locally convex spaces Problem 2-1 was settled affirmatively by Dugundji [D]. Problem 2-1 remains open for non-locally convex linear metric spaces and is one of the most resistant problems in infinite dimensional topology.

Problem 2-1 is extremely important in infinite dimensional topology because of the following two reasons :

The first reason comes from Schauder's conjecture. In 1935 Schauder proved that every compact convex set in a locally convex space has the fixed point property. Schauder conjectured that his theorem holds true without the local convexity.

2-2. Schauder's conjecture. Every compact convex set in a linear metric space has the fixed point property ?

It is of interest to know that Schauder posed Problem 2-2 in the Scottish book in 1935 and despite great efforts by topologists for more than half a century his conjecture is still unproved. Schauder's conjecture is still open even in some very special cases: For instance, it is not known whether compact convex subsets of the spaces L_p, $0 \leq p < 1$, have the fixed point property.

Let us observe the following theorem of Borsuk [BO] which has reduced Schauder's conjecture to the AR-problem :

2-3. Theorem (Borsuk 1937). Every compact AR-space has the fixed point property.

The second reason comes from the problem of topological classification of convex sets in linear metric spaces which is, in our opinion, even more important then Schauder's conjecture. It asks

2-4. Problem. (i) Is every infinite dimensional compact convex set in a linear metric space homeomorphic to the Hilbert cube $Q = [0, 1]^\infty$?

(ii) Is every complete separable infinite dimensional linear metric space homeomorphic to a Hilbert space?

Problem 2-4 was posed by Banach in early 1930's and is the most fundamental question in infinite dimensional topology: In fact infinite dimensional topology was born on the way of searching for a solution of this problem.

In the late seventies Torunczyk established very powerful characterizations of Hilbert cube manifolds and Hilbert space manifolds which reduce Problem 2 to the AR-problem.

2-5. Theorem [DTI]. (i) An infinite dimensional compact convex set X in a linear metric space is homeomorphic to the Hilbert cube if and only if X is an AR.

(ii) A complete separable linear metric space X is homeomorphic to a Hilbert space if and only if X is an AR.

3. Characterizations of ANR-spaces. The problem of determining a metric space is an ANR or not is, in general, very difficult. There are a lot of criteria for recognizing ANR-spaces, see [Bo], [BP], [Hu], [vM]. The following characterization of ANR-spaces is very useful in discovering ANR-spaces.

Let X be a metric space. For an open cover $\mathcal{U}$ of X let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U}$ equipped with *the Whitehead topology*. Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X. We say that $\{\mathcal{U}_n\}$ is a *zero sequence* if and only if

$$(*) \ \sup \{\operatorname{diam} U : U \in \mathcal{U}_n\} \to 0 \text{ as } n \to \infty.$$

We denote

$$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n \quad \text{and} \quad \mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1}).$$

For each $\sigma \in \mathcal{K}(\mathcal{U})$ we write

$$n(\sigma) = \sup \{n \in \mathrm{N} : \sigma \in \mathcal{N}(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}.$$

We say that a map $f : \mathcal{U} \to X$ is a *selection* if and only if $f(U) \in U$ for every $U \in \mathcal{U}$.

The characterization of ANR's established by the author in [N1] is simplified to the following due to observations of J. Luukkainen and K. Sakai:

3-1. Theorem. A metric space X is an ANR if and only if there exists a zero sequence of open covers $\{\mathcal{U}_n\}$ of X together with a map $f : |\mathcal{K}(\mathcal{U})| \to X$ such that $f|\mathcal{U}$ is a selection and for any sequence $\{\sigma_k\}$ with $n(\sigma_k) \to \infty$ we have diam $g(\sigma_k) \to 0$.

Let X be a closed subset of a metric space Z. For every open set $U \subset X$ we put

$$\text{Ext } U = \{x \in Z : d(x, U) < d(x, X \setminus U)\}. \tag{1}$$

Then we have, see [K]

$$\text{Ext } U \cap V = \text{Ext } U \cap \text{Ext } V. \tag{2}$$

The proof of Theorem 3-1 uses the following fact established in [Hu], see [Hu], Lemmas 4-3, 4-4, 4-5, p. 127-128.

3-2. Lemma. Let X be a closed subset of a metric space Z. Then for any sequence $\{\mathcal{U}_n\}$ of open covers of X there exist a sequence of neighbourhoods $\{W_n\}$ of X in Z and a locally finite open cover $\mathcal{V}$ of $W_1 \setminus X$ with the following properties:

(i) $d(x, X) < 1/n$ for every $x \in W_n$;

(ii) $\overline{W}_{n+1} \subset W_n$ for every $n \in \mathrm{N}$;

(hi) If $V \in \mathcal{V}$ and $V \cap \overline{W}_n \neq \varnothing$ then $V \subseteq W_{n-1}$ and there exist $\varphi(V) \in \mathcal{U}_n$ and a point $a(\varphi(V)) \in \varphi(V)$ such that $V \subset \text{Ext } \varphi(V)$ and

$$d(x, a(\varphi(V)) < 5\, d(x, X) + \text{diam } \varphi(V) \text{ for every } x \in V. \tag{3}$$

3-3. Remark. Lemma 3-2 was given in [N1], but condition (3) was stated as follows:

$$d(x, a(\varphi(V, n))) < 5\, d(x, X) \text{ for every } x \in V. \tag{4}$$

Luukkainen has observed that (4) is not correct. In fact (4) must be replaced by (3) and therefore Lemma 3-2 is a correct version of Fact 1-2 used in the proof of Theorem 1-1 given in [N1]. Because of this remark of Luukkainen we give here the proof of Theorem 3-1. However we emphasize that Theorem 3-1 is a consequence of Theorem 1-1 given in [N1] and that Theorem 1-1 of [N1] *is correct* (of course with the understanding that the sequences $\{\mathcal{U}_n\}$ in 1-1 (ii) and 1-1 (iii) of [N1] are zero sequences).

Proof of Theorem 3-1. The necessity of condition 3-1 is simple: By Arens-Eells theorem, see [BP], we may consider X as a closed subset of a normed space Z. Let W be a neighbourhood of X in Z and let $r : W \to X$ be a retraction. For each $n \in \mathrm{N}$ take an open cover $\mathcal{V}_n$ of X consisting of convex subsets in W such that:

$$\operatorname{conv} V \subset W \text{ and } \operatorname{diam} r(\operatorname{conv} V) < 2^{-n} \text{ for every } V \in \operatorname{st} \mathscr{V}_n; \tag{5}$$

$$\mathscr{V}_{n+1} < \mathscr{V}_n \text{ for every } n \in \mathrm{N}. \tag{6}$$

Denote

$$\mathscr{U}_n = \{U = V \cap X : V \in \mathscr{V}_n\}.$$

Let $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ and let $g : \mathscr{U} \to X$ be any selection. Using the convexity we extend g to a map $g : |\mathscr{K}(\mathscr{U})| \to W$. It is easy to see that the map $f = rg : |\mathscr{K}(\mathscr{U})| \to X$ satisfies the required condition.

Now we shall prove the sufficiency. Let $\{\mathscr{U}_n\}$ be a zero sequence of open covers of X satisfying the condition of Theorem 3-1. We shall show that X is an ANR.

Assume that X is a closed subset of a metric space Z. Using Lemma 3-2 we take a sequence of open neighbourhoods $\{W_n\}$ of X in Z and an open cover $\mathscr{V}$ of $W_1 \setminus X$ satisfying conditions 3-4 (i) - (iii). We show that X is a retract of W_1.

For each $V \in \mathscr{V}$, put

$$n(V) = \sup\{n : V \cap \overline{W}_n \neq \emptyset\}.$$

By 3-2 (iii) there is a $\varphi(V) \in \mathscr{U}_{n(V)}$ and $a(\varphi(V)) \in \varphi(V)$ such that $V \subset \operatorname{Ext} \varphi(V)$ and

$$d(x, a(\varphi(V))) < 5\, d(x, X) + \operatorname{diam} \varphi(V) \text{ for every } x \in V. \tag{7}$$

Observe that from (2) it follows that φ induces a simplicial map $\varphi' : \mathscr{N}(\mathscr{V}) \to \mathscr{N}(\mathscr{U})$.

Let $h : W_1 \setminus X \to \mathscr{N}(\mathscr{V})$ is the canonical map. We define a retraction $r : W_1 \to X$ by the formula:

$$r(x) = \begin{cases} x & \text{if } x \in X; \\ f(\varphi'(h(x)) & \text{if } x \in W_1 \setminus X. \end{cases}$$

We shall show that r is continuous. For every $x \in W_1 \setminus X$, say $x \in \overline{W}_{n(x)} \setminus W_{n(x)+1}$, let $\sigma = < V_1, \ldots, V_n >$ be a simplex of $\mathscr{N}(\mathscr{V})$ containing $h(x)$. It is easy to see that $\varphi(\sigma) \subset \mathscr{U}_{n(x)} \cup \mathscr{U}_{n(x)+1}$, where

$$\varphi(\sigma) = < \varphi(V_1), \ldots, \varphi(V_{n+1}) > \in \mathscr{N}(\mathscr{U}).$$

Thus we have $n(\varphi(\sigma)) \geq n(x)$. Since $f : \mathscr{U}$ is a selection from (7) we get

$$\begin{aligned} d(x, r(x)) &= d(x, f\varphi' h(x)) \\ &\leq d(x, a(\varphi(V_1)) + d(a(\varphi(V_1)), f(\varphi(V_1))) + d(f(\varphi(V_1)), f\varphi' h(x)) \\ &\leq 5\, d(x, X) + \operatorname{diam} \varphi(V_1) + \operatorname{diam} \varphi(V_1) + \operatorname{diam} f(\sigma). \end{aligned}$$

Since $n(\varphi(\sigma)) \geq n(x) \to \infty$ as $x \to x_0 \in X$ we infer that r is continuous. The theoem is proved.

4. The locally convex approximation property. In this section we shall provide some our partial answers to Problem 2-1. Our idea of attacking Problem 2-1 is to approximate convex sets in linear metric spaces by convex sets in locally convex spaces. We introduce the notion of *the locally convex approximation property* (the LCAP) for convex sets in linear metric spaces and prove that the LCAP implies the AR-property. Roughly speaking, our theorem states that if a convex set X can be "approximated", in some sense, by convex subsets in locally convex spaces then X is an AR. In the compact case the LCAP is equivalent to the notion of admissibility introduced by Klee [K1], [K2].

4-1. Definition. Let us say that a convex set X in a linear metric space is *LC-convex* if and only if for every $\varepsilon>0$ there exists a $\delta = \delta(\varepsilon, X)$ such that for every finite set $A \subset X$ with diam $A < \delta$ we have diam conv $A < \varepsilon$.

Obviously, any convex set in a locally convex space is LC-convex.

4-2. Definition. We say that a convex set X in a linear metric space Y has the *locally convex approximation property* (the LCAP) if and only if there exist an F-norm $\|\ \|$ on Y, sequence $\{X_n\}$ of LC-convex subsets of X and a sequence of continuous maps $r_n : X \to X_n$ such that for some summable sequence $\{a_n\}$ of positive numbers we have

$$\text{(LC)} \quad \lim_{n\to\infty} \inf (a_n)^{-1} \| x - r_n(x) \| = 0 \text{ for every } x \in X.$$

We have proved the following theorem which indicates the importance of the LCAP for investigating the AR-problem.

4-3. Theorem [N1]. Any convex set with the LCAP is an AR.

Our Theorem 4-3 reduces Problem 2-1 to

4-4. Problem. Has every convex set the LCAP ?

Theorem 4-3 also suggests the following problem :

4-5. Problem. Assume that X is a convex set with the LCAP. Is every convex subset of X an AR ?

Application 1. The following result is an obvious application of our Theorem 4-3.

4-6. Corollary. Any convex set which is a countable union of LC-convex subsets is an AR.

Application 2. In 1940 Krein and Milman proved the following theorem :

4-6. Theorem [KM]. Any compact convex set in a locally convex space is the closure convex hull of its extreme points.

The following question was open for a long time :

4-7. Question. Does Klein-Milman theorem holds true for non-locally convex linear metric spaces?

In 1976 Roberts constructed a striking example of a linear metric space which contains a compact convex set without any extreme points. Thus the Krein-Milman theorem does not hold true for non-locally convex linear metric spaces. One may ask whether Krein-Milman holds true for compact convex sets homeomorphic to the Hilbert cube (equivalently, an AR). However this is not the case. In fact as an application of our Theorem 4-3 we get the following which was established first in [NT1] :

4-8. Corollary. Some of compact convex sets with no extreme points constructed by Roberts has the LCAP and hence is an AR.

4-9. Question. Has every compact convex set the LCAP ?

Let us note that by Theorem 4-3 a positive answer to Problem 4-9 would also provide an affirmative solution to Schauder's conjecture.

4-10. Question. Let X denote the linear metric space constructed by Roberts [R1]. We ask :

(i) Has every convex subset of X the LCAP ?

(ii) Has *every compact convex* of X the LCAP ?

(iii) Has *every linear subspace* of X the LCAP ?

(iv) Has *the whole space* X the LCAP ?

(v) Has every compact convex set of X the fixed point property ?

5. Admissible convex sets. In this section we show that the LCAP is an extension of the notion of admissibility of Klee.

5.1. Definition ([K1], [K2]). We say that a convex set X is *admissible* if and only if for every compact subset A of X and for every $\varepsilon > 0$ there is a map f from A into a finite dimensional subset of X such that $\| x - f(x) \| < \varepsilon$ for every $x \in A$.

Observe that the LCAP gives a new definition of the admissibility of Klee for compact convex sets. In fact we have

5-2. Theorem. A compact convex set X is admissible if and only if X has the LCAP.

Proof. Assume that X is a compact convex set with the LCAP. If $\dim X < \infty$ then we take $f_\varepsilon = \mathrm{id}_X$ for every $\varepsilon > 0$. So we may assume that X is infinite dimensional. By Theorem 2-4 X is an AR. Therefore by [DT] X is homeomorphic to the Hilbert cube Q. Consequently X is admissible.

Conversely assume that X is an admissible compact convex set. Then there exists a sequence of maps $\{f_n\}$ from X into finite dimensional subsets of X such that

$$\| x - f_n(x) \| < 2^{-n} \quad \text{for every } x \in X.$$

For every $n \in \mathbf{N}$ let $\mathcal{U}_n$ be a finite open cover of $f_n(X)$ such that $\operatorname{diam} U < (\dim Y_n)^{-1}\, 2^{-n}$ for every $U \in \mathcal{U}_n$, where $Y_n = f_n(X)$; and $\operatorname{ord} \mathcal{U}_n \leq 1 + \operatorname{dhn} Y_n$.

For every $U \in \mathcal{U}_n$ select a point $x_U \in U$ and put

$$X_n = \operatorname{conv} \{x_U : U \in \mathcal{U}_n\}.$$

Let $\{\lambda_U : U \in \mathcal{U}_n\}$ denote a partition of unity inscribed into $\mathcal{U}_n$ and define a map $\varphi_n : Y_n \to X_n$ by the formular

$$\varphi_n(x) = \sum_{U \in \mathcal{U}_n} \lambda_U(x)\, x_U \quad \text{for every } x \in Y_n.$$

Then we have

$$\| \varphi_n(x) - x \| \leq \sum_{U \in \mathcal{U}_n} \| \lambda_U(x)\ (x - x_U) \| \leq \sum_{U \in \mathcal{U}_n} \| x - x_U \| <$$
$$< \dim Y_n (\dim Y_n)^{-1}\, 2^{-n} = 2^{-n}.$$

Observe that X_n is a finite dimensional convex subset of X for every $n \in \mathbf{N}$. Therefore setting $r_n = \varphi_n f_n$ we get a sequence of maps from X into finite dimensional convex subsets X_n of X such that

$$\| x - r_n(x) \| < 2^{-n+1} \quad \text{for every } x \in X.$$

Since $\{a_n\} = \{n\, 2^{-n+1}\}$ is a summable sequence we infer that X has the LCAP. The corollary is proved.

Klee [K1] [K2] showed that any convex admissible convex set X has *the compact extension property* that is any map into X defined on a compact subset of metric space extends to the whole space. Observe that Theorem 4-3 can be thought of an extension of Klee theorem.

We are not able to prove Theorem 5-2 for non-compact convex sets.

5-3. Question. Has every admissible convex set the LCAP ?

The following problem is still open :

5-4. Question. Is every convex set in a linear metric space admissible ?

Even the following special case of Question 5-4 has no answer.

5-5. Question. Assume that X is an admissible convex set in a linear metric space. Is every convex subset of X admissible ?

6. Needles points spaces. The idea of Roberts of constructing a compact convex set with no extreme points is to introduce the notion of needle point spaces.

6-1. Definition [R1] [R2]. We say that a non-zero point a in a linear metric space X is a *needle point* iff for every $\varepsilon > 0$ there exists a finite set $A(a, \varepsilon) = \{a_1, \dots, a_m\}$ satisfying the following conditions :

(i) $\| a_i \| < \varepsilon$ for every $i = 1, \dots, m$;

(ii) For every $x \in \operatorname{conv}(A(a, \varepsilon) \cup \{0\})$ there exists an $\alpha \in [0, 1]$ such that $\| x - \alpha a \| < \varepsilon$;

(iii) $a = m^{-1}(a_1 + \dots + a_m)$.

A linear metric space X is a *needle point space* iff X is a complete separable space in which every non-zero point is a needle point.

Roberts proved the following theorems :

6-2. Theorem [R2]. Every needle point space contains a compact convex set without any extreme points.

6-3. Theorem [R2]. For every $p \in [0, 1)$ the space L_p is a needle point space.

We observe that the proof of Theorem **6**-2 is quite simple. So if we have a needle point space at hand we can easily construct a compact convex set with no extreme points. However it is not easy to give an example of a needle point space: The proof of Theorem 6-3 is rather complicated.

We shall outline a proof of Theorem **6**-2. Let X be a needle point space. At first we take any point $a_0 \neq 0$ and let $A_0 = \{a_0\}$.

Assume that $A_n = \{a_1^n, \dots, a_{m(n)}^n\}$ has been defined. For every $a \in A_n$ we use Definition 2 to take $A(a, \varepsilon_{n+1})$, where

$$\varepsilon_{n+1} = 2^{-n-1} (\operatorname{card} A_n)^{-1}.$$

Put

$$A_{n+1} = \bigcup \{A(a, \varepsilon_{n+1}) : a \in A_n\} ;$$

$$A = \overline{\operatorname{conv} \bigcup_{n=0}^{\infty} A_n} \subset X.$$

Observe that A is a compact convex set in X and the only possible extreme point of A is 0. Therefore the set

$$B = \operatorname{conv}(A \cup (-A))$$

is a compact convex set without any extreme points.

For sometime it was hoped that Roberts' example would provide a counter-example to Schauder's conjecture. However this is not the case : In fact in [NT2] it is shown that *all compact convex sets* constructed by Roberts have the fixed point property. Let us observe that in [KPR] it was claimed that all compact convex sets constructed by Roberts' method have the fixed point property but no detailed proof was given.

It was proved earlier in [NT1] that every needle point space contains a compact convex AR-set with no extreme points. However let us observe that the proof given in [NT1] has not yet reached *all Roberts' compact convex sets.* Some of his compact sets were still standing away from the arguments given in [NT1].

Our result in [NT2] has settled completely the question about the fixed point property for all the compact convex sets constructed by Roberts. However we still have a problem : The result of [NT2] does not say that all Roberts' compact convex sets are AR's. It seems to the author that the AR-property for all Roberts' compact convex sets can be established by using the arguments given in [NT2]. However this has not yet been done.

After Roberts constructed his example needle point spaces became the most important area for finding a solution of Problem 2-1. It is hoped that needle point spaces (and in particular the spaces L_p, $0 \leq p < 1$) will be a good place for constructing counter-examples to Problems 2-1. The following question arises naturally :

6-4. Problem. Is every convex set in a needle point space an AR ?

7. The finite dimensional approximation property. Our aim is to search for a solution of Problem 6-4. Again we try to approximate convex sets in needle point spaces by convex sets in finite dimensional spaces. *The finite dimensinal approximation property* (the FDAP) introduced in this section is the key to this problem. Our results in Sections 3 and 4 produce linear metric spaces which contain compact convex sets with no extreme points such that all convex subsets of them are absolute retracts. This result extends and completes the earlier theorem established in [NT1].

We have been trying to find an answer to Problem 6-4. As we have seen the LCAP is quite useful for detecting the AR-property in linear metric spaces.

However it is not strong enough to attack Problem 4. For instance we are not able to show that convex subsets of a convex set with the LCAP are AR, see Question 4-5. We shall introduce the notion of the FDAP which are stronger than the LCAP. Applying the FDAP we give some partial answers to Problem 6-4.

7-1. Definition. Let X be a convex set in a linear metric space Y. We say that X has the *finite dimensional approximation property* (the FDAP) if and only if there exist an F-norm $\|\ \|$ on Y and a sequence of continuous maps r_n from X into finite dimensional subsets X_n of X such that for some summable sequence $\{a_n\}$ of positive numbers we have

$$\text{(FD)} \ \lim_{n\to\infty} \inf (a_n)^{-1} \dim X_n \| x - r_n(x) \| = 0 \quad \text{for every } x \in X.$$

Of course the FDAP is stronger than LCAP. So we also obtained the following stronger theorem:

7-2. Theorem [N3]. Let X be a convex set in a linear metric space. If X has the FDAP then every convex subset $E \subset X$ is an AR. In particular any convex set with the FDAP is an AR.

Theorem 7-2 reduces Problem 4 to

7-3. Problem. Has every needle point spaces the FDAP ?

As an application of Theorem 7-2 we obtain the following result which provides a partial answer to Problem 6-4.

7-4. Theorem. Every needle point space X contains a dense linear subspace $E \subset X$ with the following properties :

(i) E contains a compact convex set with no extreme points;

(ii) E has the FDAP, therefore every convex subset of E is an AR.

Unfortunately, we are not able to prove that $E = X$. Even the answer to the following question has not yet been found :

7-5. Question. Is there a *complete* linear metric space with properties (i) and (ii) of Theorem 7-4 ?

We are not able to take $E=X$ even for the spaces L_p, $0 \le p < 1$. However for the spaces L_p, $0 \le p < 1$, we get something better than Theorem 7-4.

7-6. Definition. We say that a subset $D \subset L_p$ is π-convex if and only if for any $f, g \in D$ and for every $\alpha \in [0, 1]$ we have $\pi_\alpha(f, g) \in D$, where $\pi_\alpha(f, g)$ is defined by

$$\pi_\alpha(f, g) = \begin{cases} f(t) & \text{if} \quad t \in [0, \alpha); \\ g(t) & \text{if} \quad t \in [\alpha, 1]. \end{cases}$$

7-7. Theorem [N1] [N2]. Every π-convex subset of L_p, $0 \le p < 1$, is an AR.

In particular we have :

7-8. Corollary. The spaces L_p, $0 \le p < 1$, are AR.

ii *Is every compact convex set* in L_p, $0 \le p < 1$, an AR ?

Our results provide new examples of convex sets with the AR-property and raise a lot of new problems for further investigation of Problem 2-1, one of the most difficult problems in infinite dimensional topology.

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ÖZET

Bu çalışmanın amacı, okuyucuya fonksiyonel analiz ve topoloji konularındaki bazı açık problemler hakkında bilgi vermektir.