DECOMPOSITION OF CURVATURE TENSOR FIELD IN A RECURRENT KAEHLERIAN SPACE

U.P. SINGH and A.K. SINGH

In the present paper the decomposition of the curvature tensor field $R^h{}_{ijk}$ of a recurrent Kaehlerian space in terms of a vector field and two tensor fields has been considered and several theorems about this decomposition have been investigated.

1. **Introduction.** An $n(=2m)$ dimensional Kaehlerian space K^n is a Riemannian space which admits a tensor field ϕ_i^h satisfying the conditions

$$\phi_i^h \phi_h^j = -\delta_{ij} \,^{1)}, \tag{1.1}$$

$$\phi_{ij} = -\phi_i^j \quad (\phi_{ij} = \phi_i^a g_{aj}), \tag{1.2}$$

and

$$\phi^h{}_{i,j} = 0 \tag{1.3}$$

where the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor field is defined by

$$R^h{}_{ijk} \stackrel{\text{def.}}{=} \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} \left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\} \,^{2)} ; \tag{1.4}$$

the Ricci tensor and scalar curvature are given by $R_{ij} = R^a{}_{aij}$ and $R = g^{ij} R_{ij}$ respectively.

It is well known that these tensors satisfy the identity [2] [3]

$$R^a{}_{ijk,a} = R_{jk,i} - R_{ik,j} \,. \tag{1.5}$$

The holomorphically projective curvature tensor $P^h{}_{ijk}$ is defined by

---

[1] All Latin indices run over the range from 1 to n.

[2] $\partial_i = \partial/\partial x^i$, where $\{x^i\}$ denotes real local coordinates.

[3] Numbers in square brackets refer to the references at the end of the paper.

$$P^h{}_{ijk} \overset{\text{def.}}{=} R^h{}_{ijk} + \frac{1}{n+2} (R_{ik}\delta_j^h - R_{jk}\,\delta_i^h + S_{ik}\,\phi_j^h - S_{jk}\,\phi_i^h + 2\,S_{ij}\,\phi_k^h) \quad (1.6)$$

where $S_{ij} = \phi_i^a\ R_{aj}$.

The Bianchi identities in K^n are given by

$$R^h{}_{ijk} + R^h{}_{jki} + R^h{}_{kij} = 0 \quad (1.7)$$

and

$$R^h{}_{ijk,a} + R^h{}_{ika,j} + R^h{}_{iaj,k} = 0\,. \quad (1.8)$$

A Kaehlerian space K^n is said to be Kaehlerian recurrent if its curvature tensor field satisfies the condition ([3])

$$R^h{}_{ijk,a} = \lambda_a\, R^h{}_{ijk} \quad (1.9)$$

where λ_a is a non-zero recurrence vector field.

The following relations follow immediately from (1.9) :

$$R_{ij,a} = \lambda_a\, R_{ij} \quad (1.10)$$

and

$$R_{,a} = \lambda_a\, R\,.$$

In the present paper, we have considered the decomposition of curvature tensor field $R^h{}_{ijk}$ in terms of a vector field and two tensor fields and several theorems have been investigated.

2. Decomposition of Curvature Tensor Fields $\mathbf{R}^h{}_{ijk}$. We consider the decomposition of recurrent curvature tensor field $R^h{}_{ijk}$ in the following form:

$$R^h{}_{ijk} = X_l^h\, v^l\, \psi_{ijk}\ , \quad (2.1)$$

where v^l is a non-zero vector field and X_l^h , ψ_{ijk} are two non-zero tensor fields such that

$$X_l^h\, \lambda_h = p_l \quad (2.2)$$

and

$$\lambda_h\, v^h = 1\,. \quad (2.3)$$

p_l is called decomposed vector field and this is a non-zero vector field.

We shall prove the following :

Theorem (2.1). Under the decomposition (2.1), the Bianchi identities for $R^h{}_{ijk}$ take the form

$$\psi_{ijk} + \psi_{jki} + \psi_{kij} = 0 \quad , \quad \psi_{ijk} = - \psi_{ikj} \tag{2.4}$$

and

$$\lambda_a \, \psi_{ijk} + \lambda_j \, \psi_{ika} + \lambda_k \, \psi_{iaj} = 0 \, . \tag{2.5}$$

Proof. In view of equations (1.10), (1.11), (1.9) and (2.1), we obtain

$$X_l^h \, v^l \, (\psi_{ijk} + \psi_{jki} + \psi_{kij}) = 0 \tag{2.6}$$

and

$$X_l^h \, v^l \, (\lambda_a \, \psi_{ijk} + \lambda_j \, \psi_{ika} + \lambda_k \, \psi_{iaj}) = 0 \, . \tag{2.7}$$

The identities (2.4) and (2.5) follow immediately from these equations and the fact $X_l^h \, v^l \neq 0$.

The following main theorems may be proved in the same way as it is proved in the recent paper [5] :

Theorem (2.2). The vector field λ_a and the tensor field X_l^h given by equations (1.9) and (2.1) behave like recurrent vector and recurrent tensor fields and their recurrent forms are given by

$$\lambda_{a,m} = \mu_m \, \lambda_a \tag{2.8}$$

and

$$X^h{}_{l,m} = \nu_m \, X_l^h \, , \tag{2.9}$$

where μ_m and ν_m are non-zero recurrence vector fields.

Theorem (2.3). Under the decomposition (2.1), the decomposed vector field p_l behaves like a recurrent vector field and its recurrent form is given by

$$p_{l,m} = (\mu_m + \nu_m) \, p_l \, . \tag{2.10}$$

Now, we prove the following :

Theorem (2.4). Under the decomposition (2.1), the vector field v^l and the tensor field ψ_{ijk} behave like recurrent vector and recurrent tensor fields.

Proof. Multiplying equation (2.5) by v^a and using relation (2.3), we obtain

$$\psi_{ijk} = \lambda_k \, \psi_{ij} - \lambda_j \, \psi_{ik} \, , \tag{2.11}$$

where $\psi_{ijk} \, v^k = \psi_{ij}$ is a tensor field.

Therefore, the relation (2.1) takes the form

$$R^h{}_{ijk} = X_l{}^h v^l (\lambda_k \psi_{ij} - \lambda_j \psi_{ik}) . \tag{2.12}$$

Differentiating equation (2.12) covariantly with respect to x^m and using equations (1.9), (2.8), (2.9), (2.12), we get

$$(\lambda_k \psi_{ij} - \lambda_j \psi_{ik}) v^l{}_{,m} = v^l \{\nu_m (\lambda_j \psi_{ik} - \lambda_k \psi_{ij}) + \mu_m (\lambda_j \psi_{ik} - \lambda_k \psi_{ij})\} . \tag{2.13}$$

Multiplying this equation by v^a, we obtain

$$(\lambda_k \psi_{ij} - \lambda_j \psi_{ik}) v^a v^l{}_{,m} = v^l v^a \{\nu_m (\lambda_j \psi_{ik} - \lambda_k \psi_{ij}) + \mu_m (\lambda_j \psi_{ik} - \lambda_k \psi_{ij})\}, \tag{2.14}$$

which yields

$$v^a v^l{}_{,m} = v^l v^a{}_{,m} . \tag{2.15}$$

Since $v^l \neq 0$, there exists a proportional non-zero vector field π_m such that

$$v^l{}_{,m} = \pi_m v^l . \tag{2.16}$$

Therefore, v^l is recurrent vector field.

Further, differentiating equation (2.1) covariantly with respect to x^m and using equations (1.9), (2.1), (2.8), (2.9) and (2.11), we obtain

$$\psi_{ijk,m} = (\lambda_m - \nu_m - \pi_m) \psi_{ijk} . \tag{2.17}$$

Hence, ψ_{ijk} is a recurrent tensor field.

If $\nu_m + \pi_m \neq 0$, we have

Corollary (2.1). Under the decomposition (2.1), the vector field $X_l{}^h v^l$ is recurrent with the recurrence vector field $(\nu_m + \pi_m)$.

Proof. Differentiating the vector field $X_l{}^h v^l$ covariantly with respect to x^m and using equations (2.9) and (2.16), we get the proof.

On the other hand, if $\nu_m + \pi_m = 0$, we have

Corollary (2.2). Under the decomposition (2.1), ψ_{ijk} will be recurrent with the same recurrence vector λ_m as the curvature tensor field $R^h{}_{ijk}$.

Proof. The proof follows immediately from equation (2.17).

Theorem (2.5). Under the decomposition (2.1), the vector field v^l and tensor fields $R^h{}_{ijk}$, R_{ij}, ψ_{ijk} satisfy the relations

$$\lambda_h R^h{}_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = p_l v^l \psi_{ijk} . \tag{2.18}$$

Proof. With the help of equations (1.5), (1.9) and (1.10) we obtain

$$\lambda_h R^h{}_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} . \tag{2.19}$$

Multiplying equation (2.1) by λ_h and using relation (2.2), we obtain

$$\lambda_h R^h{}_{ijk} = p_l v^l \psi_{ijk} . \tag{2.20}$$

From equations (2.19) and (2.20), we get the relations (2.18).

Theorem (2.6). Under the decomposition (2.1), the curvature tensor $R^h{}_{ijk}$ and holomorphically projective curvature tensor fields are equal iff

$$\delta_j^h \psi_{ik} - \delta_i^h \psi_{jk} + \psi_{ak} (\phi_j^h \phi_i^a - \phi_i^h \phi_j^a) + 2 \phi_k^h \phi_i^a \psi_{aj} = 0 . \tag{2.21}$$

Proof. Equation (1.6) may be expressed in the form

$$P^h{}_{ijk} = R^h{}_{ijk} + D^h{}_{ijk} , \tag{2.22}$$

where

$$D^h{}_{ijk} = \frac{1}{n+2} (R_{ik} \delta_j^h - R_{jk} \delta_i^h + S_{ik} \phi_j^h - S_{jk} \phi_i^h + 2S_{ij} \phi_k^h) . \tag{2.22a}$$

Contracting indices h and k in (2.1), we obtain

$$R_{ij} = X_l^k v^l \psi_{ijk} . \tag{2.23}$$

With the help of equation (2.23), we have

$$S_{ij} = \phi_i^a R_{aj} = \phi_i^a X_l^r v^l \psi_{ajr} . \tag{2.24}$$

Making use of equations (2.23) and (2.24) in (2.22a), we obtain

$$D^h{}_{ijk} = \frac{X_l^r v^l}{n+2} \{\psi_{ikr} \delta_j^h - \psi_{jkr} \delta_i^h + \psi_{akr} (\phi_j^h \phi_i^a - \phi_i^h \phi_j^a) + + 2 \phi_k^h \phi_i^a \psi_{ajr}\} . \tag{2.25}$$

From equation (2.22a), it is clear that $P^h{}_{ijk} = R^h{}_{ijk}$, iff $D^h{}_{ijk} = 0$, which in view of equation (2.25) becomes

$$\psi_{ikr} \delta_j^h - \psi_{jkr} \delta_i^h + \psi_{akr} (\phi_j^h \phi_i^a - \phi_i^h \phi_j^a) + 2 \phi_k^h \phi_i^a \psi_{ajr} = 0 . \tag{2.26}$$

Multiplying this equation by v^r and using the relation

$$\psi_{ijk} v^k = \psi_{ij} ,$$

we have the required equation.

REFERENCES

[1] TAKANO, K. : Decomposition of curvature tensor in a recurrent space, Tensor (N.S.), Vol. 18, No.3 (1967), pp. 343-347.

[2] TACHIBANA, S. : On the Bochner curvature tensor, Nat. Sci. Rep., Ochanomizu Univ. ,18(1) (1967), pp. 15-19.

[3] LAL, K.B. and SINGH, S.S. : On Kaehlerian spaces with recurrent Bochner curvature, Accademia Nazionale Dei Lincei, Series VIII, Vol. LI, No. 3-4 (1971), pp. 214-220.

[4] YANO, K. : Differential geometry on complex and almost complex spaces, Pergamon Press (1965).

[5] SINGH, U.P. and SINGH, A.K. : On decomposition of recurrent curvature tensor fields in a Kaehlerian space (To appear).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR, INDIA

ÖZET

Bu çalışmada bir tekrarlamalı Kaehler uzayının $R^h{}_{ijk}$ eğrilik tansörü alanının bir vektör alanı ile iki tansör alanı cinsinden ayrılış formülü ele alınarak bu ayrılışa dair bazı teoremler araştırılmıştır.