AN $\overset{*}{W}{}^{i}_{hj}$ GENERALISED 2-RECURRENT FINSLER SPACE [1]

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In an n-dimensional FINSLER space F_n the pseudo projective deviation tensor is $\overset{*}{W}{}^{i}_{j}(x, \dot{x}) = a\, W^i_j + b\, H^i_j$, where a and b denote scalar functions of $(x, \dot{x})$, homogeneous of degree zero in their directional arguments and W^i_j and H^i_j are the projective tensor field and BERWALD's deviation field. Then

$$\overset{*}{W}{}^{i}_{hj} = \frac{2}{3}\, \dot{\partial}_{[h}\, \overset{*}{W}{}^{i}_{j]}\,, \quad \overset{*}{W}{}^{i}_{lhj} = \dot{\partial}^{2}_{l[h}\, \overset{*}{W}{}^{i}_{j]}$$

and the FINSLER space F_n is said to be $\overset{*}{W}{}^{i}_{lhj}$ - *generalised* 2-*recurrent if*

$$\overset{*}{W}{}^{i}_{lhj(k)(m)} = \beta_m\, \overset{*}{W}{}^{i}_{lhj(k)} + \alpha_{km}\, \overset{*}{W}{}^{i}_{lhj}$$

where β_m is a recurrence vector field and α_{km} is a non-zero tensor of the second order. Under these assumptions some formulae involving the tensor fields defined above are obtained.

1. Introduction. Let us consider an n-dimensional FINSLER space F_n [1] [2] in which the BERWALD's deviation and projective deviation tensor fields are given by

$$H^i_k(x, \dot{x}) = 2\, \dot{\partial}_k G^i - \partial_h \dot{\partial}_k G^i \dot{x}^h + 2\, G^i_{kl}\, G^l - \dot{\partial}_l G^i\, \dot{\partial}_k G^l \text{ [3]} \tag{1.1}$$

and

$$W^i_k(x, \dot{x}) = H^i_k - H\, \delta^i_k - (\dot{\partial}_i H^i_k - \dot{\partial}_k H)\, \dot{x}^i/(n+1)\,. \tag{1.2}$$

The deviation tensor field W^i_j and the tensor H^i_j are homogeneous of degree two in their directional arguments, and we have the following identities and contractions [1]:

(1.3)

a) $H^i_{hji} = H_{hj}$ b) $H^i_{ijk} = H_{kj} - H_{jk}$ c) $H^i_{ji} = H_j$

d) $H^i_i = (n-1)\, H$ e) $H^i_{jkh} = -H^i_{jhk}$ f) $H^i_{kh} = -H^i_{hk}$

(1.4)

a) $H^i_{[jkh]} = 0$ b) $H^i_j(x, \dot{x})\, \dot{x}^j = 0$

c) $H_{jk}\, \dot{x}^j = H_k$ d) $\dot{\partial}_r H^r_j\, \dot{x}^j + (n-1)\, H = 0$

(1.5)

a) $W^j_{ihk} = -W^j_{ikh}$ b) $W^j_{kh} = -W^j_{hk}$ c) $W^j_{[ihk]} = 0$

d) $W^j_{hk}\, \dot{x}^h = W^j_k$ e) $W^j_{ihk}\, \dot{x}^i = W^j_{hk}$ f) $W^j_{ihk}\, \dot{x}^i\, \dot{x}^h = W^j_k$

1) Communicated by Prof. Dr. RAM BEHARI on August 7, 1975.

2) Numbers in brackets refer to the references at the end of the paper.

3) $\partial_i \equiv \partial/\partial x^i$ and $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$

$$\text{(1.6)}\quad \begin{array}{lll} \text{a)}\ W^j_k\,\dot{x}^k = 0 & \text{b)}\ \dot{\partial}_h W^j_k\,\dot{x}^k = -W^j_h & \text{c)}\ \dot{\partial}_i W^i_k = 0 \\ \text{d)}\ W^i_{ihj} = 0 & \text{e)}\ W^j_{hj} = 0 & \text{f)}\ W^j_j = 0 \end{array}$$

$$\text{(1.7)}\qquad G^s_{rln}\,\dot{x}^n = 0\,.$$

The commutation formulae involving BERWALD's curvature tensor fields are as follows:

$$\text{(1.8)}\qquad T^i_{j(h)(k)} - T^i_{j(k)(h)} = -(\dot{\partial}_r T^i_j)\,H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk}$$

and

$$\text{(1.9)}\qquad (\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k T^i_{j(h)} = T^i_r\,G^r_{jkh} - T^r_j\,G^i_{rkh}$$

where

$$\text{(1.10)}\qquad T^i_{j(h)} = \partial_h T^i_j - \dot{\partial}_m T^i_j\,G^m_h + T^m_j\,G^i_{mh} - T^i_m\,G^m_{jh}\,.$$

2. **Pseudo projective tensor fields.** In an n-dimensional FINSLER space F_n the pseudo projective deviation tensor $W^{*i}_j(x, \dot{x})$, [2] is given by

$$\text{(2.1)}\qquad W^{*i}_j(x\,,\,\dot{x}) \overset{\text{def}}{=} a\,W^i_j + b\,H^i_j$$

where a and b are scalar functions of $(x, \dot{x})$ and are homogeneous of degree zero in their directional arguments. The pseudo projective curvature tensor fields W^{*i}_{hj} and W^{*i}_{lhj} are defined by

$$\text{(2.2)}\qquad \begin{array}{ll} \text{a)} & W^{*i}_{hj} = \dfrac{2}{3}\,\dot{\partial}_{[h}\,W^{*i}_{j]} \\ \text{b)} & W^{*i}_{lhj} = \dot{\partial}_l\,W^{*i}_{hj} = \dot{\partial}^2_{l[h}\,W^{*i}_{j]}\,. \end{array}$$

Here the square brackets are used to denote the skew symmetric part with respect to the indices enclosed within them. The curvature tensor field $W^{*i}_{hj}(x, \dot{x})$ is expressed in the following form[2] :

$$\text{(2.3)}\qquad W^{*i}_{hj} = aW^i_{hj} + b\,H^i_{hj} + \frac{2}{3}\left\{\dot{\partial}_{[h}\,a\,W^i_{j]} + \dot{\partial}_{[h}\,b\,H^i_{j]}\right\}.$$

The pseudo projective deviation tensor field $W^{*i}_j(x, \dot{x})$ is positively homogeneous of degree two in its directional arguments. By virtue of the homogeneity property of $W^{*i}_j(x, \dot{x})$ we have the following identities and contractions

$$\text{(2.4)}\qquad \text{a)}\ W^{*i}_{hj}\,\dot{x}^h = W^{*i}_j \qquad \text{b)}\ W^{*i}_{lhj}\,\dot{x}^l = W^{*i}_{hj}$$

$$\text{(2.5)}\qquad \text{a)}\ W^{*i}_{lhj}\,\dot{x}^l\,\dot{x}^h = W^{*i}_j \qquad \text{b)}\ \dot{\partial}_h\,W^{*i}_j\,\dot{x}^j = -W^{*i}_h$$

and

$$\text{(2.6)}\qquad \text{a)}\ W^{*i}_i = b(n-1)\,H \qquad \text{b)}\ W^{*i}_{hi} = b\,H_h + \frac{1}{3}\left\{\dot{\partial}_h\,b(n-1)H - \dot{\partial}_i\,a\,W^i_h - \dot{\partial}_i\,b\,H^i_h\right\}$$

3. W^{*i}_{hj} **Generalised 2-recurrent** FINSLER **space. Definition (3.1).** The pseudo projective curvature tensor field W^{*i}_{lhj} is said to be recurrent pseudo projective curvature tensor field of the first order if it satisfies the relation

$$W^{*i}_{lhj(k)} \overset{\text{def}}{=} \lambda_k W^{*i}_{lhj} \tag{3.1}$$

where $\lambda_k(x)$ is a non zero recurrent vector.

Definition (3.2). A FINSLER space F_n is said to be W^{*i}_{lhj} - generalised 2-recurrent FINSLER space if the pseudo projective curvature tensor field W^{*i}_{lhj} satisfies the following relation :

$$W^{*i}_{lhj(k)(m)} \overset{\text{def}}{=} \beta_m W^{*i}_{lhj(k)} + a_{km} W^{*i}_{lhj} \tag{3.2}$$

where β_m is a recurrence vector field and a_{km} is a non zero tensor of the second order.

Transvecting (3.2) by $\dot{x}^l$ and noting (2.4b), we have

$$W^{*i}_{hj(k)(m)} = \beta_m W^{*i}_{hj(k)} + a_{km} W^{*i}_{hj}. \tag{3.3}$$

Hence the tensor field W^{*i}_{hj} is also generalised 2-recurrent in F_n. Again transvecting (3.3) by $\dot{x}^h$ and noting (2.4a), we get

$$W^{*i}_{j(k)(m)} = \beta_m W^{*i}_{j(k)} + a_{km} W^{*i}_{j} \tag{3.4}$$

Hence the pseudo projective deviation tensor field is generalised 2-recurent in the FINSLER space F_n.

Theorem (3.1). *In a generalised 2-recurrent* FINSLER *space, we have*

$$\left[\left\{ \beta_{[k} W^{*i}_{i\rangle (m)]} + W^{*i}_{i} a_{[mk]}\right\} + \frac{1}{2}(n-1)\left\{(\dot{\partial}_r b) H + b(\dot{\partial}_r H)\right\} H^r_{mk}\right] = 0. \tag{3.5}$$

Proof. Interchanging the indices m and k in (3.4) and subtracting the equation thus obtained from (3.4), we get

$$W^{*i}_{j(m)(k)} - W^{*i}_{j(k)(m)} = \beta_k W^{*i}_{j(m)} - \beta_m W^{*i}_{j(k)} + W^{*i}_{j}(a_{mk} - a_{km}). \tag{3.6}$$

Applying the commutation formula (1.8), we have

$$\begin{aligned} -(\dot{\partial}_r W^{*i}_{j}) H^r_{mk} - W^{*i}_{r} H^r_{jmk} + W^{*r}_{j} H^i_{rmk} = \beta_k W^{*i}_{j(m)} - \\ - \beta_m W^{*i}_{j(k)} + W^{*i}_{j}(a_{mk} - a_{km}). \end{aligned} \tag{3.7}$$

Contracting (3.7) with respect to the indices i and j, we get

$$\begin{aligned} -(\dot{\partial}_r W^{*i}_{j}) H^r_{mk} - W^{*i}_{r} H^r_{imk} + W^{*r}_{i} H^i_{rmk} = \beta_k W^{*i}_{i(m)} - \\ - \beta_m W^{*i}_{i(k)} + W^{*i}_{i}(a_{mk} - a_{km}). \end{aligned} \tag{3.8}$$

Applying (2.6a), (3.8) reduces to the form (3.5).

Theorem (3.2). *In a generalised 2-recurrent* FINSLER *space the recurrent tensor field* a_{mk} *is non symmetric.*

Proof. Commutating (3.2) with respect to the indices k and m, we have

$$W^{*i}_{lhj(k)(m)} - W^{*i}_{lhj(m)(k)} = \beta_m W^{*i}_{lhj(k)} - \beta_k W^{*i}_{lhj(m)} + (a_{km} - a_{mk}) W^{*i}_{lhj} . \tag{3.9}$$

Applying the commutation formula (1.8), we have

$$\left[-(\dot{\partial}_r W^{*i}_{lhj}) H^r_{km} - W^{*i}_{rhj} H^r_{lkm} - W^{*i}_{lrj} H^r_{hkm} - W^{*i}_{lhr} H^r_{jkm} + W^{*r}_{lhj} H^i_{rkm} - \beta_m W^{*i}_{lhj(k)} + \beta_k W^{*i}_{lhj(m)}\right] = (a_{km} - a_{mk}) W^{*i}_{lhj} . \tag{3.10}$$

It shows that a_{km} is non symmetric.

Theorem (3.3) *In a generalised 2-recurrent* FINSLER *space the pseudo projective curvature tensor field* W^{*i}_{lhj} *satisfies the relation*

$$\begin{aligned}\Big[(a_{km} - a_{mk})_{(n)} W^{*i}_{lhj} &+ H^r_{km(n)} (\dot{\partial}_r W^{*i}_{lhj}) + W^{*i}_{lrj} H^r_{hkm(n)} + \\ &+ W^{*i}_{rhj} H^r_{lkm(n)} + W^{*i}_{lhr} H^r_{jkm(n)} + 2\beta_n \beta_{[m} \lambda_{k]} W^{*i}_{lhj} + \\ &+ 2\lambda_n \beta_{[k} \lambda_{m]} W^{*i}_{lhj} + 2\beta_{[m} a_{k]n} W^{*i}_{lhj}\Big] \dot{x}^n = 0 .\end{aligned} \tag{3.11}$$

Proof. Differentiating (3.10) with respect to x^n (in the sense of BERWALD) and taking into account the commutation formula (1.9), we get

$$\begin{aligned}&(a_{km} - a_{mk})_{(n)} W^{*i}_{lhj} + \lambda_n W^{*i}_{lhj} (a_{km} - a_{mk}) = -H^r_{km(n)} (\dot{\partial}_r W^{*i}_{lhj}) - \\ &H^r_{km} (\lambda_n \dot{\partial}_r W^{*i}_{lhj} + W^{*s}_{lhj} G^i_{srn} - W^{*i}_{shj} G^s_{rln} - W^{*i}_{lsj} G^s_{hrn} - \\ &W^{*i}_{lhs} G^s_{jrn} - \lambda_n (W^{*s}_{lrj} H^r_{hkm} + W^{*i}_{lhr} H^r_{jkm} + W^{*i}_{rhj} H^r_{lkn} - \\ &W^{*r}_{lhj} H^i_{rkn}) - W^{*i}_{lrj} H^r_{hkm(n)} - W^{*i}_{rhj} H^r_{lkm(n)} - W^{*i}_{lhr} H^r_{jkm(n)} + \\ &W^{*r}_{lhj} H^i_{rkm(n)} - \beta_{m(n)} \lambda_k W^{*i}_{lhj} + \beta_{k(n)} \lambda_m W^{*i}_{lhj} - \\ &\beta_m (\lambda_k \beta_n W^{*i}_{lhj} + a_{kn} W^{*i}_{lhj}) + \beta_k (\beta_n \lambda_m W^{*i}_{lhj} + a_{mn} W^{*i}_{lhj}) .\end{aligned} \tag{3.12}$$

Equation (3.12) with the help of (3.10), reduces to the form

$$\begin{aligned}&(a_{km} - a_{mk})_{(n)} W^{*i}_{lhj} = -H^\gamma_{km(n)} (\dot{\partial}_\gamma W^{*i}_{lhj}) - \\ &H^r_{km} (W^{*s}_{lhj} G^i_{srn} - W^{*i}_{shj} G^s_{rln} - W^{*i}_{lsj} G^s_{hrn} - W^{*i}_{lhs} G^s_{jrn}) - W^{*i}_{lrj} H^r_{hkm(n)} - \\ &W^{*i}_{lhj} H^r_{lkm(n)} - W^{*i}_{lhr} H^r_{jkm(n)} - \beta_m (\lambda_k \beta_n W^{*i}_{lhj} + a_{kn} W^{*i}_{lhj}) + \\ &\beta_k (\beta_n \lambda_m W^{*i}_{lhj} + a_{mn} W^{*i}_{lhj}) + \lambda_n (\beta_m \lambda_k W^{*i}_{lhj} - \beta_k \lambda_m W^{*i}_{lhj}) .\end{aligned} \tag{3.13}$$

Transvecting (3.13) by $\dot{x}^n$ and using (1.7), we get (3.11).

Theorem (3.4). *In a generalised 2-recurrent* FINSLER *space the pseudo projective tensor field* W^{*i}_{hj} *satisfies the following relation* :

$$\begin{aligned}&\left[b(n-1)\left\{(\dot{\partial}_l \beta_m) H_{(k)} + (\dot{\partial}_l a_{km}) H + H_{(r)} G^r_{kln}\right\} + \right.\\ &\left.\left\{\beta_m (\dot{\partial}_l W^{*i}_{hi(k)}) \dot{x}^h + a_{km} (\dot{\partial}_l W^{*i}_{hi}) \dot{x}^h + W^{*i}_{li(k)(m)}\right\}\right] = 0 .\end{aligned} \tag{3.14}$$

Proof. Using the commutation formula (1.9) for $W^{*i}_{hj(k)}$ (k), we get

$$\begin{aligned}&\left\{\dot{\partial}_l (W^{*i}_{hj(k)})\right\}_{(m)} - \dot{\partial}_l \left\{(W^{*i}_{hj(k)})_{(m)}\right\} = W^{*i}_{rj(k)} G^r_{hlm} + \\ &W^{*i}_{hr(k)} G^r_{jlm} + W^{*i}_{hj(r)} G^r_{klm} - W^{*r}_{hj(k)} G^i_{rlm} .\end{aligned} \tag{3.15}$$

Differentiating (3.3) partially with respect to $\dot{x}^l$ and using (3.15), we get

$$\begin{aligned}&\left\{\dot{\partial}_l (W^{*i}_{hj(k)})\right\}_{(m)} - W^{*i}_{rj(k)} G^r_{hlm} - W^{*i}_{hr(k)} G^r_{jlm} - W^{*i}_{hj(r)} G^r_{k(m)} + \\ &W^{*r}_{hj(k)} G^i_{rlm} = \beta_m (\dot{\partial}_l W^{*i}_{hj(k)}) + (\dot{\partial}_l \beta_m) W^{*i}_{hj(k)} + (\dot{\partial}_l a_{km}) W^{*i}_{hj} + \\ &a_{km} (\dot{\partial}_l W^{*i}_{hj}) .\end{aligned} \tag{3.16}$$

Now, applying the commutation formula (1.9) for W^{*i}_{hj}, we get

$$\dot{\partial}_l (W^{*i}_{hj(k)}) = (\dot{\partial}_l W^{*i}_{hj})_{(k)} - W^{*i}_{rj} G^{\gamma}_{hlk} - W^{*i}_{hr} G^r_{jlk} + W^{*r}_{hj} G^i_{rlk} . \tag{3.17}$$

From (3.17), we have

$$\begin{aligned}&\left\{\dot{\partial}_l (W^{*i}_{hj(k)})\right\}_{(m)} = W^{*i}_{lhj(k)(m)} - W^{*i}_{rj(m)} G^r_{hlk} - \\ &W^{*i}_{rj} G^r_{hlk(m)} - W^{*i}_{hr(m)} G^r_{jlk} - W^{*i}_{hr} G^r_{jlk(m)} + W^{*r}_{hj(m)} G^i_{rlk} + \\ &W^{*r}_{hj} G^i_{rlk(m)} .\end{aligned} \tag{3.18}$$

With the help of (3.18) the equation (3.16) reduces to the form

$$\begin{aligned}&W^{*i}_{lhj(k)(m)} - W^{*i}_{rj(m)} G^r_{hlk} - W^{*i}_{rj} G^r_{hlk(m)} - W^{*i}_{hr(m)} G^r_{jlk} - \\ &W^{*i}_{hr} G^r_{jlk(m)} + W^{*r}_{hj(m)} G^i_{rlk} + W^{*r}_{hj} G^i_{rlk(m)} - W^{*i}_{rj(k)} G^r_{hlm} - \\ &W^{*i}_{hr(k)} G^r_{jlm} - W^{*i}_{hj(r)} G^r_{klm} + W^{*r}_{hj(k)} G^i_{rlm} = \\ &\beta_m (\dot{\partial}_l W^{*i}_{hj(k)}) + (\dot{\partial}_l \beta_m) W^{*i}_{hj(k)} + (\dot{\partial}_l a_{km}) W^{*i}_{hj} + a_{km} (\dot{\partial}_l W^{*i}_{hj}) .\end{aligned} \tag{3.19}$$

Contracting the equation (3.19) with respect to the indices i and j, then multiplying by $\dot{x}^h$ and using the relation (2.6a) we get (3.14).

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(*Manuscript received on August* 28, 1975)

ÖZET

n-boyutlu bir F_n FINSLER uzayında, a ve b, $(x, \dot{x})$ değişken takımının skalar olan ve doğrultu değişken takımı cinsinden sıfırıncı dereceden homogen fonksiyonları göstersin. W^i_j ve H^i_j aynı uzayın projektif sapma tansör alanı ve BERWALD anlamındaki sapma alanı olsun. Uzayın sözde projektif tansör alanı $\overset{*}{W}{}^i_j(x, \dot{x}) = a\, W^i_j + b\, H^i_j$ biçiminde tanımlanır ve bu tanımdan hareket edilerek

$$\overset{*}{W}{}^i_{hj} = \frac{2}{3}\, \dot{\partial}_{[h}\, \overset{*}{W}{}^i_{j]}, \quad \overset{*}{W}{}^i_{lhj} = \overset{\cdot 2}{\partial}_{l[h}\, W^i_{j]}$$

elde edilir. β_m bir indirgeme vektör alanı, α_{km} sıfır olmayan ikinci dereceden bir tansör alanı olmak üzere

$$\overset{*}{W}{}^i_{lhj(k)(m)} = \beta_m\, \overset{*}{W}{}^i_{lhj(k)} + \alpha_{km}\, \overset{*}{W}{}^i_{lhj}$$

bağıntısının gerçeklenmesi halinde, verilen F_n FINSLER uzayına $\overset{*}{W}{}^i_{lhj}$ *- genelleştirilmiş* 2 *- indirgenmiş* FINSLER *uzayı* denir. Bu tür uzaylarda yukarda tanımlanan tansörlerin sağladıkları bâzı bağıntılar elde edilmiştir.