# ON WAVE SOLUTIONS OF THE UNIFIED FIELD EQUATIONS OF FINZI IN A GENERALIZED PERES SPACE-TIME [1)]

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The wave solutions of the field equations of the unified field theory of FINZI have been investigated in a generalized PERES space-time, represented by the metric

$$ds^2 = -A\,dx^2 - B\,dy^2 - (1-E)\,dz^2 - 2E\,dz\,dt + (1+E)\,dt^2,$$

where $A$, $B$ and $E$ are functions of $(x, y, z-t)$. Also, in this set-up, we have determined two different electromagnetic fields based on the definition of GRAIFF which are null and transverse in nature.

**1. Introduction.** Several generalizations of the field equations of general relativity have been attempted since WEYL [1] showed that a fusion of electromagnetism with gravitation can be effected by enlarging the geometrical basis of the theory. Accordingly, EINSTEIN [2] developed a theory based on the geometrical interpretation of gravitation and electromagnetism by introducing a non-symmetric fundamental tensor $g_{ij}$ and a non-symmetric affinity $\Gamma^k_{ij}$ taking a priori the torsion vector $\Gamma_i \equiv \Gamma^j_{[ij]} = 0$. SCHRÖDINGER [3], BONNOR [4], BUCHDAHL [5] and many others also have given unified field theories taking a priori the torsion vector $\Gamma_i = 0$.

FINZI [6], on the other hand, without assuming the torsion vector to be zero, has given the following set of unified field equations:

$$g_{\underset{+-}{ij;k}} \equiv g_{ij,k} - g_{lj}\,\Gamma^l_{ik} - g_{il}\,\Gamma^l_{kj} = 0, \tag{1.1}$$

$$R^*_{(ij)} = 0, \tag{1.2}$$

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$$\text{rot } R^*_{[ij]} \equiv R^*_{[ij],k} + R^*_{[jk],i} + R^*_{[ki],j} = 0, \tag{1.3}$$

$$\text{div } R^{*[ij]} \equiv R^{*[ij]}_{,j} + R^{*[ij]} \Gamma^s_{(js)} = 0, \tag{1.4}$$

where $R^*_{ij} = R_{ij} + \Gamma_{i\underset{+}{j}}$ and $R_{ij}$ is the generalized RICCI tensor defined by

$$R_{ij} = \Gamma^s_{ij,s} - \Gamma^s_{is,j} - \Gamma^s_{tj} \Gamma^t_{is} + \Gamma^s_{ts} \Gamma^t_{ij}. \tag{1.5}$$

Here round and square brackets denote the symmetric and antisymmetric parts, respectively. $A$ + or — sign below an index fixes the position of the covariant index $k$ in the connection as $\Gamma^{\cdot}_{\cdot k}$ and $\Gamma^{\cdot}_{k\cdot}$ respectively. A comma preceding an index $i$ denotes partial differentiation with respect to $x^i$ and a semicolon stands for covariant differentiation with respect to $\Gamma^i_{jk}$.

IKEDA [7] in 1954 found a skew-symmetric tensor $H_{ij}$ in terms of the non-symmetric fundamental tensor $g_{ij}$ satisfying the properties

(*i*) that the total rotation of $H_{ij}$ is zero
and
(*ii*) that $H_{ij}$ has a non-zero divergence.

He named this tensor the electrcmagnetic field tensor. In order that the skew-symmetric tensor $H_{ij}$ may satisfy the property (*i*), IKEDA assumed $\Gamma_i = 0$ and defined the electromagnetic field tensor $H_{ij}$ by the relation

$$H_{ij} = (\tfrac{1}{2})\, \varepsilon_{ijkl} \sqrt{-\,|\,g_{ij}\,|}\; g^{kl}, \tag{1.6}$$

where $\varepsilon_{ijkl}$ takes the values + 1 or — 1 according to $ijkl$ is an even or odd permutation of 1234. Thus (1.6) is valid only in the case of such field equations which assume a priori $\Gamma_i = 0$. But in case of the field equations of FINZI in which $\Gamma_i \neq 0$, $H_{ij}$ given by (1.6) does not satisfy the property (*i*) and hence it cannot represent the electromagnetic field tensor in the sense of IKEDA.

In 1955 GRAIFF [8] gave two possible relations between the non-symmetric tensor $g_{ij}$ and the skew-symmetric electromagnetic field tensor $F_{ij}$ each satisfying both the properties (*i*) and (*ii*) without imposing the condition $\Gamma_i = 0$. The two forms of the electromagnetic field tensor given by GRAIFF are

$$F_{ij} = R^*_{[ij]} - \Gamma_{[i,j]} \tag{1.7}$$

and

(1.8) $$F'_{ij} = \tfrac{1}{2}\, \varepsilon_{ijkl}\, \overset{*}{R}_{[pq]}\, g^{kp}\, g^{lq} - \Gamma_{[i,j]}$$

where the symbols used are as defined above.

In this paper we propose to find out the plane wave-like solutions of the field equations (1.1) - (1.4) in a generalized PERES space-time [9], represented by the metric

(1.9) $$ds^2 = -A\, dx^2 - B\, dy^2 - (1 - E)\, dz^2 - 2E\, dz dt + (1 + E)\, dt^2,$$

where $A$, $B$ and $E$ are functions of $(x, y, Z)$; $(Z = z - t)$ and also to obtain the electromagnetic fields based on the definition of GRAIFF [8].

**2. Solution of the field equation (1.1).** In the space-time metric (1.9), the components of the non-symmetric tensor $g_{ij}$ as calculated by PRADHAN [10] are

(2.1) $$(g_{ij}) = \begin{bmatrix} -A & 0 & \rho & -\rho \\ 0 & -B & \sigma & -\sigma \\ -\rho & -\sigma & -(1-E) & -E \\ \rho & \sigma & -E & 1+E \end{bmatrix},$$

where $\rho$ and $\sigma$ are arbitrary functions of $(x, y, Z)$. The contravariant components $g^{ij}$ are given by

(2.2) $$(g^{ij}) = \begin{bmatrix} -1/A & 0 & \rho/A & \rho/A \\ 0 & -1/B & \sigma/B & \sigma/B \\ -\rho/A & -\sigma/B & -1+W & W \\ -\rho/A & -\sigma/B & W & 1+W \end{bmatrix},$$

where $W = (A\sigma^2 - B\rho^2)/AB - E$.

Equation (1.1) of FINZI's field equations is one of the equations in EINSTEIN's unified field theory which has already been solved in [10] and the components of the connection $\Gamma^k_{ij}$ have been uniquely determined.

**3. Calculation of the tensor $R^*_{ij}$.** Using the values of $\Gamma^i_{jk}$ from [10], the components of $\Gamma_i$ are computed as

$$\Gamma_1 = \Gamma_2 = 0 \text{ and } \Gamma_3 = -\Gamma_4 = -(H + I), \tag{3.1}$$

where

$$\begin{aligned} H &= (-\rho_x + \rho A_x/2A - \sigma A_y/2B)/A, \\ I &= (-\sigma_y + \sigma B_y/2B - \rho B_x/2B)/B. \end{aligned} \tag{3.2}$$

Here and in what follows the lower suffixes $x$ and $y$ attached to any function denote its partial derivatives with respect to $x$ and $y$ respectively. A single and double overhead bars stand for partial derivatives with respect to $Z$ once and twice respectively.

Putting $\Gamma_{\underset{+}{i;k}} = \theta_{ik}$, and using the values of $\Gamma_i$ and $\Gamma^i_{jk}$ from (3.1) and [10] the components of $\theta_{ik}$ are given by

$$\begin{aligned} &\theta_{31} = -\theta_{41} = -(H_x + I_x), \theta_{32} = -\theta_{42} = -(H_y + I_y), \\ &\theta_{33} = -\theta_{34} = -\theta_{43} = \theta_{44} = -(\bar{H} + \bar{I}), \text{ other } \theta_{ik} = 0. \end{aligned} \tag{3.3}$$

Substituting the values of $B_{ij}$ from [10] and calculating $\Gamma_{\underset{+}{i;k}}$ with the help of equation (3.1) we find that the symmetric components of $R^*_{ij}$ are given by

$$\begin{aligned} &R^*_{11} = -L/B, \; R^*_{(12)} = R^*_{(21)} = 0, \; R^*_{22} = -L/A, \\ &R^*_{(13)} = -B^*_{(14)} = N/B, R^*_{(23)} = -R^*_{(24)} = -M/A \\ &R^*_{33} = -R^*_{(34)} = R^*_{44} = \gamma, \end{aligned} \tag{3.4}$$

while the skew-symmetric components of $R^*_{ij}$ are given by

$$R^*_{[12]} = R^*_{[34]} = 0,$$

(3.5)

$$R^*_{[13]} = B^*_{[14]} = \alpha, \quad R^*_{[23]} = - R^*_{[24]} = \beta,$$

where

$$2L = A_{yy} + B_{xx} - \tfrac{1}{2}(A_y^2 + A_x B_x)/A - \tfrac{1}{2}(B_x^2 + A_y B_y)/B,$$

$$2M = \bar{A}_y - \tfrac{1}{2} A_y(\bar{A}/A + \bar{B}/B),$$

$$2N = - \bar{B}_x + \tfrac{1}{2} B_x(\bar{A}/A + \bar{B}/B),$$

$$2\gamma = -(\bar{\bar{A}} - \bar{A}^2/2A)/A - (\bar{\bar{B}} - \bar{B}^2/2B)/B + 2(S_x + T_y + \bar{H} + \bar{I} + H^2 + I^2)$$

$$+ b^2/AB + S(A_x/A + B_x/B) + T(A_y/A + B_y/B),$$

(3.6)

$$\alpha = H_x + \tfrac{1}{2}(H - I)\, B_x/B - \tfrac{1}{2} b_y/B - \tfrac{1}{4}(A_y/A - B_y/B)\, b/B,$$

$$\beta = I_y - \tfrac{1}{2}(H - I)\, A_y/A - \tfrac{1}{2} b_x/A + \tfrac{1}{4}(A_x/A - B_x/B)\, b/A,$$

$$b = \sigma_x + \rho_y - \rho A_y/A - \sigma B_x/B,$$

$$S = \{\tfrac{1}{2} E_x + 2\rho(-\rho_x + \rho A_x/2A)/A - \sigma(b + \rho A_y/A)/B\}/A,$$

$$T = \{\tfrac{1}{2} E_y + 2\sigma(-\sigma_y + \sigma B_y/2B)/B - \rho(b + \sigma B_x/B)/A\}/B.$$

Using (2.2) and (3.5), the skew-symmetric contravariant components $B^{*[ij]}$ of the tensor $R^*_{[ij]}$ are found to be

$$\mathrm{R}^{*[12]} = R^{*[34]} = 0, \quad R^{*[13]} = R^{*[14]} = \alpha/A,$$

(3.7)

$$R^{*[23]} = R^{*[24]} = \beta/B.$$

**4. Solutions of the field equations (1.2), (1.3) and (1.4).** Substituting the values of $R^*_{ij}$ from (3.4) in (1.2) we have

(4.1) $$L = 0,$$

(4.2) $$N = 0,$$

$$M = 0, \tag{4.3}$$

$$\gamma = 0. \tag{4.4}$$

Equation (4.1) can also be written as

$$(B_x/\sqrt{AB})_x + (A_y/\sqrt{AB})_y = 0. \tag{4.5}$$

Equations (4.2) and (4.3), after integration become

$$A_y/\sqrt{AB} = k_1, \tag{4.6}$$

$$B_x/\sqrt{AB} = k_2, \tag{4.7}$$

respectively, where in general $k_1$ and $k_2$ are functions of $(x, y)$. Equations (4.4) - (4.7) are mathematically complicated and it is difficult to get the exact solutions. However, without violating the assumptions taken for the line - element (1.9), we can take

$$A = Bf \ , \ (f = f(Z)), \tag{4.8}$$

which renders it possible to find the exact solutions of the said field equations

From (4.6) and (4.7) we get

$$B_x A_y = AB \, k_1 k_2. \tag{4.9}$$

With the help of (4.8), equation (4.9) reduces to the form

$$p\,q = \psi, (p = B_x, q = B_y, \psi = k_1 k_2 B^2), \tag{4.10}$$

which can be solved by using CHARPIT's method [11].

By virtue of (4.8), equation (4.4) reduces to

$$\begin{aligned} &2\bar{\bar{B}}/B + \bar{B}\,\bar{f}/B\,f + \bar{\bar{f}}/f - (\bar{B}/B)^2 - \tfrac{1}{2}(\bar{f}/f)^2 - 2S'B_x/B - 2T'B_y/B - \\ &2(S'_x + T'_y + H'^2 + I'^2 + \bar{H}' + \bar{I}') - b'^2/B^2 f = 0, \end{aligned} \tag{4.11}$$

where $S'$, $T'$, $H'$, $I'$ and $b'$ are the values of $S$, $T$, $H$, $I$ and $b$ respectively when (4.8) is used in the expressions for $S$, $T$, $H$, $I$ and $b$.

Thus the values of $g_{ij}$ given by (2.1) represent the plane wave-like solutions of the field equation (1.2) under conditions (4.5), (4.8) and (4.11).

It is worth mentioning here that the values of $A$ and $B$ can be found explicitly provided $k_1$ and $k_2$ are chosen properly. For instance:

*Case (i).* If $k_1$ and $k_2$ are two non-zero scalar constants, then from (4.10) the values of $A$ and $B$ satisfying (4.5), (4.6) and (4.7) are given by

$$\begin{aligned} \log B &= x\sqrt{f} + \lambda y/\sqrt{f} + \mu \\ \log A &= x\sqrt{f} + \lambda y/\sqrt{f} + \log f + \mu, \end{aligned} \tag{4.12}$$

where $\lambda$ and $\mu$ are any scalar constants.

In this case the values of $g_{ij}$ given by (2.1) represent the plane wave-like solutions of the field equation (1.2) under conditions (4.11) and (4.12).

*Case (ii).* If $k_1 = 0 = k_2$, then from (4.6) and (4.7) we find that $A_y = 0 = B_x$ in which case the $g_{ij}$ given by (2.1) with $A = A(x, Z)$, $B = B(y, Z)$ and $E = (x, y, Z)$ represent the plane wave-like solutions of the field equation (1.2) provided the condition

$$\begin{aligned} &(\overline{\overline{A}} - \overline{A}^2/2A)/A - (\overline{\overline{B}} - \overline{B}^2/2B)/B - S^* A_x/A - T^* B_y/B - \\ &2(S^*_x + T^*_y + H^{*2} + I^{*2} + \overline{H}^* + \overline{I}^*) - (\sigma_x + \rho_y)^2/AB = 0, \end{aligned} \tag{4.13}$$

holds where $S^*$, $T^*$, $H^*$ and $I^*$ are values of $S$, $T$, $H$ and $I$ respectively when $A_y = 0 = B_x$ are used in the expressions for $S$, $T$, $H$ and $I$.

Theorem 4.1. *A necessary and sufficient condition that $g_{ij}$ given by* (2.1) *be a solution of* FINZI's *field equation* (1.3) *is*

$$\alpha_y - \beta_x = 0. \tag{4.14}$$

**Proof.** Substitution of $R^*_{[ij]}$ from (3.5) in equation (1.3) gives the required condition (4.14). Conversely, if (4.14) holds then the field equation (1.3) is identically satisfied.

Theorem 4.2. *A necessary and sufficient condition that $g_{ij}$ given by* (2.1) *be a solution of* FINZI's *field equation* (1.4) *is*

$$\log \alpha + \tfrac{1}{2} \log (B/A) = k_3 \text{ and } \log \beta + \tfrac{1}{2} \log (A/B) = k_4 \tag{4.15}$$

*where $k_3$ and $k_4$ are functions of $(x, y)$.*

Proof. Substituting the values of $R^{*[ij]}$ from (3.7) and $\Gamma^i_{jk}$ from [10] in the field equation (1.4) we get the required condition (4.15). Conversely, if the condition (4.15) holds then the field equation (1.4) is identically satisfied.

5. The electromagnetic fields. Calculating the values of $\Gamma_{[i,j]}$ from (3.1) we get

$$\begin{aligned} &\Gamma_{[1,2]} = \Gamma_{[3,4]} = 0, \; \Gamma_{[1,3]} = -\Gamma_{[1,4]} = \tfrac{1}{2} (H_x + I_x), \\ &\Gamma_{[2,3]} = -\Gamma_{[2,4]} = \tfrac{1}{2} (H_y + I_y). \end{aligned} \tag{5.1}$$

Considering the form (1.7) for the electromagnetic field tensor $F_{ij}$ and substituting in it the values of $B^*_{[ij]}$ and $\Gamma_{[i,j]}$ from (3.5) and (5.1) we get

$$F_{13} = -F_{14} = U, \; F_{23} = -F_{24} = V \text{ and other } F_{ij} = 0, \tag{5.2}$$

where $U = \alpha - \frac{1}{2} (H_x + I_x)$ and $V = \beta - \frac{1}{2} (H_y + I_y)$.

Considering the form (1.8) for the electromagnetic field tensor and substituting the values of $R^*_{[ij]}$, $g^{ij}$ and $\Gamma_{[i,j]}$ from (3.5), (2.2) and (5.1) respectively in (1.8) we get

$$F'_{13} = -F'_{14} = -U', \; F'_{23} = -F'_{24} = V' \text{ and other } F'_{ij} = 0, \tag{5.3}$$

where $U' = \frac{1}{2} (\beta/B - H_x - I_x)$ and $V' = \frac{1}{2} (\alpha/A + H_y + I_y)$.

The contravariant electromagnetic field tensor $F^{ij}$ and the dual tensor $F^*_{ij}$ corresponding to $F_{ij}$ given by (5.2) are

$$F^{13} = F^{14} = U/A, \; F^{23} = F^{24} = V/B, \text{ other } F^{ij} = 0, \tag{5.4}$$

and

(5.5) $F^*_{13} = -F^*_{14} = -V\sqrt{A/B},\ F^*_{23} = -F^*_{24} = U\sqrt{B/A}$, other $F^*_{ij} = 0$.

Thus from (5.2), (5.4) and (5.5) it is easily seen that $F_{ij}\,F^{ij} = F^*_{ij}\,F^{ij} = 0$, and hence the electromagnetic field tensor given by (5.2) is null in the sense of SYNGE [12]. Moreover $F_{12} = F_{34} = 0$, the electromagnetic waves are of transverse character.

The contravariant electromagnetic field tensor $F'^{ij}$ and the dual tensor $F'^*_{ij}$ corresponding to (5.3) are given by

(5.6) $F'^{13} = F'^{14} = -U'/A,\ F'^{23} = F'^{24} = V'/B$, other $F'_{ij} = 0$,

and

(5.7) $F'^*_{13} = -F'^*_{14} = -V'\sqrt{A/B},\ F'^*_{23} = -F'^*_{24} = U'\sqrt{B/A}$, other $F'^*_{ij} = 0$.

Thus from (5.3), (5.6) and (5.7) it is clear that $F'_{ij}\,F'^{ij} = F'^*_{ij}\,F'^{ij} = 0$ and hence the electromagnetic field tensor given by (5.3) is also null. Here again $F'_{12} = F'_{34} = 0$ shows the transverse character of the electromagnetic waves.

Hence the unified field theory of FINZI in the generalized PERES space-time (1.9) gives two different electromagnetic fields given by (5.2) and (5.3) both of which are null and transverse.

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## ÖZET

FINZI'nin birleştirilmiş alan teorisinin alan denklemlerinin dalga çözümleri, $A$, $B$ ve $E$ katsayıları $x$, $y$, $z - t$ değişkenlerinin fonksiyonları olmak üzere

$$ds^2 = -A\,dx^2 - B\,dy^2 - (1 - E)\,dz^2 - 2E\,dzdt + (1 + E)\,dt^2,$$

metriği ile verilen bir genelleştirilmiş uzay-zaman evreni için incelenmiştir. Aynı çerçeve içinde, GRAIFF'in tanımına dayanarak davranışları bakımından sıfır ve çapraz olan iki farklı elektromanyetik alan belirlenmiştir.