ABOUT THE 2-SYLOW SUBGROUPS OF A Q-GROUP

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Summary : There is a long standing conjecture which asserts that for a Q-group the 2-Sylow subgroups are also Q-groups. In this note we study this conjecture for two classes of Q-groups.

BİR Q-GRUBUN 2-SYLOW ALT GRUPLARI HAKKINDA

Özet : Bu çalışmada, bir Q-grubun 2-Sylow alt gruplarının da Q-gruplar olduğuna ilişkin iddianın, iki Q-grup sınıfı için doğru olduğu ispat edilmektedir.

All groups will be finite and the notations and definitions will be those of [2].

Definition. A Q-group is a group G whose characters are rational valued.

The next theorem is a well-known result (see [1]).

Theorem 1. A group G is a Q-gruop if and only if for every $g \in G$ g is conjugate to g^m for every integer m with $(m, |g|) = 1$.

Theorem 2. Let G be a Q-group and H be a 2-Sylow subgroup of G. If for every noninvolutory $h \in H$, $N_G(< h >)$ is subnormal in G then H is also a Q-group.

Proof. Let $h \in H$ be noninvolutory. Let f be the group morphism

$$f : N_G(< h >) \longrightarrow \text{Aut}(< h >)$$

defined by $f(x)(h) = x\,h\,x^{-1}$. f is surjectiv (theorem 1), because G is a Q-group. Let z, w be a set of generators for Aut $(< h >)$ and $x, y \in N_G(< h >)$ such that $f(x) = z$ and $f(y) = w$. We can suppose that $|x| = 2^j$ and $|y| = 2^k$, because Aut $(< h >)$ is a 2-group. Since $N_G(< h >)$ is subnormal in G then

$H \cap N_G(<h>)$ is a 2-Sylow subgroup of $N_G(<h>)$ so that using Sylow theorems there exist $u, v \in N_G(<h>)$ such that $s = u x u^{-1}$, $t = v y v^{-1} \in H$. Then, clearly $f(s) = f(x) = z$ and $f(t) = f(y) = w$ so that

$$H \cap N_G(<h>)/H \cap C_G(h) \simeq \text{Aut}(<h>).$$

Theorem 3. Let G be a Q-group and H a 2-Sylow subgroup of G. Suppose that for every $H' \in \text{Syl}_2(G)$ and for every noninvolutory $h \in H \cap H'$ such that $H' \cap N_G(<h>) \in \text{Syl}_2(N_G(<h>))$ and $N_G(<h>) \cap H \subseteq H'$ there exist $g \in C_G(h)$ such that $H'^g = H$. Then H is also a Q-group.

Proof. Analogous with the proof of theorem 2, since $H' \cap N_G(<h>) \in \text{Syl}_2(N_G(<h>))$ we obtain $s, t \in H' \cap N_G(<h>)$ such that $f(s) = z$ and $f(t) = w$. Let $a = g s g^{-1} \in H$ and $b = g t g^{-1} \in H$. Then clearly $f(a) = z$ and $f(b) = w$ and H is a Q-group.

REFERENCES

[1] ISAACS, I.M. : **Character Theory of Finite Groups**, Academic Press, New York, 1976.

[2] ROSE, J.S. : **A Course in Group Theory**, Cambridge, 1978.