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ON NEARLY PARACOMPACT SPACES

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In this study a new type of weak paracompactness has been defined and some of its certain topological properties were established.

0. Introduction. The purpose of this article is to introduce a new type of paracompactness called nearly paracompactness, weaker than the usual paracompactness and to determine its basic topological properties. It has been shown that these two types of paracompactness are equivalent on semi-regular spaces. By using certain characterizations of Ernest Micheal, which could be found in [4], some necessary and sufficient conditions for the nearly paracompactness of almost regular and almost normal spaces have been obtained. Furthermore some relations between nearly paracompact spaces and new type of some covering spaces such as nearly Lindelöf, nearly compact and nearly o-compact have also been investigated. Certain sufficient conditions on the nearly paracompactness of subspaces and product topologies are obtained. Finally some invariancy conditions of nearly paracompactness under certain functions are also obtained. No seperation axioms such as being Fréchet, i.e. T_1 or Hausdorff are assumed unless otherwise stated.

1. Preliminaries. This section is devoted to some basic topological facts which have been used frequently. Let X be a topological space and $\mathring{A}$, $\overline{A}$, $\complement A$ denote the interior, the closure and the complement of A respectively. Then for any subset A following equalities hold :

$$\mathring{\overline{\mathring{\overline{A}}}} = \mathring{\overline{A}} \quad , \quad \overline{\mathring{\overline{\mathring{A}}}} = \overline{\mathring{A}} \tag{1}$$

$$\complement \overline{A} = \overset{\circ}{\widehat{\complement A}} \quad , \quad \complement \mathring{A} = \overline{\complement A}. \tag{2}$$

From (2) it can be easily deduced that for any A

$$\complement \mathring{\overline{A}} = \overline{\overset{\circ}{\widehat{\complement A}}} \quad ; \quad \complement \overline{\mathring{A}} = \overset{\circ}{\overline{\complement A}}. \tag{3}$$

Furthermore if A is open, then for any subset B

$$A \cap \overset{\circ}{B} \subseteq \overline{A \cap \overset{\circ}{B}}\,. \tag{4}$$

Consequently if B is disjoint with open A then

$$\overset{\circ}{\overline{A}} \cap \overset{\circ}{\overline{B}} = \phi\,. \tag{5}$$

A subset is called *regularly open* (*regularly closed*) if it is the interior of a closed (the closure of an open) set. By using (1) it can be easily proved that for any subset A, the necessary and sufficient condition for being regularly open (regularly closed) is $A = \overset{\circ}{\overline{A}}$ ($A = \overline{\overset{\circ}{A}}$). The complement of a regularly open (regularly closed) set is regularly closed (regularly open) since (3) holds. Every regularly open (regularly closed) set is trivially open (closed). In the topological space on $\mathbf{R}$ with the basic neighborhoods $G_x(\varepsilon) =]\,x - \varepsilon\,, \infty)$ for x where $\varepsilon > 0$, the set $K = (-\infty, 0]$ is closed but not regularly closed since $\overset{\circ}{K} = \phi \neq \overline{\overset{\circ}{K}}$. In the same topological space for any real x and positive ε, the basic neighborhood $G_x(\varepsilon)$ is open but not regularly open since $\overline{G_x(\varepsilon)} = \overset{\circ}{\overline{G_x(\varepsilon)}} = \mathbf{R}$. The necessary and sufficient condition for an open set A (closed set K) to be regularly open (regularly closed) is $\partial A = \partial \overline{A}$ ($\partial K \subseteq \overline{\overset{\circ}{K}}$). For an open set A (closed set K), the minimal regularly open set (maximal regularly closed set) contains A (contained in K) is $\overset{\circ}{\overline{A}}$ ($\overline{\overset{\circ}{K}}$). $\overset{\circ}{\overline{A}}$ ($\overline{\overset{\circ}{K}}$) is called the *regularly open envelope* (*regularly closed content*) of A (K). A simple consequence of (5) says that the regularly open envelopes of disjoint open sets are also disjoint and by using (3) it can be proved that the union of the regularly closed contents of two closed sets, whose union is the whole space, is also equal the whole space. The intersection (union) of a finite number of regularly open (regularly closed) sets is also regularly open (regularly closed). In the one dimensional Euclidian topology on $\mathbf{R}$ for any real x and positive ε, the intervals $G_x^{\,1} =]\,x - \varepsilon, x\,[$ and $G_x^{\,2} =]\,x, x + \varepsilon\,[$ are regularly open, but their union is not since

$$\overset{\circ}{\overline{G_x^{\,1} \cup G_x^{\,2}}} =]\,x - \varepsilon\,, x + \varepsilon\,[\,.$$

It follows that the union (intersection) of regularly open (regularly closed) sets is not necessarily regularly open (regularly closed). In the same topological space for any real x

$$\{x\} = \bigcap_{n=1}^{\infty}]\,x - \varepsilon_n\,, x + \varepsilon_n\,[\qquad (\varepsilon_n \downarrow 0^+),$$

i.e. the intersection (union) of a countable number of regularly open (regularly closed) sets is not necessarily regularly open (regularly closed). A topological space is called *semi-regular* if the whole regularly open neighborhoods of any point constitute the basic neighborhoods for that point [2]. For any topological space X, the family of all open sets which can be written as the union of an arbitrary number of regularly open sets of X is a topology on X, weaker then the initial topology on X and denoted by X_s. If i_s and c_s denote the interior and closure operations in X_s, then for any open set A of X it is straightforward to see that

$$\overline{A} = c_s(A) \tag{6}$$

and consequently by using the first equality of (2) one gets

$$\complement \overline{A} = \complement c_s(A) = i_s\left(\complement A\right). \tag{7}$$

Particularly for open set $\complement \overline{A}$ of X, (6) and (7) yield

$$\overline{\complement \overline{A}} = c_s\left(\complement \overline{A}\right) = c_s \circ i_s\left(\complement A\right).$$

Since the second equation of (3) can be written as

$$\complement c_s \circ i_s\left(\complement A\right) = i_s \circ c_s(A)$$

in X_s, one can easily deduce for any open set A of X that

$$\overset{\circ}{\overline{A}} = \complement \overline{\complement \overline{A}} = \complement c_s \circ i_s\left(\complement A\right) = i_s \circ c_s(A). \tag{8}$$

This result and the first equation of (3) give for any closed set K of X

$$\overline{\overset{\circ}{K}} = \complement \overset{\circ}{\overline{\complement K}} = \complement i_s \circ c_s\left(\complement K\right) = c_s \circ i_s(K). \tag{9}$$

Four basic Lemma's can be deduced from equations (8) and (9) :

Lemma 1. The regularly open envelope (regularly closed content) of any open (closed) set of X is regularly open (regularly closed) in X_s.

Lemma 2. If X is any topological space, X_s i.e. the whole family of open sets of X which can be written as the arbitrary union of regularly open sets of X, is a semi-regular topology.

Lemma 3. The topologies X and X_s are equivalent iff X is semi-regular.

Lemma 4. The necessary and sufficient conditions for being regularly open (regularly closed) in X_s is being regularly open (regularly closed) in X.

The topological space X_s is called the *semi-regularization* of X [2]. The topological space with the basic neighborhoods $G_x(\varepsilon) =]\, x - \varepsilon, \infty)$ on $\mathbf{R}$ has an indiscrete semi-regularization since all the proper open subsets are dense. Similarly the topological space on **R** with topology τ_{x_0}, generated by the whole subsets of **R** containing some fixed real number x_0, has an indiscrete semiregularization for the same reasons.

2. Nearly paracompactness and some characterizations on nearly paracompactness of certain spaces.

Definition 1. In a topological space a family of sets $\mathcal{G}$ is called *locally finite* if every point has at least one neighborhood which intersects at most a finite number of $\mathcal{G}$ [4]. A family of sets $\mathcal{G}$ is called a *refinement* of $\mathcal{U}$ if for every $G \in \mathcal{G}$ there exists a $U \in \mathcal{U}$ such that $G \subseteq U$ [4]. The family $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ is called the *precise refinement* of $\mathcal{U} = \{U_\beta : \beta \in I^*\}$ if $I = I^*$ and for every $\alpha \in I$ there exists a $\beta \in I^*$ such that $G_\alpha \subseteq U_\beta$ [4]. *A* family of sets is called open (regularly open) if all its members are open (regularly open). A topological space is called *paracompact* if every open covering admits a locally finite, open refinement. A refinement of a cover $\mathcal{U}$ always means a cover which refines $\mathcal{U}$.

Definition 2. A topological space is called *nearly paracompact* if every regularly open covering admits a locally finite open refinement. A subset is called *nearly paracompact* if the relative topology defined on it is nearly paracompact.

Theorem 1. A topological space is nearly paracompact iff every regularly open covering admits a locally finite, regularly open refinement.

Proof. Let $\mathcal{U}$ be a locally finite refinement of the open covering $\mathcal{G}$ in any nearly paracompact space. Then $\mathcal{U}^* = \left\{\overset{\circ}{\overline{U}} : U \in \mathcal{U}\right\}$, i.e. the family of regularly open envelopes of members of $\mathcal{U}$ is trivially a cover and is locally finite by (4). It is also a refinement of $\mathcal{G}$ since for any $U \in \mathcal{U}$ there exists a superset $G(U) \in \mathcal{G}$ which contains U and consequently $\overset{\circ}{\overline{U}} \subseteq G(U)$ holds. Sufficiency is straightforward (q.e.d.).

Remark 1. Every paracompact space is evidently nearly paracompact. Let X be a set with infinite points and let τ denote a topology on X such that the whole proper open subsets are dense, then the topological space determined by τ is nearly paracompact but not paracompact, since the only non empty regularly open set is X and it is impossible to define a locally finite open refinement of the covering with basic neighborhoods. The topological space with basic neighborhoods $G_x(\varepsilon) =]\, x - \varepsilon\, , \infty)$ on **R**, the topological space determined by the topology τ_{x_0} i.e. the topology all of whose open subsets of **R** contains a fixed real number x_0, cofinite and cocountable topologies on **R** fit the above description.

Theorem 2. A topological space is nearly paracompact iff its semi-regularization is paracompact.

Proof. Let X be a nearly paracompact space and let $\mathscr{G} = \{G_\alpha : \alpha \in I\}$ be an open cover for its semi-regularization space X_s. Since for every G_α there exists an index set J_α such that

$$G_\alpha = \bigcup_{\beta \in J_\alpha} W_\beta ,$$

where W_β is regularly open in X for any $\beta \in J_\alpha$. Let $\mathscr{U} = \{U_\gamma : \gamma \in I^*\}$ be a regularly open and locally finite refinement of

$$\{W_\beta : \beta \in J_\alpha , \alpha \in I\}.$$

Since for any $\gamma \in J^*$ there exists an $\alpha_\gamma \in I$ such that

$$U_\gamma \subseteq W_\beta \subseteq \bigcup_{\beta \in J_{\alpha_\gamma}} W_\beta \subseteq G_{\alpha_\gamma}$$

and since the regularly open envelope of any neighborhood disjoint with almost all U_γ has also this property, the family $\mathscr{U}$ can be accepted as the locally finite refinement of $\mathscr{G}$ in the space X_s, i.e. X_s is paracompact. Conversely let X_s be a paracompact and consequently a nearly paracompact space and let $\mathscr{G}^*$ be a regularly open cover of X. Since $\mathscr{G}^*$ is also a regularly open cover for X_s by Lemma 4, it has a regularly open, locally finite refinement $\mathscr{U}^*$ by Theorem 1. Consequently $\mathscr{U}^*$ is a regularly open, locally finite refinement of $\mathscr{G}^*$ in X by Lemma 4. So X verify the necessary and sufficient conditions for being nearly paracompact stated in Theorem 1 (q.e.d.).

Corollary. Nearly paracompactness and paracompactness are equivalent on semiregular spaces.

Theorem 3. *A* nearly paracompact Hausdorff space is paracompact iff it is semiregular.

Proof. Sufficiency is a direct consequence of Lemma 3 and Theorem 2. Since in any topological space, for any open set G

$$\overline{G} = \overline{\overset{\circ}{\overline{G}}} = \overline{\overset{\circ}{\overline{\overset{\circ}{\overline{G}}}}} = \overline{\overset{\circ}{\overline{G}}}$$

i.e. closures of G and its regularly open envelope are the same by (1), necessity follows from the regularity of paracompact Hausdorff spaces (q.e.d.).

Definition 3. A topological space is called *almost regular* if any point and regularly closed set not containing it are contained in disjoint open sets [9]. A topological space X is called *almost completely regular* if for any point x and regularly closed subset K not containing x, there exists a continuous function $f: X \to [0,1]$ such that $f(x) = 1, f(K) = 0$ [10]. A topological space is called *almost normal* if any two disjoint closed sets, one of whom is regularly closed, are contained in disjoint open sets [10]. A topological space is called *mildly normal* if any two disjoint regularly closed sets are contained in disjoint open sets [6].

Remark 3. (i) Almost normality (Almost regularity) is strictly weaker than normality (regularity).

(ii) Every almost normal space is not necessarily almost regular.

Proof. Let the basic neigborhoods of any $x \in]0,1]$ be the same as the basic neighborhoods of x in the relative topology on $[0,1]$ determined by $\mathbf{R}^1$, and let the basic neighborhood system of $x = 0$ be

$$\{[0, \varepsilon[- A : \varepsilon > 0, A = \{2^{-n} : n \in \mathbf{N}\}\} .$$

The topology on $[0,1]$ with this set of basic neighborhoods determines an almost regular but not regular space. The cofinite and cocountable topologies on $\mathbf{R}$ determine almost normal but not normal spaces. The topology

$$\{\phi, X, \mathbf{N}_1, \mathbf{N}_2, \{2\}\}$$

on the set of first three positive integers $X = \mathbf{N}_3$ determines an almost normal but not almost regular space since there are no disjoint open sets one of whom containing $x = 1$ and the other containing $\mathbf{N}_3 - \mathbf{N}_1$, see [9] and [10].

Lemma 5. A topological space is almost regular iff its semi-regularization is regular.

Proof. Let X be an almost regular space, G be an open set in X_s and $x \in G$. By the definition of semi-regularization spaces, almost regularity of X and equation (6) yields the existence of two regularly open neighborhoods U_x and W_x of x in X such that

$$c_s(\overset{\circ}{\overline{U}_x}) = \overline{\overset{\circ}{\overline{U}_x}} = \overline{U}_x \subseteq W_x \subseteq G$$

i.e. X_s is regular. A similar proof can be given for sufficiency (q.e.d.).

Theorem 4. Every nearly paracompact Hausdorff space is almost regular, even almost completely regular and mildly normal.

Proof. Every nearly paracompact Hausdorff space has regular and even normal Hausdorff semi-regularization space by Theorem 2, since every paracompact Hausdorff space has this property [4] (q.e.d.).

The almost normality of nearly paracompact Hausdorff spaces remains as an open question.

Lemma 6. In any topological space the union of a locally finite, regularly closed family is regularly closed.

Proof. Let $\mathscr{K}$ be a locally finite, regularly closed family. Then

$$\overline{\widehat{\bigcup_{K \in \mathscr{K}} K}^{\circ}} \subseteq \bigcup_{K \in \mathscr{K}} K$$

since the union of a locally finite, closed family is closed. The reverse relation is a consequence of

$$K \subseteq \overline{\widehat{\bigcup K}^{\circ}} \qquad (K \in \mathscr{K})$$

since $\mathscr{K}$ is a regularly closed family (q.e.d.).

Lemma 7. Let $\mathscr{G} = \{G_\alpha : \alpha \in I\}$ be any family and $\mathscr{K} = \{K_\beta : \beta \in J\}$ a locally finite regularly closed covering of a topological space X. If every K_β intersects at most a finite number of G_α then for every $\alpha \in I$ there exists a regularly open $U(G_\alpha)$ such that $G_\alpha \subseteq U(G_\alpha)$ and the family $\mathscr{U} = \{U(G_\alpha) : \alpha \in I\}$ is locally finite.

Proof. Since

$$G_\alpha \cap \bigcup_{\beta \in J_\alpha} K_\beta = \phi$$

where $J_\alpha = \{\beta \in J : K_\beta \cap G_\alpha = \phi\}$ then

$$G_\alpha \subseteq \complement \bigcup_{\beta \in J_\alpha} K_\beta = U(G_\alpha) \qquad (\alpha \in I).$$

$U(G_\alpha)$ is regularly open by Lemma 6. For every point $x \in X$ there exists a positive integer $n(x)$ and a basic neighborhood G_x of x such that

$$G_x \subseteq \bigcup_{i \leq n(x)} K_{\beta_i}$$

since $\mathscr{K}$ is a locally finite covering. The hypothesis and the equivalency of $K_\beta \cup G_\alpha = \phi$ with $K_\beta \cap U(G_\alpha) = \phi$ yields the locally finiteness of $\mathscr{U}$ (q.e.d.).

Lemma 8. In any topological space if a regularly open covering $\mathscr{G}$ has a locally finite, regularly open refinement, then $\mathscr{G}$ has a locally finite regularly open precise refinement.

Proof. Let $\mathscr{U} = \{U_\beta : \beta \in J\}$ be a locally finite, regularly open refinement of $\mathscr{G} = \{G_\alpha : \alpha \in I\}$. By using the axiom of choice it is possible to choose a unique $\alpha = \alpha(\beta) \in I$ for any $\beta \in J$ such that $U_\beta \subseteq G_\alpha$. Define a map $f : J \to I$ as $f(\beta) = \alpha(\beta)$. The family of open sets

$$(G_\alpha \supseteq)\ W_\alpha = \bigcup_{f(\beta)=\alpha} U_\beta \qquad (\alpha \in I)$$

is locally finite since a basic neighborhood intersecting only $U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}$ intersects only $W_{f(\beta_1)}, W_{f(\beta_2)}, \ldots, W_{f(\beta_n)}$. The family of regularly open covers of W_α's is a covering and locally finite by (4) (q.e.d.).

Theorem 5. An almost regular space is nearly paracompact iff every regularly open covering admits a locally finite, regularly closed refinement.

Proof. Let X be a nearly paracompact, almost regular space, $\mathscr{G}$ its regularly open covering, G_x the member of $\mathscr{G}$ which contains x and W_x the regularly open neigborhood of x such that $\overline{W_x} \subseteq G_x$, the existence of which is a consequence of the necessary and sufficient condition for almost regularity of the space [9]. If $\{U_x : x \in X\}$ is a locally finite and regularly open precise refinement of $\{W_x : x \in X\}$ then $\{\overline{U_x} : x \in X\}$ is a locally finite, regularly closed refinement of $\mathscr{G}$. Conversely let $\mathscr{G}^*$ be a regularly open covering of X, $\mathscr{K}$ its locally finite, regularly closed refinement and W_x a basic neighborhood of a point x disjoint with almost all $K \in \mathscr{K}$. Since $W_x \cap \overset{\circ}{K} = \phi$ for any such $K \in \mathscr{K}$, it follows that $\overset{\circ}{\overline{W_x}} \cap K = \overset{\circ}{\overline{W_x}} \cap \overline{\overset{\circ}{K}} = \phi$ and there exists a locally finite, regularly closed refinement of $\left\{\overset{\circ}{\overline{W_x}} : x \in X\right\}$. Lemma 7 yields the existence of regularly open family $\mathscr{U}^* = \{U(K) : U(K) \supseteq K \in \mathscr{K}\}$ and since there exists a $G^*(K) \in \mathscr{G}^*$ such that $G^*(K) \supseteq K$, the regularly open covering

$$\mathscr{G}^{**} = \{G^*(K) \cap U(K) : K \in \mathscr{K}\}$$

is locally finite and a refinement of $\mathscr{G}^*$ i.e. the necessary and sufficient condition of Theorem 1 is established (q.e.d.).

Theorem 6. An almost regular space is nearly paracompact iff every regularly open covering admits a refinement that can be a union of countable locally finite, regularly open families.

Proof. Let X be a topological space satisfying the hypothesis of sufficiency. Then its semi-regularization space is paracompact, since every open covering of

X_s has a refinement that can be union of a countable locally finite open families [4]. Necessity is trivial (q.e.d.).

Theorem 7. An almost normal space is nearly paracompact iff every regularly open covering admits a locally finite, regularly closed refinement.

Proof. Let X be a nearly paracompact, almost normal space, $\mathcal{G}=\{G_\alpha : \alpha \in I\}$ its locally finite, regularly open covering. Let $<$ denote the well ordering defined in I and let α_0 be the first element of I according to $<$. Without loss of generality we can assume that there exists a point $x \in X$ contained in $G_{\alpha_0} \cap \bigcup_{\alpha_0 \neq \alpha} \complement G_\alpha$ (Otherwise every point of G_{α_0} contained in another G_α $(\alpha \neq \alpha_0)$ i.e. G_{α_0} would be an unnecessary member of covering and could be removed from $\mathcal{G}$). The necessary and sufficient condition for almost normality [10] and the inclusion relation

$$\phi \neq \complement \bigcup_{\alpha_0 \neq \alpha} G_\alpha = \complement \bigcup_{\alpha_0 < \alpha} G_\alpha \subseteq G_{\alpha_0}$$

yields the existence of a non empty regularly open set W_{α_0} such that

$$\complement \bigcup_{\alpha_0 < \alpha} G_\alpha \subseteq W_{\alpha_0} \subseteq \overline{W}_{\alpha_0} \subseteq G_{\alpha_0}$$

subsequently

$$X = W_{\alpha_0} \cup \complement \bigcup_{\alpha_0 < \alpha} G_\alpha .$$

Let $\alpha \in I$ and assume

$$G_\alpha \cap \bigcup_{\alpha \neq \beta} \complement G_\beta \neq \phi \tag{A}$$

and for all $\overline{\alpha} < \alpha$ let all sets W_β $(\beta \leq \overline{\alpha})$ be chosen so that

$$\left.\begin{aligned} X &= \bigcup_{\beta \leq \overline{\alpha}} W_\beta \cup \bigcup_{\overline{\alpha} < \gamma} G_\gamma , \\ \phi &\neq \overline{W}_\beta \subseteq G_\beta \quad (\beta \leq \overline{\alpha}) . \end{aligned}\right\} \tag{B}$$

It is easy to prove that

$$X = \bigcup_{\beta < \alpha} W_\beta \cup \bigcup_{\alpha \leq \gamma} G_\gamma . \tag{C}$$

In fact if $G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_n}$ are the only members of the locally finite family $\mathcal{G}$ containing the point x, and if

$$\sup_{k \leq n(x)} \alpha_k = \alpha(x) < \alpha$$

then

$$x \in \bigcup_{\beta \leq \alpha(x)} W_\beta \subseteq \bigcup_{\beta < \alpha} W_\beta$$

since the first equation of (B) was true for $\bar{\alpha} = \alpha(x)$.

Since (A) and (B) give

$$\phi \neq G_\alpha \cap \bigcap_{\alpha \neq \beta} \complement G_\beta \subseteq G_\alpha \cap \Big(\bigcap_{\beta < \alpha} \complement W_\beta \cap \bigcap_{\alpha < \beta} \complement G_\beta \Big),$$

almost normality of X and (C) yield the existence of a non empty regularly open W_α such that

$$X = \bigcup_{\beta \leq \alpha} W_\beta \cup \bigcup_{\alpha < \gamma} G_\gamma . \tag{D}$$

Transfinite induction assures the existence of W_α for all $\alpha \in I$ such that condition (D) is satisfied. $\{\overline{W}_\alpha : \alpha \in I\}$ is a locally finite refinement of $\mathscr{G}$ since $\mathscr{G}$ is locally finite and (D) is true. Sufficiency is easily achieved along the lines of the sufficiency proof of Theorem 5 (q.e.d.).

Lemma **9.** In any topological space if a cover $\mathscr{G}$ has a refinement such that the arbitrary union of members of the refinement are regularly closed, then $\mathscr{G}$ has a precise refinement with the same property.

Proof. Similar to that of Lemma 8 (q.e.d.).

Theorem **8.** An almost normal T_1 space is nearly paracompact iff every regularly open covering has a refinement such that the arbitrary union of members of the refinement is regularly closed.

Proof. Necessity is a direct consequence of Theorem 7 and Lemma 6. Conversely let X be a space with the sufficiency hypothesis and let $\mathscr{G} = \{G_\alpha : \alpha \in I\}$ be its regularly open covering and $<$ denote the well ordering defined in I. Subsequently every subset of I contains its first element. Let us prove first for every positive integer n that there exists a precise refinement of

$$\mathscr{K}_n = \{K_{\alpha,n} : K_{\alpha,n} \subseteq G_\alpha , \alpha \in I\}$$

such that

$$K_{\beta,n+1} \cap K_{\alpha,n} = \phi \qquad (\alpha < \beta) \tag{*}$$

and the arbitrary unions of members of $\mathscr{K}_n$ are regularly closed. In fact the sufficiency hypothesis and Lemma 9 assure the existence of the family $\mathscr{K}_1$. Now suppose all families $\mathscr{K}_k$ $(k \leq n)$ have been defined. Then all the sets

$$W_{\alpha,n+1} = G_\alpha \cap \complement \bigcup_{\beta<\alpha} K_{\beta,n} \qquad (\alpha \in I)$$

are regularly open since

$$x \in W_{\alpha(x),n+1} = G_{\alpha(x)} \cap \complement \bigcup_{\alpha<\alpha(x)} K_{\alpha,n}$$

where $\alpha(x) = \min \{\alpha \in I : x \in G_\alpha\}$, the family $\{W_{\alpha,n+1} : \alpha \in I\}$ is a covering of X. Again the sufficiency hypothesis and Lemma 9 yield the precise refinement $\mathscr{K}_{n+1}$, such that the arbitrary unions of members of $\mathscr{K}_{n+1}$ are regularly closed. Furthermore the definition of $W_{\beta,n+1}$ gives

$$K_{\beta,n+1} \subseteq W_{\beta,n+1} \subseteq \complement \bigcup_{\alpha<\beta} K_{\alpha,n} \subseteq \complement K_{\alpha,n}$$

when $\alpha < \beta$. The induction process is over. Now let us define

$$U_{\alpha,n} = \complement \bigcup_{\beta \neq \alpha} K_{\beta,n} \qquad (\alpha , n) \in I \times \mathbf{N}$$

we get

$$U_{\alpha,n} \subseteq K_{\alpha,n} \subseteq G_\alpha , \qquad (\alpha , n) \in I \times \mathbf{N}$$

$$U_{\alpha,n} \cap U_{\beta,n} = \phi \qquad n \in \mathbf{N}$$

since $\mathscr{K}_n$ is a cover. If $\alpha_n(x)$ denotes the first element of $\{\alpha \in I : x \in K_{\alpha,n}\}$ then $x \in U_{\alpha_k,k+1}$ where $\alpha_k = \min_n \alpha_n(x)$, since $x \notin K_{\beta,k+1}$ when $\alpha_k < \beta$ by (*) and $x \notin K_{\beta,k+1}$ when $\beta < \alpha_k$ by the definition of α_k. This means that $\{U_{\alpha,n} : (\alpha , n) \in I \times \mathbf{N}\}$ is a regularly open refinement of $\mathscr{G}$. Let $\mathscr{K}^*$ be the precise refinement of this cover such that the arbitrary unions of members of $\mathscr{K}^*$ are regularly closed and let

$$A_n = \left\{x \in X : \exists\, V_x \in \mathscr{N}_x , \exists\, \alpha_x \in I ; V_x \cap \bigcup_{\alpha \neq \alpha_x} G_\alpha = \phi\right\}$$

where $\mathscr{N}_x$ is the filter of the neighborhoods of x. Then A_n is open and

$$\bigcup_\alpha B_{\alpha,n} \subseteq \bigcup_\alpha U_{\alpha,n} \subseteq A_n \qquad (n \in \mathbf{N})$$

where $\mathscr{K}^* = \{B_{\alpha,n} : (\alpha , n) \in I \times \mathbf{N}\}$. The necessary and sufficient condition for almost normality [10] yields the existence of a regularly open G_n such that

$$\bigcup_\alpha B_{\alpha,n} \subseteq G_n^* \subseteq \overline{G_n^*} \subseteq A_n \qquad (n \in \mathbf{N}).$$

The definition of A_n and the inclusion relation $\overline{G}_n^* \subseteq A_n$ assures the local finiteness of the regularly open family

$$\mathscr{G}_n^* = \{G_n^* \cap U_{\alpha,n} : \alpha \in I\}.$$

Since the countable union

$$\bigcup_{n=1}^{\infty} \mathscr{G}_n^* = \{G_n^* \cap U_{\alpha,n} : (\alpha, n) \in I \times \mathrm{N}\}$$

is a refinement of $\mathscr{G}$ and since every almost normal T_1 space is almost regular, sufficiency follows from Theorem 6 (q.e.d.).

Theorem 9. A Hausdorff space with normal semi-regularization is nearly paracompact iff every regularly open covering has a refinement such that the arbitrary unions of members of the refinement are regularly closed.

Proof. The necessary and sufficient condition for paracompactness of a normal Hausdorff space is the existence of a refinement of a given open covering, such that the arbitrary unions of members of the refinement are closed. This can be shown just as in Theorem 8. This proves the sufficiency. Necessity follows from Lemma 6 and Theorem 5 since a Hausdorff space with normal semi-regularization is almost regular (q.e.d.).

Definition 4. In any topological space the intersection of arbitrary numbers of regularly closed sets is called *star-closed* [5].

Theorem 10. A Hausdorff space with disjoint open sets which contains disjoint star-closed sets is nearly paracompact iff every regularly open covering has a refinement such that, arbitrary unions of members of the refinement are regularly closed.

Proof. Direct consequence of Theorem 9 since this space has a normal semi-regularization (q.e.d.).

3. Nearly paracompactness of subspaces and product spaces.

Definition 5. In a topological space a subset is called *N-closed* if every covering of A with the regularly open subsets of the space has a finite sub-covering [3]. A topological space X is called *nearly compact* if X is *N-closed* [7]. A topological space X is called *locally nearly compact* if every point has a neighborhood with N-closed closure [3].

Remark 4. it is known that a topological space is nearly compact (locally nearly compact) iff its semi-regularization is compact (locally compact), see Theorem 4.1 and Thorem 4.5 of [3].

Remark 5. The product space of two nearly paracompact spaces isn't necessarily nearly paracompact. In fact $\mathbf{R}_u^1$ i.e. the upper limit topology on $\mathbf{R}$ is paracompact consequently nearly paracompact but the product $\mathbf{R}_u^1 \times \mathbf{R}_u^1$ is not [4]. $\mathbf{R}_u^1 \times \mathbf{R}_u^1$ is not nearly paracompact by Corollary of Theorem 2 since $\mathbf{R}_u^1$ and consequently $\mathbf{R}_u^1 \times \mathbf{R}_u^1$ is regular and therefore semiregular.

Theorem 11. The product of a nearly paracompact and a nearly compact space is nearly paracompact.

Proof. If X is nearly paracompact and Y is nearly compact then $X_s \times Y_s$ is paracompact [4]. The equality

$$\overset{\circ}{\overline{W}}_x \times \overset{\circ}{\overline{U}}_y = \overset{\circ}{\overline{W_x \times U_y}}$$

says that

$$X_s \times Y_s = (X \times Y)_s$$

i.e. the semi-regularization space of $X \times Y$ is nearly paracompact, where W_x and U_y are any basic neighborhoods of x and y in the spaces X and Y respectively (q.e.d.).

Theorem 12. In a nearly paracompact Hausdorff space an open subset which can be written as a countable union of regularly closed sets is nearly paracompact.

Proof. Let A be an open set in a topological space X and let i and c denote the interior and closure operations in the relative topology defined on A. Then for any $B \subseteq A$

$$i \circ c(B) = i(\overline{B} \cap A) = i(\overline{B} \cap A) \cap A = \overset{\circ}{\widehat{\overline{B} \cap A}} = \overset{\circ}{\overline{B}} \cap A$$

is satisfied. If X is nearly paracompact and Hausdorff and A is written as the union of a countable number of regularly closed K_n's, then for any regularly open covering $\{i \circ c(G_\alpha) : \alpha \in I\}$ of the relative topology on A by

$$A \subseteq \bigcup_{\alpha \in I} \overset{\circ}{\overline{G}}_\alpha$$

and subsequently for any positive integer n, the regularly open covering

$$\bigcup_{\alpha \in I} \overset{\circ}{\overline{G}}_\alpha \cup \complement K_n$$

of X has a locally finite, regularly open refinement $\{W_{\beta,n} : \beta \in J_n\}$. All families

$$\mathscr{G}_n^* = \{W_{\beta,n} \cap A : W_{\beta,n} \cap K_n \neq \phi, \beta \in J_n\}$$

are locally finite in the relative topology on A and furthermore they are regularly open since

$$i \circ c\,(W_{\beta,n} \cap A) = \overline{\overset{\circ}{W_{\beta,n} \cap A}} \cap A = W_{\beta,n} \cap A \subseteq i \circ c\,(W_{\beta,n} \cap A)\,.$$

For any regularly open set $i \circ c\,(G_x) = \overset{\circ}{\overline{G}}_x \cap A$ of A which contains $x \in A$, the closure of $i \circ c\,(A \cap W_x)$ on A satisfies

$$c\left(i \circ c\,(A \cap W_x)\right) = c\,(\overset{\circ}{\overline{A \cap W_x}} \cap A) = \overline{\overset{\circ}{\overline{A \cap W_x}} \cap A} \cap A$$

$$\subseteq \overline{\overset{\circ}{\overline{W}}_x} \cap A = \overline{W}_x \cap A \subseteq \overset{\circ}{\overline{G}}_x \cap A = i \circ c\,(G_x)$$

where W_x is the neighborhood of x in X such that $\overline{W}_x \subseteq \overset{\circ}{\overline{G}}_x$. Thus the relative topology on A is almost regular and since $\bigcup_{n=1}^{\infty} \mathscr{G}_n{}^*$ is the refinement of $\{i \circ c\,(G_\alpha) : \alpha \in I\}$, A is nearly paracompact by Theorem 6 (q.e.d.).

Corollary. A clopen set of a nearly paracompact Hausdorff space is nearly paracompact.

4. Invariancy of nearly paracompactness under certain functions and partition of unity.

Definition 6. A function $f: X \to Y$ is called *almost continuous* if the inverse images of regularly open sets of Y are open in X [8]. A function is called *almost open* (*almost closed*) if the images of regularly open (regularly closed) sets are open (closed) [8]. An almost continuous and almost closed function $f: X \to Y$ is called *almost proper* if for all $y \in Y$, the fiber $f^{-1}(y)$ is N-closed.

Theorem 13. An almost continuous, almost open and almost closed image of a nearly paracompact Hausdorff space is nearly paracompact.

Proof. Let X be a nearly paracompact and Hausdorff and let $f: X \to Y$ be an almost continuous, almost open and almost closed surjection. Since the image of a regularly closed set under almost open and almost closed function is regularly closed [5], $f: X_s \to Y_s$ is a closed surjection. Furthermore $f: X_s \to Y_s$ is continuous since the inverse image of a regularly open set under almost continuous, almost open and almost closed function is regularly open [5]. Hence $Y_s = f(X_s)$ is paracompact (q.e.d.).

Thorem 14. If X is almost regular, Y is nearly paracompact and $f: X \to Y$ is an almost proper and almost open surjection, then X is nearly paracompact.

Proof. Let $\mathscr{G} = \{G_\alpha : \alpha \in I\}$ be any regularly open covering of X. For arbitrary $y \in Y$, N-closedness yields the existence of a positive integer $n(y)$ such that

$$f^{-1}(y) \subseteq \bigcup_{k \leq n(y)} G_{\alpha_k} = G(y) .$$

Since X is almost regular, for any $x \in f^{-1}(y)$ there exists a regularly open neighborhood $W_x = W_x(\alpha_{k(x)})$ such that

$$W_x(\alpha_{k(x)}) \subseteq \overline{W_x(\alpha_{k(x)})} \subseteq G_{\alpha_{k(x)}}(y) \subseteq G(y)$$

subsequently for a suitable positive integer $m(y)$

$$f^{-1}(y) \subseteq \bigcup_{i \leq m(y)} W_{x_i} \subseteq \overset{\circ}{\widehat{\bigcup_{i \leq m(y)} \overline{W}_{x_i}}} \subseteq G(y) .$$

Let $\mathscr{V} = \{V_y : y \in Y\}$ be the locally finite regularly open, precise refinement of

$$U_y = \complement_Y f\left(\complement_X \overset{\circ}{\overline{\bigcup_{i \leq m(y)} W_{x_i}}}\right) (\ni y)$$

Since

$$f^{-1}(V_y) \subseteq \overset{\circ}{\overline{\bigcup_{i \leq m(y)} W_{x_i}}} \subseteq G(y)$$

the regularly open cover

$$\mathscr{G}^* = \{f^{-1}(V_y) \cap G_{\alpha_k}(y) : k \leq n(y),\ y \in Y\}$$

is a refinement of G. Since $\mathscr{V}$ is locally finite in Y it is easy to see that $\mathscr{G}^*$ is locally finite in X (q.e.d.).

Theorem 15. An almost proper and almost open injection into a nearly paracompact Hausdorff space can only be defined from a nearly paracompact Hausdorff space.

Proof. Since the range of an almost open an almost closed function is an open-closed and subsequently nearly paracompact by the Corollary of Theorem 12, the claim follows from Theorem 14 (q.e.d.).

Theorem 16. Every reguraly open covering $\mathscr{G}$ of a nearly paracompact Hausdorff space has a partition of unity subordinated to $\mathscr{G}$.

Proof. By Theorem 4, Theorem 5 and Lemma 9 the regularly open covering $\mathscr{G} = \{G_\alpha : \alpha \in I\}$ of the nearly paracompact Hausdorff space X has a locally finite, regularly open refinement $\mathscr{U}=\{U_\alpha : \alpha \in I\}$ such that $\overline{U_\alpha} \subseteq G_\alpha$ for all $\alpha \in I$ and similarly $\mathscr{U}$ has a refinement $\mathfrak{W}=\{W_\alpha : \alpha \in I\}$ with the same property so $\overline{W_\alpha} \subseteq U_\alpha$ for all $\alpha \in I$. Since $\overline{W_\alpha}$ and $\complement U_\alpha$ are regularly closed in X and since X_s is normal and Hausdorff and a continuous function on the semi-regularization space X_s is also continuous on X, there exists a continuous $g: X \to [0,1]$ such that $g_\alpha(\overline{W_\alpha})=1$ and $g_\alpha\left(\complement U_\alpha\right)=0$ (Take $g_\alpha \equiv 0$ if $U_\alpha=\phi$).

$$\overline{\{x\in X : g_\alpha(x) \neq 0\}} = \overline{g_\alpha^{-1}(]0,1])} \subseteq G_\alpha \qquad (\alpha \in I)$$

since $g_\alpha^{-1}(]0,1]) \cap \complement U_\alpha = \phi$ for all $\alpha \in I$, so the support of g_α is contained in G_α for each α. Local finiteness of $\mathfrak{W}$ guaranties that for any $x \in X$ at least one and at most a finite number of g_α are not zero. Therefore $\sum_\alpha g_\alpha$ is a well defined continuous real valued function and the family

$$f_\alpha = \frac{g_\alpha}{\sum_\alpha g_\alpha} \qquad (\alpha \in I)$$

is the required partition of unity (q.e.d.).

5. Nearly paracompactness and some relations with certain weak covering spaces.

Definition 7. A topological space is called *nearly Lindelöf* if every regularly open covering admits a countable subcovering. A topological space is called *nearly countably compact* if every countable regularly open covering admits a finite subcovering.

Remark 6. The topology τ_{x_0} on **R** defined in Remark 1 is nearly Lindelöf but not Lindelöf. Similarly for a fixed integer k_0 the topology τ_{k_0} on **Z**, the set of all integers is nearly countably compact but not countably compact.

Theorem 17. A topological space is nearly Lindelöf (nearly countably compact) iff its semi-regularization is Lindelöf (countably compact).

Proof. Similar to proof of Theorem 2 (q.e.d.).

Theorem 18. Almost regularity and nearly paracompactness are equivalent on nearly Lindelöf Hausdorff spaces.

Proof. Sufficiency follows from Theorem 4, necessity obtained from Theorem 6 and Definition 7 (q.e.d.).

Theorem 19. *A* seperable nearly paracompact space is nearly Lindelöf.

Proof. Any locally finite open family $\mathscr{G}$ in a seperable space has at most a countable number of members, otherwise at least one point of the countable dense set of the space would be contained in an infinite number of members of $\mathscr{G}$ (q.e.d.).

Lemma 10. A locally nearly compact Hausdorff space is almost regular.

Proof. The semi-regularization of a locally nearly compact Hausdorff space is locally compact and Hausdorff so regular (q.e.d.).

Theorem 21. A nearly Lindelöf, locally nearly compact Hausdorff space is nearly paracompact.

Proof. A simple consequence of Theorem 18 and Lemma 10 (q.e.d.).

Definition 8. A topological space is called *lightly compact* if every locally finite open family has at most a finite number of members [1]. A topological space is called *Micheal* if every open covering has a refinement which can be written as the union of a countable number of locally finite open families [1]. A topological space is called *nearly Micheal* if every regularly open covering has a refinement which can be written as the union of a countable number of locally finite open families.

Remark 7. The topological space determined by τ_{x_0} on **R** defined in Remark 1 is nearly Micheal but not Micheal.

Theorem 22. In a topological space every locally finite regularly open family has at most a finite number of members iff every countable regularly open covering has a finite, dense subfamily.

Proof. In any topological space X satisfying the necessity conditions, at least one of the members of any regularly open covering contains an infinite number of points of any subset with infinite points. In fact if $\overline{G}_n \cap A$ were finite for all positive integers, where $\mathscr{G} = \{G_n : n \in \mathbf{N}\}$ is any regularly open covering and A any infinite set, then by choosing a point $x_{n_1} \in A \cap \complement \overline{G}_1$ and a covering member G_{n_1} which contains x_{n_1}, $U_1 = G_{n_1} \cap \complement \overline{G}_1$ would be a non empty regularly open set. Having chosen the positive integers $n_1 < n_2 < \dots < n_k$ and the points $x_{n_i} \in G_{n_i} \cap A$ and constructed $U_i = G_{n_i} \cap \complement \overline{G}_i$ $(i \leq k)$, there would be a point

$$x_{n_{k+1}} \in A \cap \complement \bigcup_{i \leq n_k} \overline{G_i} \qquad (n_k < n_{k+1})$$

and if $G_{n_{k+1}}$ is the member of $\mathscr{G}$ which contains $x_{n_{k+1}}$ then

$$U_{k+1} = G_{n_{k+1}} \cap \bigcap_{i \leq n_k} \complement \overline{G_i} \neq \phi$$

is regularly open and this procedure never stops in any finite steps. Sinse $\mathscr{G}$ is a covering, there exists a positive integer $n(x)$ for any $x \in X$ such that $x \in G_{n(x)} \in \mathscr{G}$ so for $n_k > n(x)$ $G_{n(x)} \cap U_{k+1} = \phi$ holds i.e. $\{U_n : n \in \mathbf{N}\}$ is locally finite and bas a infinite number of members since for $k < p, n_k \leq n_{p-1}$ and so

$$U_k \cap U_p \subseteq G_{n_k} \cap \bigcap_{i \leq n_{p-1}} \complement \overline{G_i} = \phi .$$

That would be a contradiction to the necessity conditions. Consequently if, in any topological space X which satisfying the necessity conditions, countable regularly open covering has no finite dense sub-family then by choosing an infinite number of

$$x_n^{(\circ)} \in \complement \bigcup_{k \leq n} \overline{G_k} \qquad (n \in \mathbf{N})$$

one reaches easily the following contradiction to the above conclusion : for any positive integer n, $x_i^{(\circ)} \notin \overline{G}_n$ if $i \geq n$. Conversely let $\mathscr{G}^* = \{G_n^* : n \in \mathbf{N}\}$ be a locally finite, countable regularly open family. Then the family of

$$W_n = \complement \bigcup_{k=n}^{\infty} \overline{G_k}^* \qquad (n \in \mathbf{N})$$

is regularly open by Lemma 6 and furthermore is a covering since $\mathscr{G}^*$ is locally finite. Since $W_n \subseteq W_{n+1}$ and $G_k^* \cap W_n = \phi$ $(n \leq k)$ consequently $G_k^* \cap \overline{W}_n = \phi$ $(n \leq k)$ for any positive integer n, the family W_n's has no finite dense sub-family (q.e.d.).

Theorem 23. In a topological space X every locally finite regularly open family has at most a finite number of members iff X is lightly compact.

Proof. Sufficiency is clear. Let X be a topological space with the necessity hypothesis and let $\{G_n : n \in \mathbf{N}\}$ be any countable open covering of X. Then the regularly open covering $\left\{\overset{\circ}{\overline{G}}_n : n \in \mathbf{N}\right\}$ has a finite dense sub-family as

$$X = \bigcup_{k \leq N} \overline{\overset{\circ}{\overline{G}}}_{n_k}$$

by Theorem 22. But this means that

$$X = \bigcup_{k \leq N} \overline{G}_{n_k}$$

since all G_{n_k} $(k \leq N)$ are open. This is the necessary and sufficient condition for being lightly compact for X [1] (q.e.d.).

Theorem 24. A topological space is nearly compact iff it is lightly compact and nearly paracompact.

Proof. A nearly compact space is lightly compact by Theorem 22 and Theorem 23 and clearly is nearly paracompact. Sufficiency is straightforward (q.e.d.).

Theorem 6′. Nearly paracompactness and nearly Michealness are equivalent on almost regular spaces.

Proof. See Theorem 6 and Definition 8 (q.e.d.).

Theorem 25. Nearly Lindelöfness and nearly Michealness are equivalent on lightly compact spaces.

Proof. Sufficiency is a direct consequence of the definition of nearly Michealness and Theorem 23. The converse is clear since every nearly Lindelöf space is nearly Micheal (q.e.d.).

Theorem 26. A topological space is nearly compact iff it is nearly countably compact and nearly Lindelöf (nearly Micheal).

Proof. Every nearly countably compact space is lightly compact by Theorem 22 and Theorem 23. The remainder statements are clear (q.e.d.).

Theorem 27. An almost regular space is nearly compact iff it is lightly compact and nearly Lindelöf (nearly Micheal).

Proof. Sufficiency follows from Theorem 25, Theorem 6′ and Theorem 24. The necessity is clear (q.e.d.).

Theorem 28. Nearly paracompactness and nearly Lindelöfness are equivalent on locally nearly compact, lightly compact Hausdorff spaces.

Proof. Necessity follows from Lemma 10, Theorem 6′ and Theorem 25. Sufficiency is a consequence of Theorem 21 (q.e.d.).

Lemma 11. A nearly countabley compact and 1° countable Hausdorff space is almost regular.

Proof. Let X be a nearly countably compact, 1° countable Hausdorff space, x any point of X, G_x a regularly open neighborhood of x and $\mathscr{B}_x = \{W_x(n) : n \in \mathbb{N}\}$ be the local base at x with monotonically decreasing members. Since $\{x\} = \bigcap_{n=1}^{\infty} W_x(n)$ and X is nearly countably compact

$$X = G_x \cup \bigcup_{k \leq n} \complement \overline{W}_x(k)$$

or equivalently

$$G_x \supseteq \complement \bigcup_{k \leq n} \complement \overline{W}_x(k) = \bigcap_{k \leq n} \overline{W}_x(k) = \overline{W}_x(n) = \overline{\overset{\circ}{\overline{W}}_x(n)}$$

satisfied by a suitable positive integer n, i.e. X is almost regular (q.e.d.).

Theorem **29.** Nearly paracompactness and nearly Lindelöfness are equivalent on nearly countably compact, 1° countable Hausdorff spaces.

Proof. Since every nearly countably space is lightly compact by Theorem 22 and Theorem 23 and since any topological space with the necessary condition is almost regular by Lemma 11, necessity follows from Theorem 6′ and Theorem 25. Sufficiency can be proved similarly (q.e.d.).

After Lemma 10 and Lemma 11, Theorem 28 and Theorem **29** are the special cases of the following Theorem :

Theorem 30. Nearly compactness, nearly paracompactness, nearly Lindelöfness and nearly Michealness are equivalent on almost regular, lightly compact spaces.

Proof. If X is almost regular, lightly compact and nearly paracompact then it is nearly compact by Theorem 24, consequently it is nearly Lindelöf by Theorem 26, so it is nearly Micheal by Theorem 25 and finally it is nearly paracompact by Theorem 6′ (q.e.d.).

Definition 9. A topological space X is called σ-compact if X is the union of countable number of its compact subsets [4]. X is called nearly σ-compact if X is the union of countable number of its N-closed subsets.

Remark 8. The topological space on **R** determined by τ_{x_0} defined in Remark 1 clearly is nearly σ-compact but it is not σ-compact since a subset is compact in τ_{x_0} iff it is finite.

Theorem 31. Nearly σ-compactness and nearly Lindelöfness are equivalent on locally nearly compact spaces.

Proof. Since the necessary and sufficient condition for N-closedness of $A \subseteq X$ in X is the compactness of the relative topology on A determined by the X_s, a topological space is nearly σ-compact iff its semi-regularization space is σ-compact. So the Theorem follows from Theorem 17 since σ-compactness and Lindelöfness are equivalent on locally compact spaces (q.e.d.).

Theorem 32. Nearly paracompactness and nearly σ-compactness are equivalent on locally compact, 2° countable Hausdorff spaces.

Proof. Lemma 10, Theorem 31 and Theorem 18 prove the sufficiency, Theorem 19 and Theorem 31 prove the necessity since 2° countable spaces are seperable (q.e.d.).

Definition 10. Let the topological spaces X_α $(\alpha \in I)$ be pairwise disjoint (Otherwise take the topological equivalent $X_\alpha' = \{\alpha\} \times X_\alpha$ of X_α). The weak topology on $X = \bigcup_\alpha X_\alpha$ determined by the canonical mappings $i_\alpha : X_\alpha \to X$ is called the *free union* of the family X_α $(\alpha \in I)$ and denoted by $X = \sum_\alpha X_\alpha$ [4]. A subset $A \subseteq X_\alpha$ is open iff $A \cap X_\alpha$ is open in X_α for all $\alpha \in I$ [4].

Lemma 12. In a free union space $X = \sum_\alpha X_\alpha$ a subset $A \subseteq X$ is regularly open iff $A \cap X_\alpha$ is regularly open in X_α for all $\alpha \in I$.

Proof. Since any X_α satisfies the necessary and sufficient condition for being open in X, all of the X_α's are open-closed subsequently regularly open and regularly closed. If i_α and c_α denote the interior and closure operations respectively on X_α, i.e. on the relative topology at X_α determined by $X = \sum_\alpha X_\alpha$ then for any $A \subseteq X$

$$i_\alpha \circ c_\alpha(A \cap X_\alpha) = i_\alpha(\overline{A \cap X_\alpha} \cap X_\alpha) = \overset{\circ}{\overline{A \cap X_\alpha}} \cap X_\alpha = \overset{\circ}{\overline{A \cap X_\alpha}}.$$

So if $A \subseteq X$ is regularly open then for any $\alpha \in I$

$$i_\alpha \circ c_\alpha(A \cap X_\alpha) = \overset{\circ}{\overline{A \cap X_\alpha}} = A \cap X_\alpha,$$

i.e. all $A \cap X_\alpha$ are regularly open in X_α. Conversely if for any $\alpha \in I$, $A \cap X_\alpha$ is regularly open in X_α then A is open in X and $A \subseteq \overset{\circ}{\bar{A}}$. Furthermore for any $x \in \overset{\circ}{\bar{A}}$ there exists a $\beta \in I$ with $x \in X_\beta$, so by using (4) in Section 1 one gets

$$x \in \overset{\circ}{\bar{A}} \cap X_\beta \subseteq \overset{\circ}{\overline{A \cap X_\beta}} = i_\beta \circ c_\beta(A \cap X_\beta) = A \cap X_\beta\,,$$

and this means $x \in A \supseteq \overset{\circ}{\bar{A}}$, i.e. A is regularly open in X (q.e.d.).

Theorem 33. A locally nearly compact Hausdorff space is nearly paracompact iff it is the free union of nearly σ-compact spaces.

Proof. Since every locally compact, nearly σ-compact space is nearly Lindelöf any locally nearly compact Hausdorff space will be nearly paracompact if it is nearly o-compact by Lemma 10 and Theorem 18. Now let $\mathscr{G} = \{G_\beta : \beta \in J\}$ be any regularly open covering of the free union of the nearly paracompact spaces X_α $(\alpha \in I)$, and let $\mathscr{G}^*$ be the regularly open refinement of $\{X_\alpha \cap G_\beta : \beta \in J\}$ in X_α. Since $\bigcup\limits_\alpha \mathscr{G}_\alpha{}^*$ constitute a regularly open refinement for $\mathscr{G}$, the free union space $\sum\limits_\alpha X_\alpha$ of nearly paracompact spaces X_α is nearly paracompact. So a locally nearly compact Hausdorff space X is nearly paracompact if X is the free union of nearly σ-compact spaces X_α $(\alpha \in I)$, because any X_α is locally nearly compact and Hausdorff since it is regularly closed in locally nearly compact $X = \sum\limits_\alpha X_\alpha$. Conversely let X be a locally nearly compact and nearly paracompact space and let $\mathscr{U} = \{U_x : x \in X\}$ be the precise, locally finite open refinement of the covering determined by the regularly open neighborhoods with N-closed closures. Clearly every U_x has a non empty intersection with at at most a finite number U_y $(y \in X)$. Let $x\mathscr{R}y$ iff $x \in U_{x_1}, y \in U_{x_n}$ and $U_{x_k} \cap U_{x_{k+1}} \neq \phi$ for all positive integers $k \leq n-1$. $\mathscr{R}$ is an equivalence relation on X. If $\mathscr{R}[x]$ denotes the equivalence class which contains x then

$$X = \bigcup_{x \in X} \mathscr{R}[x]$$

is a separation and $\mathscr{R}[x]$ is open for any x. By taking any $\mathscr{R}[x]$ define $K_1 = U_{x_0} \subseteq \mathscr{R}[x]$ and $K_n = \bigcup \{U_y : U_y \cap K_{n-1} \neq \phi\}$. Every K_n is N-closed since K_n contains at most a finite number of U_y. The definition of $\mathscr{R}$ yields

$$\mathscr{R}[x] = \bigcup_{n=1}^{\infty} K_n$$

and consequently

$$\mathscr{R}[x] = \bigcup_{n=1}^{\infty} \overline{K_n}$$

since $\mathscr{U}$ is an open covering, so every $\mathscr{R}[x]$ is nearly σ-compact (q.e.d.).

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ÖZET

Bu çalışmada yeni bir tür zayıf parakompaktlık tanımlanmakta ve bu türün bazı belirli topolojik özellikleri belirlenmektedir.