ON A LORENTZIAN MANIFOLD $V_{\mathscr{L}}^{2n+1}$ ENDOWED WITH A PRINCIPAL CONNECTION

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Let $V_{\mathscr{L}}^{2n+1}$ be a Lorentzian manifold of odd dimension $2n+1$, let $\{p, \xi_A; A = 1, \dots, 2n+1; p \in V_{\mathscr{L}}^{2n+1}\}$ be a quasi - orthonormal vector basis associated with $p \in V_{\mathscr{L}}^{2n+1}$ and let $\{\alpha^A\}$ be the dual basis. If $\{\xi_a ; a = 1, 2n+1\}$ are the two null (real) vectors of this basis, then the submanifold $W \subset V_{\mathscr{L}}^{2n+1}$ defined by $\alpha^{2n+1} = 0$, $\alpha^{2n} = 0$ is degenerate. That is if $T_{p|W}$ and $N_{p|W}$ are the tangent space and the normal space at $p|W$ of W, then one has $T_{p|W} \cap N_{p|W} \neq 0$.

The present paper is concerned with the study of such degenerate submanifolds W (denoted by $V_{(1)}^{2n-1}$ because the defect of the inclusion $x: W \to V_{\mathscr{L}}^{2n+1}$ is 1) of a $V_{\mathscr{L}}^{2n+1}$ manifold structured by a principal connection [2]. Different properties of the tangential (and normal) null field ξ_1 are obtained. If the spatial tangential connection forms associated with $x: V_{(1)}^{2n-1} \to V_{\mathscr{L}}^{2n+1}$ vanish then one defines on $V_{(1)}^{2n-1}$ a conformal cosymplectic structure having ξ_1 as canonical field.

1. Let $V_{\mathscr{L}}^{2n+1}$ be a Lorentzian C^∞-manifold of odd dimension $2n+1$ and let $\mathscr{F} = \cup \{p, \xi_A; A = 1,2,\dots, n+1\}$ be the principal fiber bundle of the *quasi-orthonormal frames* $\{p, \xi_A\}$ and $\alpha_B^A = l_{BC}^A \alpha^C$ the connection forms on $\mathscr{F}$, where α^C is the dual basis of the vector basis ξ_C.

We denote by $a, b = 1, 2n+1$ and $r, s = 2,3,\dots 2n$ the isotropic indices and the spatial indices respectively of the vector basis ξ_A. The basis ξ_A being normed, we have [1]

$$< \xi_a , \xi_r > = 0, \; < \xi_r , \xi_s > \; = - \delta^r_s ,$$

(1)
$$< \xi_a , \xi_b > = \begin{cases} 1 & a = b \\ 0 & a \neq b , \end{cases}$$

where δ^r_s is the Kronecker symbol.

The connection ∇ which structures $V^{2n+1}_{\mathscr{L}}$ is given by

(2)
$$\nabla \xi_A = \alpha^B_A \otimes \xi_B$$

and, making use of (1), we obtain

(3)
$$\alpha_1^{2n+1} = 0 = \alpha^1_{2n+1} , \quad \alpha^1_1 + \alpha^{2n+1}_{2n+1} = 0 ,$$
$$\alpha_r^{2n+1} - \alpha^r_1 = 0 \quad , \quad \alpha^r_{2n+1} - \alpha^1_r = 0 ,$$
$$\alpha^r_s + \alpha^s_r = 0 .$$

The connection ∇ being torsion free (a Levi-Civita connection), the two groups of structure equations (É. CARTAN) are

(4)
$$d \wedge \alpha^A = \alpha^B \wedge \alpha^A_B ,$$

(4')
$$d \wedge \alpha^A_B = \alpha^C_B \wedge \alpha^A_C + \Omega^A_B .$$

In (4') Ω^A_B are the curvature 2-forms and the connection form $\alpha^1_1 (= - \alpha^{2n+1}_{2n+1})$ is the unique *homothety component* of ∇.

In [2] the connection forms $(\alpha^r_{2n+1} , \alpha^{2n+1}_r)$ have been called *mixed connection forms* of ∇ and iff

(5)
$$\alpha^r_1 = \bar{k}_r \, \alpha^r , \quad \alpha^1_r = k_r \, \alpha^r \quad \text{(no summation over r)}$$

holds, the connection is called *principal* [2]. Such a connection is denoted by ∇_P and the Lorentzian manifold structured with a ∇_P connection is denoted by V^{2n+1}_P.

This paper is concerned with the study of some properties of a degenerate manifold of defect 1 and codimension 2 included in V_P^{2n+1}.

2. Let

$$dp = \alpha^A \otimes h_A$$

be the line element (dp is a canonical 1-form on a Lie group G) of the manifold V_P^{2n+1}. A degenerate submanifold V^{2n-1} of defect 1 and of codimension 2 can be considered as an integral submanifold of the system

$$\alpha^{2n+1} = 0 ,$$

$$\alpha^{2n} \quad = 0 .$$

If we denote by $\tilde{\alpha}_B^A$, $\tilde{\alpha}^C$, $\tilde{k}$,... the forms and functions induced by the inclusion $x : V^{2n-1} \to V_P^{2n+1}$, we can write for the induced line element on V^{2n-1}

$$dx(p) = \tilde{\alpha}^1 \otimes \xi_1 + \tilde{\alpha}^i \otimes \xi_i , \qquad i = 2,3,\dots 2n-1. \tag{6}$$

Taking account of (1), the metric of V^{2n-1} is of spatial type and has the form

$$\widetilde{ds}^2 = -\sum_i (\tilde{\alpha}^i)^2 . \tag{7}$$

The vector ξ_1 being both tangent and normal to V^{2n-1}, the normal space $N_{x(p)}$ ie determined by ξ_1 and ξ_{2n}. If $T_{x(p)}$ is the tangent space to V^{2n-1} at $x(p)$, we can split it as

$$T_{x(p)} = \{\xi_1\} \otimes S_{x(p)} ,$$

where $S_{x(p)}$ is the spatial component of $T_{x(p)}$.

Clearly one has

$$T_{x(p)} \cap N_{x(p)} \neq 0 ,$$

and this points out the degenerate character of the immersion x. Furthermore, the rank of the matrix (7) being equal to $2n-2$, it follows that the defect of V^{2n-1} is equal to 1. Such a manifold will be denoted by $V_{(1)}^{2n-1}$.

So, taking account of (5), we see that exterior differentiation of the isotropic covector gives no new equations. (This is in accordance with the pair of associated coisotropic hypersurfaces which always carries a V_p^{2n+1} manifold [3]). Exterior differentiation of the spatial covector gives then (by means of the Cartan's lemma) the $2n-2$ linear equations

$$\tilde{\alpha}_i^{2n} = m_{ik} \tilde{\alpha}^k, \; m_{ik} = m_{ki}, \qquad i, k = 2,3,\dots 2n-1.$$

Thus by the virtue of a known theorem, one can see that this system is always completely integrable and this proves the existence of the degenerate manifold $V_{(1)}^{2n-1}$.

Denote now by

$$\eta = \tilde{\alpha}^1 \wedge \tilde{\alpha}^2 \wedge \dots \wedge \tilde{\alpha}^{2n-1}$$

the volume element of $V_{(1)}^{2n-1}$ and by

$$\eta_s = \tilde{\alpha}^2 \wedge \tilde{\alpha}^3 \wedge \dots \wedge \tilde{\alpha}^{2n-1}$$

the simple form which corresponds to $S_{x(p)}$ and which will be called the *spatial simple* $(2n-2)$ *— form*.

Making use of (4) and (5) we get

$$d \wedge \eta_s = \tilde{\bar{k}}\, \eta_s .$$

Hence the vector space $S_{x(p)}$ is integrable and therefore, one can say that the distribution $S_{x(p)}$ is ***involutive***.

Further, by means of the fundamental formula of the variation calculus

$$\mathscr{L}_X = d\, i_X + i_X\, d,$$

we get, making use of (4),

$$\mathscr{L}_{\xi_1} \eta = (div\ \xi_1)\, \eta \Rightarrow div\ \xi_1 = \tilde{\bar{\bar{k}}}$$

and

$$\mathscr{L}_{\xi_1} \eta_s = \tilde{\bar{\bar{k}}}\, \eta_s\, .$$

Thus, the condition that ξ_1 be an infinitesimal automorphism of the volume element of $V_{(1)}^{2n-1}$ (or *div* $\xi_1 = 0$) is equivalent to the condition that ξ_1 be an *infinitesimal automorphism* of the spatial simple $(2n - 2)$ — form.

Finally, making use of (2), we get, after calculation

$$\nabla_{\xi_1} \xi_1 = \tilde{l}_{11}^1\, \xi_1$$

(in the following we shall call ξ_1 the *characteristic field* of $V_{(1)}^{2n-1}$).

This relation proves the property of the vector to be *geodesic* also in the 2-codimensional case.

By means of a calculation which we shall not develope here, one can see that this property holds also in the case of a degenerate manifold of arbitrary codimension in V_P^{2n+1}.

Remark. The necessary and sufficient condition that the characteristic field ξ_1 be an invariant integral relation for the homothety component $\tilde{\alpha}_1^1$ is

$$\tilde{l}_{11}^1 = 0\, .$$

We shall call $\tilde{l}_{11}^1$ the *geodesic scalar* associated with the vector ξ_1.

Theorem. *Being given a Lorentzian manifold* V_P^{2n+1} *structured with a principal connection* ∇_p, *every degenerate submanifold of defect* 1 *(denoted by* $V_{(1)}^{2n-1}$*) and of codimension* 2 *is determined by a linear system always completely integrable. If* ξ_1 *and* η_s *are the characteristic vector field and the spatial simple* $(2n - 2)$*-form respectively, then :*

(i) the distribution defined by the spatial tangent space $S_{x(p)}$ *having as dual* η_s *is involutive;*

(ii) the necessary and sufficient condition that ξ_1 *be an infinitesimal automorphism of* η_s *is* $div\, \xi_1 = 0$.

3. Taking the star-operator of the induced line element $dx(p)$, we get

$$\ast\, dx(p) = \xi_{2n+1}\, \eta_s + \tilde{\alpha}^1 \wedge \sum_i (-1)^{i-2}\, \tilde{\alpha}^2 \wedge \ldots \wedge \widehat{\tilde{\alpha}^i} \wedge \ldots \wedge \tilde{\alpha}^{2n-1}\, \xi_i \,.$$

By exterior differentiation and taking account of (4) and (5), we find

$$d \wedge (\ast\, dx(p)) = -\, \tilde{\alpha}_1^1 \wedge dx(p) + (\tilde{k}\, \xi_1 + \mathrm{tr}\, \varphi\, \xi_{2n})\, \eta \,.$$

Let us now call the vector

$$\mathscr{H}_m = \tilde{k}\, \xi_1 + tr\, \varphi\, \xi_{2n}$$

the *mean curvature vector* of $V_{(1)}^{2n-1}$ associated with x. Consequently if

$$\tilde{k} = tr < dx(p),\ \nabla\, \xi_{2n+1} > = 0 \,,$$

then we shall say that $V_{(1)}^{2n-1}$ is minimal in the direction of the transversal isotropic vector ξ_{2n+1}. Thus, if $\mathscr{H}_m = 0$, the line element $dx(p)$ is $\ast$-completely integrable [2].

Recalling that a *Hoffmann manifold* is a manifold for which the mean curvature vector is parallel, let us search for the conditions under which we have

$$\nabla\, \mathscr{H}_m = 0 \,.$$

Making use of (2), we obtain

$$\begin{aligned} & d\tilde{k} + \tilde{k}\, \tilde{\alpha}_1^1 = 0 \,, \\ (8) \qquad & \tilde{\alpha}_i^{2n} \approx \tilde{\tilde{k}}_i\, \tilde{\alpha}^i \,, \quad \text{(no summation over } i) \\ & d\, tr\, \varphi = 0 \,. \end{aligned}$$

The first relation (8) shows ($\tilde{\alpha}_1^1$ is exact) that the connection ∇_p is *equiaffine* [4]; the second relation (8) shows that the two fundamental quadratic forms are conformal and therefore the Chern's arithmetic invariant associated with x is equal to 1; the third relation (8) shows that the mean curvature in the direction of the spatial normal vector is constant.

Theorem. *Let $\mathscr{H}_m$ be the mean curvature vector associated with the inclusion $x : V_{(1)}^{2n-1} \to V_p^{2n+1}$. Then the necessary and sufficient condition that the line element of $V_{(1)}^{2n-1}$ be $\ast$-completely integrable is that $\mathscr{H}_m$ be identical zero. If $V_{(1)}^{2n-1}$ is a Hoffmann manifold, then :*

(*i*) *the connection ∇_p is equiaffine;*

(*ii*) *Chern's aritmetic invariant associated with x is equal to* 1;

(*iii*) *the mean curvature in the direction of the spatial normal vector is constant.*

4. Suppose that the considered hyperbolic space $V_{\mathscr{L}}^{2n+1}$ is a Minkowski space and consider the map

$$\tilde{p} \to p + f\,\xi_1 \quad , \quad f \in \mathscr{D}(V_{(1)}^{2n+1}). \tag{9}$$

By means of (2) and (6) we get for the line element of the manifold $\tilde{V}_{(1)}^{2n-1}$ generated by $\tilde{p}$ the following expression

$$d\tilde{p} = (\tilde{\alpha}^1 + f\,\tilde{\alpha}_1^1 + df)\,\xi_1 + \sum_r \xi_r(1 + f\bar{\bar{k}}_r)\,\tilde{\alpha}^r\,. \tag{10}$$

One can easily see from the above expression of $d\tilde{p}$, that for every value of f we obtain a new manifold which is of the same type as the first manifold; this fact generalizes a property already outlined in [5]

One can also see that, if the first manifold is umbilical in the direction of the characteristic vector ξ_1, the every mapping of type (9) is *conformal.*

By means of (9), a spatial manifold of codimension 3 is defined by

(11) $$\tilde{\alpha}^1 + f\,\tilde{\alpha}_1^1 + df = 0.$$

By exterior differentiation, we obtain

(12) $$(\tilde{\alpha}^1 + df) \wedge \tilde{\alpha}_1^1 = 0$$

and one sees that (since $\tilde{\alpha}_1^1$ is closed) the system (11) + (12) is completely integrable.

In the particular case when the conformity scalar is equal to $-1/f$, then the hypersurface $\tilde{V}_{(1)}^{2n-1}$ is reduced to an isotropic curve.

Theorem. *Let f be an arbitrary function on $V_{(1)}^{2n-1}$. Then every mapping having as generator the chacteristic field ξ_1 offers the possibility to obtain a manifold of the same definition. If the departure manifold $V_{(1)}^{2n-1}$ is umbilical in the direction of the characteristic vector ξ_1, then the mapping is conformai. One can always determine a function f such that the manifold $\tilde{V}_{(1)}^{2n-1}$ be spatial and of codimension 3. If the conformity scalar is equal to $-1/f$, then the manifold $\tilde{V}_{(1)}^{2n-1}$ is reduced to an isotropic curve. Finally, one may cancel all the coefficients and then the point $\tilde{p}$ is a fixed point.*

5. Suppose that the spatial tangential connection forms are zero, i.e.:

(i3) $$\tilde{\alpha}_k^i = 0.$$

In this case we get from (4)

(14) $$d \wedge \tilde{\alpha}^i = \tilde{\bar{k}}_i\, \tilde{\alpha}^1 \wedge \tilde{\alpha}^i, \quad \text{(no summation over } i)$$

that is all the spatial dual 1-forms are completely integrable.

Consider now on $V_{(1)}^{2n-1}$ the pre-symplectic [6] 2-form

$$\Omega = \tilde{\alpha}^2 \wedge \tilde{\alpha}^3 + \ldots + \tilde{\alpha}^{2n-2} \wedge \tilde{\alpha}^{2n-1}.$$

It is easy to see that the rank of this form is equal to $2n - 2$, that is $\dim \ker \Omega = 1$.

Since

$$\tilde{\alpha}^1 \wedge (\wedge\, \Omega)^{n-2} \neq 0,$$

one can say that the chacteristic covector $\tilde{\alpha}^1$ and Ω define an *almost cosymplectic structure* $\mathbf{C}(\Omega, \tilde{\alpha}^1)$ on $V_{(1)}^{2n-1}$. Making use of (14), we obtain in the case of hypothesis (13)

$$d \wedge \Omega = \tilde{\bar{k}}\, \tilde{\alpha}^1 \wedge \Omega,$$

which shows that the structure $\mathbf{C}(\Omega, \tilde{\alpha}^1)$ is *conformal cosymplectic*, having ξ_1 as *canonical field* (or Reeb's field).

A simple calculation gives

$$\mathscr{L}_{\xi_1} \Omega = \tilde{\bar{k}}\, \Omega$$

and therefore α_{I} is a conformal infinitesimal transformation of Ω. On the other hand one has also

$$\mathscr{L}_{\xi_1} \tilde{\alpha}^1 = \tilde{\alpha}_1^1$$

and if $\tilde{\alpha}_{\text{I}}^1 = f\, \tilde{\alpha}^1$ then $\tilde{\alpha}^1$ is closed. In this case one has a *semi-cosymplectic structure* [7] and ξ_{I} is a conformal cosymplectic infinitesimal transformation for this structure.

Theorem. *If the spatial dual forms are completely integrable (case of* (13)*) then one can define on* $V_{(1)}^{2n-1}$ *a conformal cosymplectic structure having as canonical field the characteristic field of* $V_{(1)}^{2n+1}$. *If the structure is semi-cosymplectic then this field defines a conformal cosymplectic infinitesimal transformation.*

REFERENCES

[1] CAGNAÇ, FR. : *Géométrie des hypersurfaces isotropes*, C.R. Acad. Sci. PARIS, **201** (1965), 30 - 45.

[2] ROSCA, R. AND VANHECKE, L. : *Sur les variétés lorentziennes munies d'une connexion principale*, Ann. Inst. H. Poincaré, **22**, 3 (1975), 201 - 216.

[3] ROSCA, R. : *Sur les hypersurfaces isotropes de défaut 1 incluses dans un espace hyperbolique de type normal*, Ann. Inst. H. Poincaré, 10, 3 (1974), 237 - 243.

[4] GHEORGHIEV, GH. MIRON, R. AND PAPUC, D. : *Geometrie analitică şi diferenţială*, Ed. did. şi ped. Bucureşti (1969).

[5] ROSCA, R. : *Varietăţi isotrope şi pseudoisotrope incluse într-o varietate relativistă*, Ed. Acad. RSR, Bucureşti (1972).

[6] SOURIAU, J.M. : *Structure des systèmes dynamiques*. DUNOD, PARIS (1970).

[7] ROSCA, R. AND VANHECKE, L. : *Sous - variétés remarquables d'un espace pseudo - riemannien ou pseudo - euclidien*, (Manuscript).

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ÖZET

$V_{\mathcal{L}}^{2n+1}$ ile $2n+1$ gibi tek boyutlu bir Lorentz varyetesi, $\{p, \xi_A; A = 1, \ldots, 2n+1;\ p \in V_{\mathcal{L}}^{2n+1}\}$ bir $p \in V_{\mathcal{L}}^{2n+1}$'e bağlı bir kuaziortogonal vektör bazı ve $\{\alpha^A\}$ bu vektör bazının dual bazı olsun. $\{\xi_a\,;\, a = 1,\ 2n+1\}$ bu bazın iki (reel) sıfır vektörünü gösterecek olursa $\alpha^{2n+1} = 0$ ve $\alpha^{2n} = 0$ ile tanımlanan $W \subset V_{\mathcal{L}}^{2n+1}$ alt varyetesi dejeneredir. Bu deyimle, W alt varyetesinin $p|W$ noktasındaki sırası ile $T_{p|W}$ ve $N_{p|W}$ ile gösterilen teğet ve normal uzayları için $T_{p|W} \cap N_{p|W} \neq 0$ olduğu kastedilmektedir.

Bu araştırma, bir esas bağımlılıkla yapılandırılmış bir $V_{\mathcal{L}}^{2n+1}$ varyetesinin, $x : W \to V_{\mathcal{L}}^{2n+1}$ daldırma tasvirinin defekti 1'e eşit olmasından dolayı $V_{(1)}^{2n-1}$ şeklinde gösterilen W dejenere alt varyeteleri konu olarak alınmıştır. ξ_1 teğetsel (ve normal) sıfır alanının çeşitli özellikleri elde edilmiştir. $x : V_{(1)}^{2n-1} \to V_{\mathcal{L}}^{2n+1}$ tasvirine bağlı uzaysal teğetsel bağımlılık formları sıfır olmaları halinde $V_{(1)}^{2n-1}$ üzerinde ξ_1'i kanonik alan olarak kabul eden bir konform ko-semplektik yapı tanımlanır.