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ON ONE MATHEMATICAL METHOD FOR SOLUTION OF DYNAMICAL VISCOELASTICITY PROBLEMS

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Summary: Effective mathematical method for solution of nonstationary dynamical problems of linear anisotropic viscoelasticity at arbitrary difference and nondifference hereditary kernels is developed in the paper. In case when theratio of kernels of relaxation is independent on time, the known theorems are generalized and the new theorems, by means of which the solution of fundemental dynamical problems of anisotropic inhomogeneous viscoelasticity are reduced to solutions of corresponding problems of elastodynamics and to solutions of some one-dimensional mixed value problems for integro-differential equation in partial derivatives hyperbolic type, are proved. The solutions of all necessary problems for arbitrary hereditary kernels are constructed. These results represent the principle of correspondence of nonstationary dynamical problems solutions of elasticity and viscoelasticity theories in originals.

DİNAMİK VİSKOZESNEKLİK PROBLEMLERİNİN ÇÖZÜMÜ İLE İLGİLİ BİR MATEMATİKSEL YÖNTEM HAKKINDA

Özet: Bu çalışmada, keyfi çekirdekli anizotrop viskozesnek cisimler kuramının stasyoner olmayan dinamik problemlerinin analitik çözüm yöntemi verilmektedir.

1. STATEMENT OF THE PROBLEM

Let us consider the state equations of anisotropic linear viscoelastic medium

$$\sigma_{ij}(x, t) = R_{ijkl}(x)\left[\varepsilon_{kl} - \int_0^t \Gamma(t, \xi)\, \varepsilon_{kl}(x, \xi)\, d\xi\right], \tag{1}$$

where $R_{ijkl}(x)$ are known functions of three independent variables x_1, x_2, x_3, $x = (x_1, x_2, x_3)$, Γ is kernel of relaxation, σ_{ij} and ε_{kl} are components of stress and strain tensors. In response to these relations and the dependence of deformations with the displacement

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \quad (k, l = 1, 2, 3)$$

into the equations of motions in the stress

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad (i, j = 1, 2, 3),$$

we obtain the equations of motion of inhomogeneous anisotropic viscoelastic medium in the displacement written in the vector form

$$L(\mathbf{u}) - \int_0^t \Gamma(t, \xi) L(\mathbf{u}) d\xi = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \tag{2}$$

where $\mathbf{u}(x, t) = \{u_1, u_2, u_3\}$ is a displacement vector, $\rho(x)$ is density, L is an operator of the inhomogeneous anisotropic elastic medium. In isotropic case

$$L(\mathrm{u}) = (\lambda + 2\mu) \nabla (\nabla, \mathrm{u}) - \mu [\nabla, [\nabla, \mathrm{u}]] + \nabla\lambda (\nabla, \mathrm{u}) + 2 (\nabla\mu, \nabla) \mathrm{u} + [\nabla\mu, [\nabla, \mathbf{u}]],$$

where $\lambda(x)$ and $\mu(x)$ are Lame's elastic coefficients, ∇ is Hamiltonian operator.

Assume that a viscoelastic body is bounded by the surface S. We shall consider the motion of this body when on the surface S is given either the force or the displacement, i.e.

$$\sigma_{mj} n_j |_S = L_{1m}(\mathbf{u}) |_S - \int_0^t \Gamma(t, \xi) L_{1m}(\mathrm{u}) d\xi |_S = f_m(x) |_S a_m(t) \quad (i, j, m = 1, 2, 3), \tag{3}$$

$$u_m |_S = F_m(x) |_S b_m(t), \tag{4}$$

where L_{1m} is a linearly differential operator corresponding to the equations of state of anisotropic inhomogeneous elastic medium, n_m are the projections of the outward unit normal vector to S (here by the index m is not summed).

It is required to find $\mathbf{u}(x, t)$ from the equation of motion (2) and one of the boundary conditions (3) and (4) with the inital conditions

$$\mathbf{u}\Big|_{t=0} = 0, \quad \frac{\partial \mathbf{u}}{\partial t}\Big|_{t=0} = 0. \tag{5}$$

For the solution of these problems it is necessary to have the solutions of some auxiliary problems.

The first auxiliary problem is the following:

$$L(\mathbf{u}_i) = \rho \frac{\partial^2 \mathbf{u}_i}{\partial \tau^2}, \quad \mathbf{u}_i = \{u_{i1}, u_{i2}, u_{i3}\}, \mathbf{u}_i = 0, \frac{\partial \mathbf{u}_i}{\partial \tau} = 0, \quad \tau = 0, \tag{6}$$

$$L_{1m}(\mathrm{u}_i) |_S = f_m(x) |_S \delta_{mi} \delta(\tau), \tag{7}$$

where $\delta(\tau)$ is the Dirac function, δ_{mi} are the Kronecker symbols.

The second auxiliary problem is the previous problem with the last boundary condition substituted to

$$u_{mi}|_S = F_m(x)|_S \, \delta_{mi} \, \delta(\tau).$$

As is seen these problems are the transient dynamical problems for elastic medium subjected to the action of impulsive influence.

The third auxiliary problem is the one-dimensional problem for the homogeneous viscoelastic medium

$$\frac{\partial^2 w_i}{\partial \tau^2} - \int_0^t \Gamma(t, \xi) \frac{\partial^2 w_i(\xi, \tau)}{\partial \tau^2} - \frac{\partial^2 w_i}{\partial t^2} = 0,$$

$$w_i \Big|_{t=0} = 0, \quad \frac{\partial w_i}{\partial t}\Big|_{t=0} = 0, \tag{8}$$

$$w_i(t, 0) - \int_0^t \Gamma(t, \xi) \, w_i(\xi, 0) \, d\xi = a_i(t); \quad w_i \to 0, \tau \to \infty. \tag{9}$$

The fourth auxiliary problem is the previous problem with the last condition substituted to $w_i(t, 0) = b_i(t)$.

2. RESULTS AND DISCUSSION

Theoremt 1. The solution of the problem (2), (3), (5) is

$$\mathbf{u}(x, t) = \int_0^\infty \mathbf{u}_i^1(x, \tau) \, w_{1i}(t, \tau) \, d\tau \quad (i = 1, 2, 3), \tag{10}$$

where $\mathbf{u}_i^1(x, \tau)$ and $w_{1i}(t, \tau)$ are solutions of first and third auxiliary problems.

Proof. Substituting (10) in (2), and assuming the possibility of twice differentiation with respect to co-ordinates under the integral sign, we obtain

$$\int_0^\infty L(\mathbf{u}_i^1(x, \tau)) \, w_{1i}(t, \tau) \, d\tau - \int_0^t \mathrm{T}(t, \xi) \int_0^\infty L(\mathbf{u}_i^1(x, \tau)) \, w_{1i}(\xi, \tau) \, d\tau \, d\xi =$$

$$= \rho \int_0^\infty \mathbf{u}_i^1(x, \tau) \frac{\partial^2 w_{1i}(t, \tau)}{\partial t^2} \, d\tau.$$

Since the function $\mathbf{u}_i^1$ satisfies the equation (6) the preceding equation can be written in the form

$$\int_0^\infty \frac{\partial^2 \mathbf{u}_i^1}{\partial \tau^2} w_{1i}(t, \tau)\, d\tau - \int_0^t \Gamma(t, \xi) \int_0^\infty \frac{\partial^2 \mathbf{u}_i^1}{\partial \tau^2} w_{1i}(\xi, \tau)\, d\tau\, d\xi = $$
$$= \int_0^\infty \mathbf{u}_i^1 \frac{\partial^2 w_{1i}(t, \tau)}{\partial t^2}\, d\tau .$$

Noting that

$$\int_0^\infty w_{1i}(t, \tau) \frac{\partial^2 \mathbf{u}_i^1(x, \tau)}{\partial \tau^2}\, d\tau = \int_0^\infty \mathbf{u}_i^1(x, \tau) \frac{\partial^2 w_{1i}(t, \tau)}{\partial \tau^2}\, d\tau,$$

which is obtained by integrating the first integral by parts considering the conditions of boundedness

$$w_{1i} \frac{\partial \mathbf{u}_i^1}{\partial \tau} - \mathbf{u}_i^1 \frac{\partial w_{1i}}{\partial \tau} \to 0 \quad \text{when} \quad \tau \to \infty,$$

we find

$$\int_0^\infty \mathbf{u}_i^1 \left[\frac{\partial^2 w_{1i}}{\partial \tau^2} - \int_0^t \Gamma(t, \xi) \frac{\partial^2 w_{1i}(\xi, \tau)}{\partial \tau^2}\, d\xi - \frac{\partial^2 w_{1i}}{\partial t^2} \right] d\tau = 0.$$

On the basis of (8), the last equation is satisfied identically.

The initial conditions are satisfied on the basis of initial conditions on the function $w_i(t, 0)$.

Let us show that the boundary condition is satisfied too. Substituting (10) in (3), we obtain

$$\int_0^\infty L_{1m}(\mathbf{u}_i^1)|_S\, w_{1i}(t, \tau)\, d\tau - \int_0^t \Gamma(t, \xi) \int_0^\infty L_{1m}(\mathbf{u}_i^1)|_S\, w_{1i}(\xi, \tau)\, d\tau\, d\xi = f_m(x)|_S\, a_m(t).$$

According to the condition (7) this equation may be written in the form

$$f_m(x)|_S \left[w_{1m}(t, 0) - \int_0^t \Gamma(t, \xi)\, w_{1m}(\xi, 0)\, d\xi \right] = f_m(x)|_S\, a_m(t).$$

According to the condition (9) the last equation is satisfied identically.

The theorem is proved.

Theorem 2. The solution of the problem (2), (4), (5) is

$$\mathbf{u}(x, t) = \int_0^\infty \mathbf{u}_i^2(x, \tau)\, w_{2i}(t, \tau)\, d\tau \quad (i = 1, 2, 3), \tag{11}$$

where $\mathbf{u}_i^2$ and w_{2i} are solutions of second and fourth auxiliary problems.

This theorem is proved as the previous.

If the displacements are given over the part of the surface S_2 and the externel forces over the remainder S_1, the solution of the considered problem is given by the sum of the formulas (10) and (11), where the function $f_m(x)$ is given on S_1 and $F_m(x)$ is on S_2.

In witness of these theorems assignment of the boundary conditions is compulsory in the form of the production of two functions, one of which depends on the time, and the others on the space co-ordinates. On the contrary we shall act as following: Assume that on the part S_1 of the surface the stressvector $\mathbf{P} = \mathbf{f}(x, t)$ and on the other part S_2 the displacement vector $\mathbf{u} = \mathbf{F}(x, t)$ is given. Considering that f and F are twice differentiable functions we expand them to series

$$\mathbf{p}|_{S_1} = \sum_i \mathbf{f}_i(x)|_{S_1} a_i(t), \quad \mathbf{u}|_{S_2} = \sum_i \mathbf{F}_i(x)|_{S_2} b_i(t). \tag{12}$$

Theorem 3. The solution of the problem (2), (5), (12) is

$$\mathrm{u}(x, t) = \sum_i \int_0^\infty [\mathbf{u}_i^1(x, \tau)\, w_{1i}(t, \tau) + \mathbf{u}_i^2(x, \tau)\, w_{2i}(t, \tau)]\, d\tau, \tag{13}$$

where $\mathbf{u}_i^1$, u_i^2, w_{1i} and w_{2i} are solutions of the first, second, third and fourth auxiliary problems correspondingly.

The formulas (10), (11) and (13) are principles of correspondence of the originals of solution of nonstationary dynamical problems for elastic and visco-elastic mediums.

3. THE SOLUTIONS OF ONE-DIMENSIONAL AUXILIARY PROBLEMS

The solutions of these problems will be constructed for difference function of kernel relaxation. Then it is not difficult to see that the solutions of these problems are obtained from the solution of the next problem of impact to the thin semiinfinite viscoelastic rod

$$\frac{\partial^2 u}{\partial x^2} - \int_0^t \Gamma(t-\xi) \frac{\partial^2 u(x, \xi)}{\partial x^2} d\xi = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2},$$

$$u = 0, \frac{\partial u}{\partial t} = 0 \quad \text{when } t = 0; \tag{14}$$

$$\sigma(0, t) = \sigma_0 H(t); \quad \sigma \to 0 \quad \text{when} \quad x \to \infty,$$

where $u(x, t)$ is displacement, $\sigma(x, t)$ is stress, $H(t)$ is Heaviside's unit function and c is velocity of wave.

With the help of the Laplace integral transformation on the time we get the following solution of this problem in representation

$$\bar{\sigma}(x, p) = \frac{\sigma_0}{p} \exp\left[-\frac{xp}{c} \Big/ \sqrt{1 - \bar{\Gamma}}\right], \tag{15}$$

where the line above the functions shows their representations, p is parameter of transformation.

The final solution is connected with the calculation of the original with the use of Mellin formula

$$\sigma(x, t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{pt} \bar{\sigma}(x, p)\, dp.$$

In case of simple relations between strain and stress this integral is calculated. Solutions of many problems are connected with this case from which we can show the works [1, 3–6]. The contour integral used here becomes very difficult even in case of the smallest complications depending $\bar{\Gamma}(p)$ on p. Therefore the method of contour integration becomes unfit for more real relations between stresses and strains. It is explained by this that to last time numerous practically important problems for real bodies were not investigated. Here we reduce a new solution method of indicated problems developed in [2,7] which completely excludes the named difficulties.

As $\mathrm{Re}(p/\sqrt{1 - \bar{\Gamma}}) > 0$, the formula (15) may be presented in the form

$$\bar{\sigma} = \frac{2\sigma_0}{\pi p} (1 - \bar{\Gamma}) \int_0^\infty \frac{\lambda \sin(\lambda x/c)}{p^2 + \lambda^2 - \lambda^2 \bar{\Gamma}}\, d\lambda\,. \tag{16}$$

Considering the inequality $|p^2 \bar{\Gamma}/(\lambda^2 + p^2)(1 - \bar{\Gamma})| < 1$, which is valed for great values $|p|$, the integrand expression in (16) is expanded on series

$$\bar{\sigma} = \frac{2\sigma_0}{\pi p} \int_0^\infty \sum_{n=0}^{\infty} \frac{(-p^2 \bar{\Gamma})^n}{(\lambda^2 + p^2)^{n+1} (1 - \bar{\Gamma})^n}\, \lambda \sin\left(\frac{\lambda x}{c}\right) d\lambda\,. \tag{17}$$

As integrand series uniformly converges in $0 \leq \lambda < \infty$, and the terms of series are continuous functions λ in indicated domain, then calculating the integrals and taking into account $\bar{\Gamma}(1 - \bar{\Gamma})^{-1} = \bar{K}$, where $\bar{K}(p)$ is the description of creeping kernel $K(t)$, we find

$$\bar{\sigma}(x,p)=\frac{\sigma_0}{p}e^{-\frac{px}{c}}\left[i+\sum_{n=0}^{\infty}\frac{(-\bar{K})^n}{2^{2n}n!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!}{k!\,(n-k-i)!}\left(\frac{2xp}{c}\right)^{k+1}\right]. \quad (18)$$

Considering the representation in the series with Mcdonald's function $K_{n-\frac{1}{2}}(z)$ the formula (18) may be written in more simple form

$$\bar{\sigma}(x,p)=\frac{\sigma_0}{p}e^{-\frac{px}{c}}+\sqrt{\frac{2x}{\pi pc}}\,\sigma_0\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{xp\bar{K}}{2c}\right)^n K_{n-\frac{1}{2}}\left(\frac{px}{c}\right).$$

It is easy to establish that the ratio of absolute values of terms of series (18) $|a_{n+1}/a_n|$ for all x and great $|p|$ is smaller than a unit, i.e. series (18) is absolutely convergent for great $|p|$ uniformly with respect to $x\in[0,\infty)$. Let us estimate the remainder term of this series. From (17) we obtain for the remainder term R_{n+1} the expression

$$R_{n+1}=\frac{2\sigma_0}{\pi p}\int_0^{\infty}\sum_{m=n+1}^{\infty}\frac{(-p^2\bar{K})^m}{(\lambda^2+p^2)^m}\lambda\sin\left(\frac{\lambda x}{c}\right)d\lambda.$$

Transforming it by the following way

$$R_{n+1}=\frac{2\sigma_0}{\pi p}\int_0^{\infty}\frac{(-p^2\bar{K})^{n+1}}{(\lambda^2+p^2)^{n+1}}\sum_{m=0}^{\infty}\frac{(-p^2\bar{K})^m}{(\lambda^2+p^2)^m}\lambda\sin\left(\frac{\lambda x}{c}\right)d\lambda$$

and substituting there the sum of integrand series, we find

$$R_{n+1}=\frac{2\sigma_0}{\pi p}(-p^2\bar{K})^{n+1}\int_0^{\infty}\frac{\lambda\sin\left(\frac{\lambda x}{c}\right)d\lambda}{(\lambda^2+p^2)^{n+1}(\lambda^2+p^2+p^2\bar{K})}.$$

As it is seen, the remainder term has the sign of the first rejected term of the series. It is easy to establish the inequality

$$\left|\frac{p^{2(n+1)}}{(\lambda^2+p^2)^{n+1}(\lambda^2+p^2+p^2\bar{K}}\right|<r^{2(n+1)}(\lambda^4+r^4)^{-1-\frac{n}{2}},$$

where $r=|p|$. Considering this in expression R_{n+1} and calculating the integral, we find

$$|R_{n+1}|<|\bar{K}|^{n+1}|\sigma_0|/|p|.$$

The first term of formula (18) is the description of solution of corresponding problem for elastic rod and the others arise as the expence of viscoelastic property of the material. In [7] formula (18) is obtained by another way. It may be obtained also by expanding on series the exponent in (15) and by applying to every term of series the formula for fractional degree of operators.

The series in formula (18) converges absolutely and its terms are continuous functions of parameter p, therefore one can calculate termwise the inverse transformation; as a result we have

$$\sigma(x,t)=\sigma_0 H\left(t-\frac{x}{c}\right)\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}n!}\sum_{m=0}^{n-1}\frac{(2n-m-2)!}{m!(n-m-1)!}\left(\frac{2x}{c}\right)^{m+1}K_n^{(m)}\left(t-\frac{x}{c}\right)\right], \tag{19}$$

where $K_n(t)$ are iterated kernels

$$K_1(t)=K(t),\ K_n(t)=\int_0^t K_1(t-\tau)K_{n-1}(\tau)\,d\tau,\quad K_n^{(m)}(t)=\frac{d^m}{dt^m}K_n(t).$$

Theorem 4. Formula (19) is the prescise solution of problem (14) for any difference kernel $\Gamma(t-\tau)$.

The theorem is proved by direct substitution.

Let $K(t)=\dfrac{E}{\mu}H(t)$, where E is a Young modulus, μ is a coefficient of viscosity. Then formula (19) is reduced to the form

$$\sigma(x,t)=\sigma_0 H\left(t-\frac{x}{c}\right)\left[e^{-\frac{Ex}{2\mu c}}+\frac{Ex}{2\mu c}\int_{x/c}^{t}e^{-\frac{E\tau}{2\mu}}\frac{I_1\left(\frac{E}{2\mu}\sqrt{\tau^2-x^2c^{-2}}\right)}{\sqrt{\tau^2-x^2c^{-2}}}d\tau\right], \tag{20}$$

where $I_\nu(z)$ is a modified Bessel function.

This is a known solution for Maxwell model [3].

Then $K(t)=[K_0+\phi(t)]$, where $K_0\neq 0$, $\Phi(0)=0$. Formula (19) is reduced to the form

$$\sigma(x,t)=\sigma_0 H\left(t-\frac{x}{c}\right)\left[e^{-\frac{K_0x}{2c}}+\frac{K_0x}{2c}\int_{x/c}^{t}e^{-\frac{K_0\tau}{2}}\frac{I_1\left(\frac{K_0}{2}\sqrt{\tau^2-x^2c^{-2}}\right)}{\sqrt{\tau^2-x^2c^{-2}}}d\tau+\right.$$
$$\left.+\frac{x}{c}\sum_{m=1}^{\infty}\frac{1}{2^m m!}\varphi_m(x,t)\right], \tag{21}$$

$$\varphi_m(x,t)=\int_{x/c}^{t}\Pi_m(t-\tau)\,F_m(x,\tau)\,d\tau\,,\quad \Pi_1(t)=-\frac{d\Phi(t)}{dt}\,,$$

$$\Pi_m(t)=\Pi_1(0)\,\Pi_{m-1}(t)+\int_0^t\Pi_{m-1}(t-\tau)\,d\Pi_1(\tau)\,,$$

$$F_m(x,t)=\left(\frac{K_0}{2}\right)^{1-m}e^{-\frac{K_0 t}{2}}\left(t^2-\frac{x^2}{c^2}\right)^{\frac{m-1}{2}}I_{m-1}\left(\frac{K_0}{2}\sqrt{t^2-\frac{x^2}{c^2}}\right).$$

In [7] it is proved that formula (21) is a precise solution of formulated problem for indicated creeping kernel, absolutely converging for any finite values of time. This solution generalizes the results of problem [6], which are found approximately.

Let $K(t)$ be an Abel kernel $K(t)=At^{-\alpha}$, $A=\text{const}>0$, $0\le\alpha<1$. Then formula (19) has the form

$$\sigma(x,t)=\sigma_0\,H\left(t-\frac{x}{c}\right)\Bigg\{1-\frac{Ax}{2c}\left(t-\frac{x}{c}\right)^{-\alpha}+\ldots+$$
$$+\frac{[-A\,\Gamma(1-\alpha)]^n}{2^{2n}\,n!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!}{k!\,(n-k-1)!}\times \tag{22}$$
$$\times\left(\frac{2x}{c}\right)^{k+1}[\Gamma(n-n\alpha-k)]^{-1}\left(t-\frac{x}{c}\right)^{(1-\alpha)n-k-1}+\ldots\Bigg\}.$$

Here all terms are crossed out with entire negatives $(1-\alpha)\,n-k$ and $\Gamma(z)$ is Euler's gamma function.

For small $t-x/c$ we obtain from (22)

$$\sigma(x,t)\approx\sigma_0\,H\left(t-\frac{x}{c}\right)\sum_{n=0}^{\infty}\frac{1}{n!\,\Gamma(1-n\alpha)}\left[-\frac{A\,\Gamma(1-\alpha)\,x}{2c}\left(t-\frac{x}{c}\right)^{-\alpha}\right]^n.$$

Here, taking into account the formula

$$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_\gamma(-\xi)^z\,e^{-\xi}\,d\xi\,,$$

where the contour γ goes around the positive semi-axis, we obtain

$$\sigma(x,t)\approx\sigma_0\,H\left(t-\frac{x}{c}\right)\frac{i}{2\pi}\int_\gamma(-\xi)^{-1}\exp\left[-\xi-\frac{A\,\Gamma(1-\alpha)\,x}{2c}\left(t-\frac{x}{c}\right)^{-\alpha}(-\xi)^{\alpha}\right]d\xi\,.$$

Introducing the notations

$$(-\xi)^{-1} = f(\xi), \left(t - \frac{x}{c}\right)^{-\alpha} = \lambda, \frac{A\,\Gamma(1-\alpha)\,x}{2c}(-\xi)^{\alpha} + \frac{\xi}{\lambda} = S(\xi),$$

the last formula is written in the form

$$\sigma(x, t) \approx \sigma_0 H\left(t - \frac{x}{c}\right)\frac{i}{2\pi}\int_{\gamma} f(\xi)\, e^{-\lambda S(\xi)}\, d\xi\,.$$

Since for small $t - x/c$ the parameter λ becomes great, then for calculation of integral one can use a saddle-point method. Without reducing a standard scheme of this method we shall write the principal term of asymptotic expansion

$$\sigma(x, t) \approx \sigma_0 H\left(t - \frac{x}{c}\right)[2\pi(1-\alpha)]^{-1/2}\left[\frac{2c}{A\,\Gamma(1-\alpha)\,\alpha x}\right]^{\frac{1}{2(1-\alpha)}}\left(t - \frac{x}{c}\right)^{\frac{\alpha}{2(1-\alpha)}} \times$$

$$\times \exp\left\{-\frac{1-\alpha}{\alpha}\left[\frac{A\,\Gamma(1-\alpha)\,\alpha x}{2c}\right]^{\frac{1}{1-\alpha}}\left(t - \frac{x}{c}\right)^{-\frac{\alpha}{1-\alpha}}\right\}, \quad x > 0. \qquad (23)$$

Curved dependences (23) are reduced in fig. 1-3. From these graphs it follows that the solution in front of the wave has no jump, i.e. in spite of the form

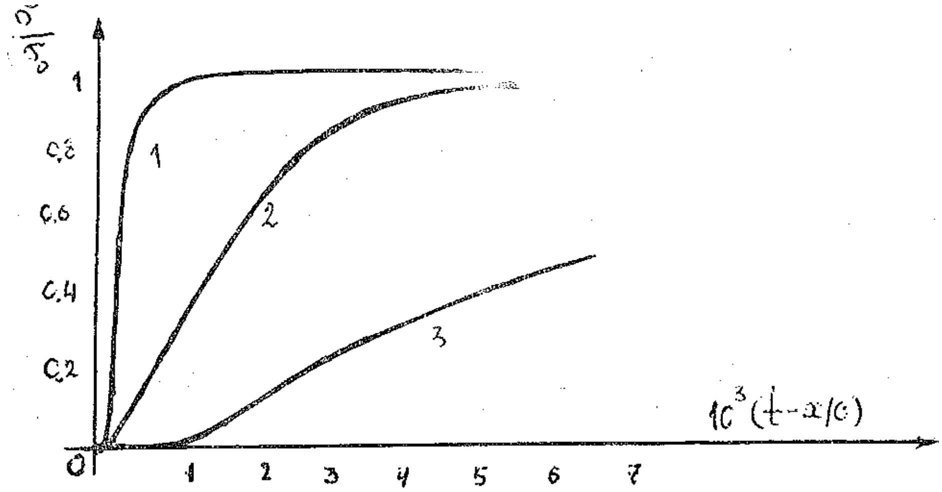

Fig. 1. $A = 0.8 \cdot 10^3\ s^{1/2}$, $c = 3.3 \cdot 10^3$ m/s, $\alpha = 0.5$. The fiist curve corresponds to $x = 0.1\ m$ the second 0.25 m, the third 0.5 m.

of boundary condition the front of strongdiscontinuity waves does not exist. It is easy to show that this is connected with infinite growth of dissipation function for $t - x/c \to 0$. With receding from the end $x = 0$ the length of region on which the dissipation function is infinity increases. Fig. 2 illustrates the influence of parameter of A kernel on dependence on solution from difference $t - x/c$ at a fixed point $x = 0.25$ m. It is seen, that the more A, the stronger

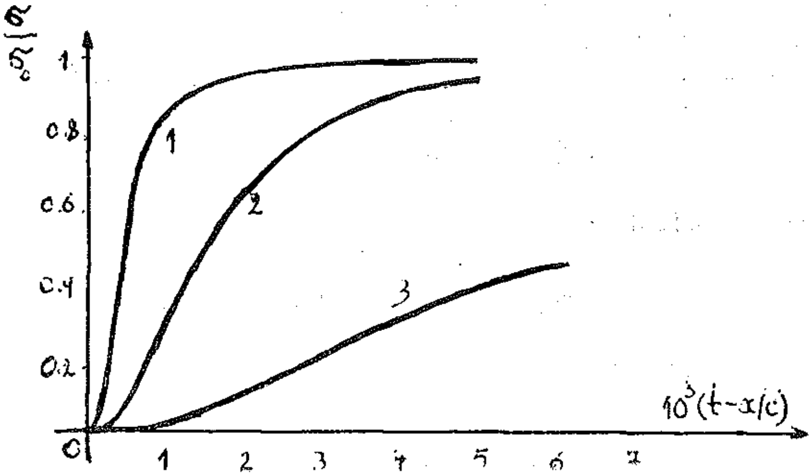

Fig. 2. $\alpha = 0.3$, $x = 0.25$ *m*. The firstcurve $A = 400\ s^{1/2}$, the second $800\ s^{1/2}$ the third $1600\ s^{1/2}$.

of the aboveindicated effect appears. In Fig. 3 the influences of parameter α on the solution are reduced. The nearer α to a unit the steepness of the wave front is eroded; at convergence of α to zero the front becomes steeper (for $\alpha=0$ Maxwell's model is obtained).

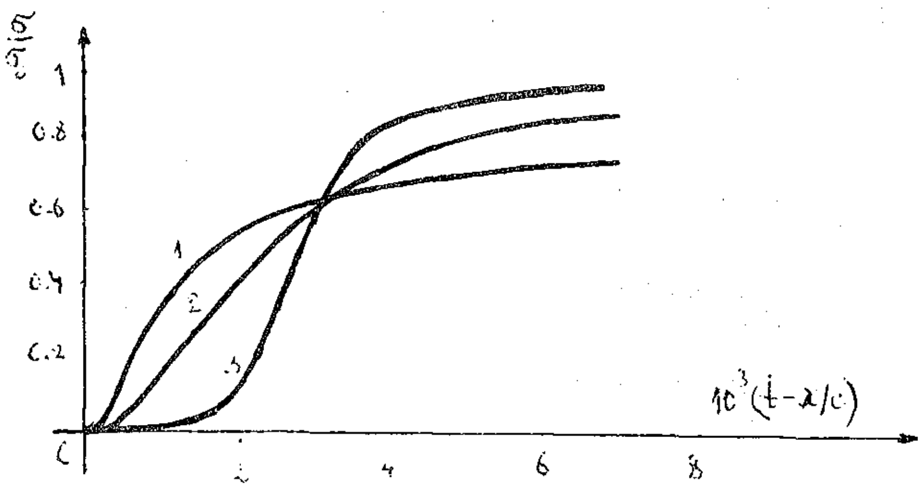

Fig. 3. $A = 800\ s^{1/2}$, $x = 0.25$ *m*. The first curve $\alpha = 0.3$, the second 0.5, the third 0.7.

Formula (23) is obtained by another method in [8].

Finally note that by means of formula (19) under corresponding choice of kernel $K(t)$, initial and boundary conditions, one can obtain all known solutions of one-dimensional problems obtained by other authors.

Let us introduce the denotation $w(t, \tau) = \sigma(x, t) \mid \sigma_0, x = \tau, c = 1$. Then the solution of the fourth auxiliary problem for difference relaxation kernel is written in the form

$$w_i(t, \tau) = \int_0^t w(t - s, \tau)\, d\, b_i(s).$$

If in this formula the function $b_i(t)$ is substituted for

$$a_i(t) + \int_0^t K(t - s)\, a_i(s)\, d\, s,$$

then one can obtain the solution of the third auxiliary problem.

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