ON PROJECTIVE TRANSFORMATION OF PSEUDO PROJECTIVE SYMMETRIC SPACES

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Summary : In this paper we study the projectve transformation of pseudo projective symmetric spaces $(PWS)_n$.

PSÖDO PROJEKTİF SİMETRİK UZAYLARIN PROJEKTİF DÖNÜŞÜMÜ HAKKINDA

Özet : Bu çalışmada psödo projecktif simetrik uzayların projektif dönüşümü incelenmektedir.

1. INTRODUCTION

Let M_n be an $n(>2)$ dimensional Riemannian space of class upto any necessary order with its metric g. Let M_n^* be another Riemannian space with metric g^* such that M_n^* is obtained by a projective transformation of the M_n, that is, the M_n and the M_n^* are in geodesic correspondence. The projective transformation is characterized by the relation of the Christoffel symbols of the M_n and the M_n^* as follows

$$\left[{}^{*h}_{ij}\right] = \left\{{}^{h}_{ij}\right\} + \delta_i^h \phi_j + \delta_j^h \phi_i \tag{1.1}$$

where Latin indices take values 1,2,..., n, the symbol * denotes the quantity of the M_n^* and the ϕ_i is a gradient vector field associated with the projective transformation [1].

If the ϕ_i vanishes, the transformation is affine.

Let R^h_{ijk}, R_{ij} and R denote respectively the curvature tensor, Ricci tensor and the scalar curvature of the M_n. Under the projective transformation (1.1), as is well known, the projective curvature tensor W^h_{ijk} is invariant, that is

$$\overset{*}{W}{}^h_{ijk} = W^h_{ijk} \tag{1.2}$$

where $W^h_{ijk} = R^h_{ijk} - \dfrac{1}{n-1}(\delta^h_k R_{ij} - \delta^h_j R_{ik})$ and satisfies

$$W^h_{ijk} = - W^h_{ikj} \tag{1.3}$$

$$W^a_{aij} = W^a_{iaj} = W^a_{ija} = 0 \tag{1.4}$$

$$W^h_{ijk} + W^h_{jki} + W^h_{kij} = 0 \tag{1.5}$$

$$W_{hijk} + W_{hjki} + W_{hkij} = 0. \tag{1.6}$$

In a recent paper Chaki introduced a type of non-flat Riemannian space whose projective curvature tensor W^h_{ijk} satisfies the condition

$$W^h_{ijk,l} = 2\,\lambda_l\, W^h_{ijk} + \lambda^h\, W_{lijk} + \lambda_i\, W^h_{ijk} + \lambda_j\, W^h_{ilk} + \lambda_k\, W^h_{ijl} \tag{1.7}$$

where λ_l is a non-zero vector. Such a space has been called a pseudo projective symmetric space [2] and was denoted by $(PWS)_n$. If λ_l becomes zero then the space reduces to a projective symmetric space [3]. In this paper, we shall study the case where the M_n and M^*_n are pseudo projective symmetric spaces. In section 2, we shall discuss the case of $\lambda_l = \lambda^*_l$, that is, the associated vectors of M_n and M^*_n are same. In section 3, we shall devote the case of $\lambda_l \neq \lambda^*_l$, that is, the associated vector of the M^*_n is different from the one of the M_n.

2. THE CASE OF $\lambda_l = \lambda^*_l$

Differentiating (1.2) covariantly and making use of (1.1), we have

$$W^{*h}_{ijk;l} = W^h_{ijk,l} - 2\,\phi_l\, W^h_{ijk} + \delta^h_l\, \phi_a\, W^a_{ijk} - \phi_i\, W^h_{ljk} - \phi_j\, W^h_{ilk} - \phi_k\, W^h_{ijl} \tag{2.1}$$

where ; and , denote covariant differentiation with respect to g^* and g respectively. Then from (2.1) by virtue of (1.7) we have

$$2\,\phi_j\, W^h_{ijk} - \delta^h_l\, \phi_a\, W^a_{ijk} + \phi_i\, W^h_{ijk} + \phi_j\, W^h_{ijk} + \phi_k\, W^h_{ijl} = 0. \tag{2.2}$$

On contraction we get by virtue of (1.4)

$$\phi_a\, W^a_{ijk} = 0 \qquad (n > 2). \tag{2.3}$$

Substituting (2.3) into (2.2), we get

$$2\,\phi_l\, W^h_{ijk} + \phi_i\, W^h_{ijk} + \phi_j\, W^h_{ilk} + \phi_k\, W^h_{ijl} = 0 \tag{2.4}$$

and consequently,

$$2\,\phi_l\, W_{hijk} + \phi_l\, W_{hljk} + \phi_j\, W_{hilk} + \phi_k\, W_{hijl} = 0. \tag{A}$$

But K. Amur and P. Desai showed in their paper [3] that the equation (A) implies either the space is Einstein or the ϕ_l is a null vector. Hence we can state the following theorem:

Teorem 2.1. Let M^*_n be a projective transformation of an M_n. If the projective curvature tensors satisfy the condition (A), then the M_n and the M^*_n are Einstein spaces or the transformation is affine.

First we assume that the M_n and M_n^* are both projective symmetric spaces. Then we have $W^h_{ijk,l} = 0$ and $\overset{*}{W}{}^h_{ijk;l} = 0$. Since the condition (A) is satisfied, from Theorem (2.1) we have

Theorem 2.2. If a projective symmetric space is transformed into another projective symmetric one by a projective transformation (1.1), then the spaces are Einstein or the transformation is affine.

Next we assume that the M_n and the M_n^* are both pseudo projective symmetric spaces with the same associated vector λ_l. Then we have from (1.7)

$$\overset{*}{W}{}^h_{ijk;l} = 2\lambda_l \overset{*}{W}{}^h_{ijk} + \lambda^h \overset{*}{W}_{lijk} + \lambda_i \overset{*}{W}{}^h_{ljk} + \lambda_j \overset{*}{W}{}^h_{ilk} + \lambda_k \overset{*}{W}{}^h_{ijl}\,.$$

Consequently the condition (A) is satisfied owing to (1.2). Thus from theorem (2.1) we have

Theorem 2.3. If a pseudo projective symmetric space is transformed into another pseudo projective symmetric space with the same associated vector by a projective transformation (1.1), then the spaces are Einstein or the transformation is affine.

3. THE CASE OF $\lambda_l \neq \lambda_l^*$

We assume that the M_n and M_n^* are both pseudo projective symmetric spaces with the different associated vectors. Then we have

$$W^h_{ijk,l} = 2\lambda_l W^h_{ijk} + \lambda^h W_{lijk} + \lambda_i W^h_{ljk} + \lambda_j W^h_{ilk} + \lambda_k W^h_{ijl}$$

and

$$\overset{*}{W}{}^h_{ijk;l} = 2\overset{*}{\lambda}_l \overset{*}{W}{}^h_{ijk} + \lambda^{*h} \overset{*}{W}_{lijk} + \overset{*}{\lambda}_i \overset{*}{W}{}^h_{ljk} + \overset{*}{\lambda}_j \overset{*}{W}{}^h_{ilk} + \overset{*}{\lambda}_k W^h_{ijl}\,.$$

Substituting these equations into (2.1) and using (1.2) we get

$$\begin{aligned} 2(\overset{*}{\lambda}_l - \lambda_l) W^h_{ijk} &+ (\lambda^{*h} - \lambda^h) W_{lijk} + (\overset{*}{\lambda}_i - \lambda_i) W^h_{ljk} + \\ &+ (\overset{*}{\lambda}_j - \lambda_j) W^h_{ilk} + (\overset{*}{\lambda}_k - \lambda_k) W^h_{ijl} = -2\phi_l W^h_{ijk} + \\ &+ \delta^h_l \phi_a W^a_{ijk} - \phi_i W^h_{ljk} - \phi_j W^h_{ilk} - \phi_k W^h_{ijl}\,. \end{aligned} \tag{3.1}$$

Transvecting (3.1) with g^{lj} we have

$$\begin{aligned} 2(\overset{*}{\lambda}_l - \lambda_l) W^l_k &+ (\lambda^{*h} - \lambda^h) W_{lk} + (\lambda^{*j} - \lambda^j) W^h_{ljk} + \\ &+ (\lambda^{*i} - \lambda^i) W^h_{ilk} + (\overset{*}{\lambda}_k - \lambda_k) W^h_l = -2\phi_l W^h_k + \\ &+ \delta^h_l \phi_a W^a_k - \phi^j W^h_{ljk} - \phi^i W^h_{ilk} - \phi_k W^h_l \end{aligned} \tag{3.2}$$

where

$$W^h_k = g^{ij} W^h_{ijk} = \frac{1}{(n-1)} (n R^h_k - R \delta^h_k),$$

$$W_{ij} = g_{ia} W^a_j = W_{ji}$$

or

$$\begin{aligned} 2 (\overset{*}{\lambda}_l - \lambda_l) W_{hk} + (\overset{*}{\lambda}_h - \lambda_h) W_{lk} + (\lambda^{*a} - \lambda^a) W_{hlak} + \\ + (\lambda^{*a} - \lambda^a) W_{halk} + (\overset{*}{\lambda}_k - \lambda_k) W_{hl} = - 2 \phi_l W_{hk} + \\ + g_{hl} \phi_a W^a_k - \phi^a W_{hlak} - \phi^a W_{halk} - \phi_k W_{hl} . \end{aligned} \quad (3.3)$$

Interchanging h and k in (3.3) and substracting it from (3.3) and then transvecting with g^{hl} it is found

$$(\lambda^{*a} - \lambda^a) W_{ka} = (n-1) \phi^a W_{ka} . \quad (3.4)$$

Contracting (3.1) we get

$$3 (\overset{*}{\lambda}_a - \lambda_a) W^a_{ijk} = (n-2) \phi_a W^a_{ijk} . \quad (3.5)$$

From (3.4) and (3.5) we get

$$\phi^a W_{ak} = 0. \quad (3.6)$$

Transvecting (3.1) with $3 (\overset{*}{\lambda}_h - \lambda_h)$ and applying (3.5) ve get

$$\begin{aligned} 2 (n-2) (\overset{*}{\lambda}_l - \lambda_l) \phi_a W^a_{ijk} = - 3 (\overset{*}{\lambda}_h - \lambda_h) (\lambda^{*h} - \lambda^h) W_{lijk} - \\ - (n-2) (\overset{*}{\lambda}_i - \lambda_i) \phi_a W^a_{ljk} - (n-2) (\overset{*}{\lambda}_j - \lambda_j) \phi_a W^a_{ilk} - \\ - (n-2) (\overset{*}{\lambda}_k - \lambda_k) \phi_a W^a_{ijl} - 2 (n-2) \phi_l \phi_a W^a_{ijk} - \\ - (n-2) \phi_i \phi_a W^a_{ljk} - (n-2) \phi_j \phi_a W^a_{ilk} - \\ - (n-2) \phi_k \phi_a W^a_{ijl} + 3 (\lambda_l - \lambda_l) \phi_a W^a_{ljk} . \end{aligned} \quad (3.7)$$

Again using (1.3), (1.5) and (1.6) we get from (3.1)

$$\begin{aligned} 2 (\overset{*}{\lambda}_l - \lambda_l) W^a_{ijk} + 2 (\overset{*}{\lambda}_j - \lambda_j) W^a_{ikl} + 2 (\overset{*}{\lambda}_k - \lambda_k) W^a_{ilj} = \\ = - 2 (\overset{*}{\lambda}_j - \lambda_j) W^a_{ilk} - 2 (\overset{*}{\lambda}_k - \lambda_k) W^a_{ijl} - \\ - 2 (\overset{*}{\lambda}_l - \lambda_l) W^a_{ikj} - \delta^a_l \phi_h W^b_{ijk} - \delta^a_j W^b_{ikl} - \delta^a_k \phi_b W^b_{ilj} . \end{aligned} \quad (3.8)$$

Now multiplying (3.8) by $(n-2) \phi_a$ and using (3.7) and transvecting with g^{ij} we get

$$[3 (\lambda^{*i} - \lambda^i) + (n-2) \phi^i] W^a_{ikl} = 0 \qquad \text{[by (3.6)]}. \quad (3.9)$$

Then either

$$\phi_a W'^a_{ikl} = 0 \quad (3.10)$$

or

$$3(\lambda^{*i} - \lambda^i) + (n-2)\,\phi^i = 0. \tag{3.11}$$

Now using (3.6) and (3.10) in (3.3) it follows that

$$(\overset{*}{\lambda}_a - \lambda_a)\,\phi^a\,W_{lk} = 0. \tag{3.12}$$

Thus we have the following

Lemma. If a pseudo projective symmetric space M_n with the associate vector λ_l is transformed into another pseudo projective symmetric space M_n^* with the associated vector λ_l^* by a projective transformation (1.1), then one of the following cases occurs

$$3(\lambda^{*i} - \lambda^i) + (n-2)\,\phi^i = 0 \tag{3.13}$$

$$(\overset{*}{\lambda}_a - \lambda_a)\,\phi^a = 0 \tag{3.14}$$

$$W_{lk} = 0, \quad \text{that is, the } M_n \text{ is an Einstein space.} \tag{3.15}$$

Now we consider the equation (3.13). From (3.5) and (3.13) we get

$$2(n-2)\,\phi_a\,W^a_{ijk} = 0$$

or

$$\phi_a\,W^a_{ijk} = 0, \quad \text{since} \quad n > 2\,.$$

This relation holds in (3.10). Hence either (3.14) or (3.15) holds.

Thus we can state the following

Theorem 3.1. If a pseudo projective symmetric space M_n with the associated vector λ_l is transformed into another pseudo projective symmetric space M_n^* with the associated vector λ_l^* by a projective transformation (1.1), then one of the following cases occurs:

(i) Either M_n is an Einstein space or

(ii) $(\overset{*}{\lambda}_a - \lambda_a)\,\phi^a = 0$.

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