# ON $|\bar{N}, p_n|$ SUMMABILITY FACTORS

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In this paper an analogue of a theorem of Kogbetliantz (1925) on absolute Cesàro summability factors of an infinite series is established, which includes results by Mohanty (1950) and Kishore and Hotta (1970) as special cases.

1. **Introduction.** Let $\Sigma a_n$ be an infinite series with partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants real or complex such that

$$P_n = p_0 + p_1 + \dots + p_n ; \; P_{-1} = p_{-1} = 0 .$$

The sequence-to-sequence transformation

$$\bar{t}_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \, (P_n \neq 0),$$

defines the $(\bar{N}, p_n)$-mean of the sequence $\{s_n\}$, or of the series $\Sigma a_n$. If the sequence $\{\bar{t}_n\}$ is of bounded variation, i.e.

$$\sum_{n=1}^{\infty} | \bar{t}_n - \bar{t}_{n-1} | < \infty,$$

then the series is said to be absolutely summable $(\bar{N}, p_n)$, or simply summable $|\bar{N}, p_n|$.

In the particular case when $p_n = 1$, the $(\bar{N}, p_n)$-mean reduces to $(C, 1)$-mean; and if we take $p_n = \frac{1}{n}$, then it is known that $\left|\bar{N}, \frac{1}{n}\right|$ is equivalent to $|R, \log n, 1|$.

2. Concerning absolute CESÀRO summability factors KOGBETLIANTZ [2] proved the following theorem.

**Theorem A.** *If* $\Sigma a_n$ *is summable* $|C, \delta|$, *then* $\Sigma \dfrac{a_n}{n^{\delta-\gamma}}$ *is summable* $|C, \gamma|$ *for* $\gamma \leqslant \delta$; $\delta, \gamma > 0$.

MOHANTY ([3] see also TATCHELL [4]) has established the following result.

**Theorem B.** *If* $\Sigma a_n$ *is summable* $|R, \log n, 1|$, *then the series* $\Sigma \dfrac{a_n}{\log n}$ *is summable* $|C, 1|$.

KISHORE and HOTTA [1] recently proved the following analogue of Theorem *A* for $|\bar{N}, p_n|$-summability.

**Theorem C.** *If* $\Sigma a_n$ *is* $|\bar{N}, p_n|$*-summable, then* $\Sigma \dfrac{a_n p_n Q_n}{q_n P_n}$ *is* $|\bar{N}, q_n|$ *summable provided* $\{p_n\}$ *and* $\{q_n\}$ *are positive sequences such that the sequences* $\left\{\Delta\left(\dfrac{Q_n}{q_n}\right)\right\}$, $\left\{\dfrac{Q_n p_n}{P_n q_n}\right\}$ *and* $\left\{\dfrac{Q_{n+1}}{q_{n+1}} \dfrac{\Delta p_n}{p_n}\right\}$ *are bounded.*

The object of this paper is to establish an analogue of theorem *A* with general factor $\varepsilon_n$ which includes, amongst others, theorems *B* and *C* in special cases. We prove the following theorem :

**Theorem.** *If* $\Sigma a_n$ *is* $|\bar{N}, p_n|$ *summable then* $\Sigma a_n \varepsilon_n$ *is* $|\bar{N}, q_n|$ *summable, provided* $\{p_n\}$ *and* $\{q_n\}$ *are positive sequences such that, as* $n \to \infty$,

(*i*) $\dfrac{p_n}{P_n} = O\left(\dfrac{q_n}{Q_n}\right)$,

(*ii*) $\dfrac{q_n P_n}{p_n Q_n} \varepsilon_n = O(1)$,

(*iii*) $\dfrac{P_n}{p_n} \Delta\varepsilon_n = O(1)$.

**3. Proof of theorem.** Let $\bar{t}_n$ denote the $(\bar{N}, p_n)$-mean of the $\Sigma a_n$. Then

$$\bar{t}_n = \frac{1}{P_n} \sum_{v=0}^{n} P_v s_v = \frac{1}{P_v} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v .$$

If $\bar{T}_n$ denotes the $(\bar{N}, q_n)$-mean of $\Sigma a_n \varepsilon_n$, we similarly have

$$\bar{T}_n = \frac{1}{Q_n} \sum_{v=0}^{n} (Q_n - Q_{v-1}) a_v \varepsilon_v$$

and

$$\bar{T}_{n+1} - \bar{T}_n = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n+1} Q_{v-1} \varepsilon_v a_v$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \Delta[Q_{v-1} \varepsilon_v] s_v + \frac{q_{n+1}}{Q_{n+1}} \varepsilon_{n+1} s_{n+1}$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{1}{p_v} \Delta(Q_{v-1} \varepsilon_v) s_v p_v +$$

$$+ \frac{q_{n+1}}{Q_{n+1} p_{n+1}} \varepsilon_{n+1} s_{n+1} p_{n+1}$$

$$= - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{1}{p_v} \Delta(Q_{v-1} \varepsilon_v) \Delta(P_{v-1} \bar{t}_{v-1}) -$$

$$- \frac{q_{n+1}}{P_{n+1} Q_{n+1}} \varepsilon_{n+1} \Delta(P_n \bar{t}_n)$$

$$= - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{1}{p_v} \Delta(Q_{v-1} \varepsilon_v) (\bar{t}_{v-1} \Delta P_{v-1} + P_v \Delta \bar{t}_{v-1}) -$$

$$- \frac{q_{n+1}}{P_{n+1} Q_{n+1}} \varepsilon_{n+1} (\bar{t}_n \Delta P_n + P_{n+1} \Delta \bar{t}_n)$$

$$= \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \Delta(Q_{v-1}\,\varepsilon_v)\,\bar{t}_{v-1} - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{P_v}{p_v} \times$$

$$\times\ \Delta(Q_{v-1}\,\varepsilon_v)\,\Delta\bar{t}_{v-1} + \frac{q_{n+1}}{Q_{n+1}}\,\varepsilon_{n+1}\,\bar{t}_n -$$

$$- \frac{q_{n+1}\,P_{n+1}}{p_{n+1}\,Q_{n+1}}\,\varepsilon_{n+1}\,\Delta\bar{t}_v$$

$$= - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n-1} Q_v\,\varepsilon_{v+1}\,\Delta\bar{t}_{v-1} - \frac{q_{n+1}\,Q_n}{Q_n Q_{n+1}}\,\varepsilon_{n+1}\,\bar{t}_{n-1} -$$

$$- \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{P_v}{p_v}\,\Delta(Q_{v-1}\,\varepsilon_v)\,\Delta\bar{t}_{v-1} + \frac{q_{n+1}}{Q_{n+1}}\,\varepsilon_{n+1}\,\bar{t}_n -$$

$$- \frac{q_{n+1}\,P_{n+1}}{p_{n+1}\,Q_{n+1}}\,\varepsilon_{n+1}\,\Delta\bar{t}_n\,.$$

$$= - \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n-1} Q_v\,\varepsilon_{v+1}\,\Delta\bar{t}_{v-1} - \frac{q_{n+1}}{Q_{n+1}}\,\varepsilon_{n+1}\,\Delta\bar{t}_{n-1} -$$

$$- \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{P_v}{p_v}\,\Delta(Q_{v-1}\,\varepsilon_v)\,\Delta\bar{t}_{v-1} - \frac{q_{n+1}\,P_{n+1}}{p_{n+1}\,Q_{n+1}}\,\varepsilon_{n+1}\,\Delta\bar{t}_n$$

$$\bar{T}_n - \bar{T}_{n+1} = \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} Q_v\,\varepsilon_{v+1}\,\Delta\bar{t}_{v-1} +$$

$$+ \frac{q_{n+1}}{Q_n Q_{n+1}} \sum_{v=0}^{n} \frac{P_v}{p_v}\,\Delta(Q_{v-1}\,\varepsilon_v)\,\Delta\bar{t}_{v-1} +$$

$$+ \frac{q_{n+1}\,P_{n+1}}{p_{n+1}\,Q_{n+1}}\,\varepsilon_{n+1}\,\Delta\bar{t}_n\,.$$

Consequently,

$$\sum_{n=0}^{\infty} |\bar{T}_n - \bar{T}_{n+1}| \leq \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{v=0}^{n} Q_v \, \varepsilon_{v+1} \, \Delta \bar{t}_{v-1} \right| +$$

$$+ \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{v=0}^{n} \frac{P_v}{p_v} \Delta(Q_{v-1} \, \varepsilon_v) \, \Delta \bar{t}_{v-1} \right| +$$

$$+ \sum_{n=0}^{\infty} \frac{q_{n+1} \, P_{n+1}}{p_{n+1} \, Q_{n+1}} |\varepsilon_{n+1}| \, |\Delta \bar{t}_n|$$

$$= \sum_1 + \sum_2 + \sum_3 \quad \text{(say)}.$$

Now,

$$\sum_1 = \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{v=0}^{n} Q_v \, \varepsilon_{v+1} \, \Delta \bar{t}_{v-1} \right|$$

$$\leq \sum_{v=0}^{\infty} Q_v \, |\varepsilon_{v+1}| \, |\Delta \bar{t}_{v-1}| \sum_{n=v}^{\infty} \left( \frac{1}{Q_n} - \frac{1}{Q_{n+1}} \right)$$

$$= \sum_{v=0}^{\infty} |\varepsilon_{v+1}| \, |\Delta \bar{t}_{v+1}|$$

$$= O(\Sigma \, |\Delta \bar{t}_{v-1}|) \text{ by hypotheses } (i) \text{ and } (ii)$$

$$= O(1).$$

and also $$\sum_3 = \sum_{n=0}^{\infty} \frac{q_{n+1} \, P_{n+1}}{p_{n+1} \, Q_{n+1}} |\varepsilon_{n+1}| \, |\Delta \bar{t}_n|$$

$$= O(\Sigma \, |\Delta \bar{t}_n|) \text{ by hypothesis } (ii)$$

$$= O(1).$$

Now,

$$\sum_2 = \sum_{n=0}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}} \left| \sum_{v=0}^{n} \frac{P_v}{p_v} \Delta(Q_{v-1} \varepsilon_v) \Delta \bar{t}_{v-1} \right|$$

$$\leq \sum_{v=0}^{\infty} \frac{P_v}{p_v} \left| \Delta(Q_{v-1} \varepsilon_v) \right| \left| \Delta \bar{t}_{v-1} \right| \sum_{n=v}^{\infty} \frac{q_{n+1}}{Q_n Q_{n+1}}$$

$$+ \sum_{v=0}^{\infty} \frac{P_v}{p_v Q_v} \left| \Delta(Q_{v-1} \varepsilon_v) \right| \left| \Delta \bar{t}_{v-1} \right|$$

$$= \sum_{v=0}^{\infty} \frac{P_v q_v}{p_v Q_v} \left| \varepsilon_v \right| \left| \Delta \bar{t}_{v-1} \right| + \sum_{v=0}^{\infty} \frac{P_v Q_v}{p_v Q_v} \left| \Delta \varepsilon_v \right| \left| \Delta \bar{t}_{v-1} \right|$$

$$= \sum_{v=0}^{\infty} \frac{P_v q_v}{p_v Q_v} \left| \varepsilon_v \right| \left| \Delta \bar{t}_{v-1} \right| + \sum_{v=0}^{\infty} \frac{P_v}{p_v} \left| \Delta \varepsilon_v \right| \Delta \bar{t}_{v-1} \left|\right.$$

$= O(\Sigma \left| \Delta \bar{t}_{v-1} \right|)$ hy hypotheses *(ii)* and *(iii)*

$= O(1)$.

Thus the proof of the theorem is completed.

## REFERENCES

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## ÖZET

Bu çalışmada Kogbetliantz'ın bir sonsuz serinin mutlak Cesàro toplanabiliriik çarpanları hakkında 1925 de verniş olduğu teoremin bir benzeri verilmekte olup, bu teorem Mohanty (1950) ve Kishore ile Hotta (1970) nu sonuçlarını özel haller olarak ihtiva etmektedir.