EFFECT OF THE PRESENCE OF A CRACK ON THE THERMAL STRESSES IN A LONG INSULATED CYLINDER

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This paper deals with the problem of determining thermal stresses in a long insulated cylinder in the vicinity of a penny-shaped ctack. The solution is obtained in terms of Dini series.

1. Introduction. The concentration of high intensity stresses around a crack in elastic media has been the subject of many investigations in recent years mainly owing to its importance in Fracture Mechanics. For cylindrical bodies such isothermal problems were considered by Collins (1962), Sneddon et al. (1963) and more recently by Atsumi et al. (1972). Solutions for similar thermoelastic problems were put forward by Das (1968, 69). All the above mentioned authors used the integral equation-technique to solve the problems. We propose to report here the solution in terms of Dini series, of a thermoelastic crack problem concerned with a long circular cylinder whose fraction-free curved surface is insulated. It may be noted that the Dual-series-equation-technique is much simpler than the Dual-integral equation-technique.

2. Formulation of the Problem. Let us consider a circular cylinder of radius «a» having a penny-shaped crack of unit radius. We assume, for the present problem, that there is symmetry along the z-axis which is taken along the axis of the cylinder. The problem of determining the stress-field in the cylinder $0 \leqq r \leqq a$, $-\infty < z < \infty$ with a crack occupying the region $r = 1$, $z = 0$ may then be considered as equivalent to that of finding the stresses in the semi-infinite cylinder $0 \leqq r \leqq a$, $z \geqq 0$. The thermal and mechanical conditions are described as follows :

$$T(r, 0) = f(r), \; 0 \leqq r < 1 \tag{2.1}$$

$$\frac{\partial T(r, 0)}{\partial z} = 0 \; , \; 1 < r \leqq a \tag{2.2}$$

$$\sigma_{rz}(r, 0) = 0\,, \quad 0 \leqq r \leqq a \tag{2.3}$$

$$\sigma_{zz}(r, 0) = 0\,, \quad 0 \leqq r < 1 \tag{2.4}$$

$$u_z(r, 0) = 0\,, \quad 1 < r \leqq a \tag{2.5}$$

and

$$\frac{\partial T(a, z)}{\partial r} = 0\,, \quad z \geqq 0 \tag{2.6}$$

$$\sigma_{rz}(a, z) = 0\,, \quad z \geqq 0 \tag{2.7}$$

$$u_r(a, z) = 0\,, \quad z \geqq 0 \tag{2.8}$$

where $f(r)$ is a given function of r and $T = T(r, z)$ is the temperature at any point (r, z).

3. Heat Conduction Problem. In the steady state and in the absence of heat sources the Fourier heat conduction equation is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0\,, \quad 0 \leqq r \leqq a\,, \quad z \geqq 0. \tag{3.1}$$

A solution of the equation (3.1) satisfying the condition (2.6) is taken in the form

$$T(r, z) = \sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n r)\, e^{-\lambda_n z} \tag{3.2}$$

where $\{\lambda_n\}$ is the sequence of positive roots of equation $J_1(\lambda a) = 0$. Insertion of (3.2) into (2.1) and (2.2) yields the following dual series equations :

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n r) = f(r)\,, \quad 0 \leqq r < 1 \tag{3.3}$$

$$\sum_{n=1}^{\infty} a_n J_0(\lambda_n r) = 0 \qquad , \quad 1 < r \leqq a\,. \tag{3.4}$$

To solve the above equations we assume

$$\sum_{n=1}^{\infty} a_n J_0(\lambda_n r) = -\frac{1}{r}\frac{\partial}{\partial r}\int_r^1 \frac{t\, g_1(t)}{(t^2 - r^2)^{1/2}}\, dt,\ [g_1(0) = 0]\,,\ 0 \leqq r < 1. \tag{3.5}$$

By employing the technique of finding the Dini-coefficients, we find

$$a_n = \frac{2\lambda_n^2}{\lambda_n^2 a^2 J_0^2(a\lambda_n)} \int_0^1 J_0(r\lambda_n)\, dr \frac{\partial}{\partial r} \int_r^1 \frac{t\, g_1(t)}{(t^2 - r^2)^{1/2}}\, dt\,.$$

An interchange of the order of integration and use of the results

$$\int_0^1 J_0(\lambda_n r)\, dr \frac{\partial}{\partial r} \int_r^1 t\, g_1(t)\, (t^2 - r^2)^{-\frac{1}{2}}\, dt =$$

$$= \sqrt{\frac{\pi}{2}}\, \lambda_n \int_0^1 g_1(t)\, t^{1/2} J_{-\frac{1}{2}}(\lambda_n t)\, dt \tag{3.6}$$

[Ref. Srivastava (1964), p.166]

and

$$J_{-\frac{1}{2}}(\lambda_n t) = \sqrt{\frac{2}{(\pi t\, \lambda_n)}} \cos(\lambda_n t) \tag{3.7}$$

lead to

$$a_n = \frac{2}{a^2 J_2^2(a\lambda_n)} \int_0^1 g_1(t) \cos(\lambda_n t)\, dt\,. \tag{3.8}$$

Substituting (3.8) into (3.3), one gets

$$\sum_{n=1}^{\infty} \lambda_n^{-1} J_0(\lambda_n r) \frac{2}{a^2 J_2^2(a\lambda_n)} \int_0^1 g_1(t) \cos(\lambda_n t)\, dt = f(r)\,,\ 0 \leq r < 1. \tag{3.9}$$

Following Srivastava (1964), we have

$$S_{\nu,\alpha,\beta,r}(r, t\,;\, a) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_\alpha(r\lambda_n)\, J_\beta(t\lambda_n)\, (\lambda_n)^r}{J^2_{\nu+1}(a\lambda_n)}$$

$$= \int_0^\infty J_\alpha(rx)\, J_\beta(tx)\, x^{1+r}\, dx + \frac{2}{\pi} \sin\left\{\frac{\pi}{2}(\alpha + \beta + r - 2\nu)\right\}.$$

$$\cdot \int_0^\infty \frac{K_0(ay)}{I_0(ay)} I_\alpha(ry)\, I_\beta(ty)\, y^{1+r}\, dy \tag{3.10}$$

and from Erdelyi (1954),

$$\int_0^\infty x^{\nu-\delta+\frac{1}{2}} J_\sigma(rx)\, J_\nu(tx)\, (tx)^{1/2}\, dx$$

$$= \frac{2^{\nu-\delta+1}\, t^{\nu+\frac{1}{2}}}{\Gamma_{(\delta-\nu)}\, r^{\delta}} (r^2 - t^2)^{\sigma-\nu-1} H(r - t)\,,\ (r > 0,\ -1 < \nu < \delta). \tag{3.11}$$

Hence

$$\frac{2}{a^2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)\,\lambda_n^{-1}\cos(\lambda_n t)}{J_2^2(a\lambda_n)}=\sqrt{\frac{\pi t}{2}}\,S_{1,0,-\frac{1}{2},-\frac{1}{2}}(r,t;a)$$

$$=\frac{H(r-t)}{(r^2-t^2)^{1/2}}+\frac{2}{\pi}\int_0^{\infty}\frac{K_1(ay)}{I_1(ay)}\cosh(ty)\,I_0(ry)\,dy\,. \tag{3.12}$$

Now inverting the order of summation and integration in (3.9), we obtain, with the help of (3.12),

$$\int_0^r\frac{g_1(t)}{(r^2-t^2)^{1/2}}\,dt+\int_0^1 g_1(u)\,du\,.\,\frac{2}{\pi}\int_0^{\infty}\frac{K_1(ay)}{I_1(ay)}\cosh(uy)\,.\,I_0(ry)\,dy=f(r)\,.$$

Taking Abel's inversion, we obtain the following Fredholm integral equation for $g_1(t)$:

$$g_1(t)+\int_0^1 g_1(u)\,K_1(t,u)\,du=h_1(t) \tag{3.13}$$

where

$$h_1(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{rf(r)}{(t^2-r^2)^{1/2}}\,dr \tag{3.14}$$

and

$$K_1(t,u)=\frac{4}{\pi^2}\int_0^{\infty}\frac{K_1(ay)}{I_1(ay)}\cosh(ty)\cosh(uy)\,dy\,. \tag{3.15}$$

4. Thermoelastic Problem. Let $\psi(r,z)$ be the thermoelastic displacement potential which satisfies the following equations [Nowacki (1962)]

$$u_i=\psi_{,i} \tag{4.1}$$

$$\sigma_{ij}=2\mu(\psi_{,ij}-\delta_{ij}\,\psi_{,KK}) \tag{4.2}$$

$$\psi_{,KK}=mT,\ m=\frac{1+\eta}{1-\eta}\,\alpha_t\,, \tag{4.3}$$

u_i being the displacement components, η the Poisson ratio, μ the modulus of rigidity and α_t the coefficient of linear thermal expansion.

Equation (4.3) with (3.1) yields the biharmonic equation

$$\nabla^4 \psi(r, z) = 0 .$$

A solution of this equation is taken in the form

$$\psi(r, z) = -\frac{m}{2} \sum_{n=1}^{\infty} a_n \lambda_n^{-3} (1 + \lambda_n z) e^{-\lambda_n z} J_0(\lambda_n r) \tag{4.4}$$

where a_n is given by (3.8).

The complementary stresses and displacements for an isothermal elastic problem can be obtained from the following equations [Sneddon (1951)]

$$\sigma_{rz} = \frac{\partial}{\partial r} \left[(1 - \eta) \nabla^2 X - \frac{\partial^2 X}{\partial z^2} \right] \tag{4.5}$$

$$\sigma_{zz} = \frac{\partial}{\partial z} \left[(2 - \eta) \nabla^2 X - \frac{\partial^2 X}{\partial z^2} \right] \tag{4.6}$$

$$u_z = \frac{1}{2\mu} \left[2(1 - \eta) \nabla^2 X - \frac{\partial^2 X}{\partial z^2} \right] \tag{4.7}$$

$$u_r = - \frac{i}{2\mu} \frac{\partial^2 X}{\partial r \, \partial z} , \tag{4.8}$$

where $X(r, z)$ is an axisymmetric biharmonic function and is defined suitably for this problem in the form

$$X(r, z) = -2\mu \sum_{n=1}^{\infty} \lambda_n^{-3} b_n (2\eta + \lambda_n z) e^{-\lambda_n z} J_0(\lambda_n r) . \tag{4.9}$$

Substituting (4.4) into (4.1) and (4.2), and (4.9) into (4.5) - (4.8), we have, by the principle of superposition,

$$u_r = \frac{m}{2} \sum_{n=1}^{\infty} a_n \lambda_n^{-2} (1 + \lambda_n z) e^{-\lambda_n z} J_1(\lambda_n r) -$$

$$- \sum_{n=1}^{\infty} \lambda_n^{-1} b_n (i - 2\eta - \lambda_n z) e^{-\lambda_n z} J_1(\lambda_n r) \tag{4.10}$$

$$u_z = \frac{m}{2} \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \lambda_n^{-1} z J_0(\lambda_n r) +$$

$$+ \sum_{n=1}^{\infty} b_n \lambda_n^{-1} (2 - 2\eta + \lambda_n z) e^{-\lambda_n z} J_0(\lambda_n r) \tag{4.11}$$

$$\sigma_{rz} = -m\mu z \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} J_1(\lambda_n r) - 2\mu z \sum_{n=1}^{\infty} \lambda_n b_n e^{-\lambda_n z} J_1(\lambda_n r) \tag{4.12}$$

$$\sigma_{zz} = -m\mu z \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} J_0(\lambda_n r) - m\mu \sum_{n=1}^{\infty} \lambda_n^{-1} a_n e^{-\lambda_n z} J_0(\lambda_n r)$$

$$- 2\mu \sum_{n=1}^{\infty} b_n(1 + \lambda_n z) e^{-\lambda_n z} J_0(\lambda_n r) . \tag{4.13}$$

It is clear from (4.12) that the condition (2.3) is automatically satisfied. The conditions (2.7) and (2.8) are also satisfied because $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of $J_1(\lambda a) = 0$. The remaining conditions (2.4) and (2.5) lead to a pair of dual series equations :

$$\sum_{n=1}^{\infty} \lambda_n^2 B_n J_0(\lambda_n r) = F(r) , \ 0 \leqq r < 1 \tag{4.14}$$

$$\sum_{n=1}^{\infty} \lambda_n B_n J_0(\lambda_n r) = 0 , \quad 1 < r \leqq a \tag{4.15}$$

in which

$$F(r) = -\frac{m}{2} f(r) \quad \text{and} \quad \lambda_n^2 B_n = b_n . \tag{4.16}$$

To solve the above equations we assume

$$\sum_{n=1}^{\infty} \lambda_n B_n J_0(\lambda_n r) = \int_r^1 \frac{g(t)}{(t^2 - r^2)^{1/2}} dt, \ [g(0) = 0] , \ 0 \leqq r < 1 , \tag{4.17}$$

where

$$B_n \lambda_n^2 = \frac{2}{a^2 J_2^2(a\lambda_n)} \int_0^1 g(t) \sin(\lambda_n t)\, dt$$

$$\left[\because \int_0^t r J_0(\lambda_n r) (t^2 - r^2)^{-1/2} dr = \lambda_n^{-1} \sin(\lambda_n t) \right] . \tag{4.18}$$

Substituting (4.18) into (4.14) and using the results (3.7) and (3.10) one may obtain

$$- \sqrt{\frac{\pi}{2}}\, g(1)\, S_{1,0,-\frac{1}{2},-\frac{1}{2}}(r, 1; a) + \sqrt{\frac{\pi}{2}} \int_0^1 g'(t)\, t^{\frac{1}{2}}\, S_{1,0,-\frac{1}{2},-\frac{1}{2}}(r, t; a)\, dt$$

$$= F(r)\,, \quad 0 \leqq r < 1\,. \tag{4.19}$$

By the use of (3.11) and (3.12), equation (4.19) may be rewritten as

$$\int_0^r \frac{g'(t)}{(r^2 - t^2)^{1/2}}\, dt + \frac{2}{\pi} \int_0^1 g'(t)\, dt \int_0^\infty \frac{K_1(ay)}{I_1(ay)}\, I_0(ry) \cosh(ty)\, dy$$

$$= F(r) + \frac{2}{\pi}\, g(1) \int_0^\infty \frac{K_1(ay)}{I_1(ay)}\, I_0(ry) \cosh y\, dy\,. \tag{4.20}$$

Taking Abel's inversion we obtain the Fredholm integral equation

$$g'(t) + \int_0^1 L(u, t)\, g'(u)\, du = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{rF(r)}{(t^2 - r^2)^{1/2}}\, dr + g(1)\, L(t, 1) \tag{4.21}$$

where

$$L(u, t) = \frac{4}{\pi^2} \int_0^\infty \frac{K_1(ay)}{I_1(ay)} \cosh(ty) \cosh(uy)\, dy\,. \tag{4.22}$$

Applying the method of integration by parts to the integral in the left hand side of (4.21), one gets

$$g'(t) - \int_0^1 \left[\frac{4}{\pi^2} \int_0^\infty y\, \frac{K_1(ay)}{I_1(ay)} \cosh(ty) \sinh(ay)\, dy\right] g(u)\, du$$

$$= \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{rf(r)}{(t^2 - r^2)^{1/2}}\, dr\,. \tag{4.23}$$

Integrating (4.23) with respect to t, we are lead to the equation

$$g(t) - \int_0^1 K(u, t)\, g(u)\, du = h(t) \tag{4.24}$$

where

$$K(u, t) = \frac{4}{\pi^2} \int_0^\infty \frac{K_1(ay)}{I_1(ay)} \sinh(ty) \sinh(uy)\, dy \tag{4.25}$$

and

$$h(t) = \frac{2}{\pi} \int_0^t \frac{rF(r)}{(t^2 - r^2)^{1/2}}\, dr\,. \tag{4.26}$$

5. Determination of Stress Intensity Factor. The normal stress on the plane of the crack is given by

$$\sigma_{zz}(r, 0) = -m\mu \sum_{n=1}^{\infty} a_n \lambda_n^{-1} J_0(\lambda_n r) - 2\mu \sum_{n=1}^{\infty} b_n J_0(\lambda_n r)\,.$$

For

$$r > 1\,,\ \sum_{n=1}^{\infty} a_n \lambda_n^{-1} J_0(\lambda_n r) = \int_0^1 \frac{g_1(t)}{(r^2 - t^2)^{1/2}}\, dt + \frac{2}{\pi} \int_0^1 g_1(t)\, dt\,.$$

$$\cdot \int_0^{\infty} \frac{K_1(ay)}{I_1(ay)} \cosh(ty)\, I_0(ry)\, dy$$

and

$$\sum_{n=1}^{\infty} b_n J_0(\lambda_n r) = -g(1)\left[\frac{1}{(r^2-1)^{1/2}} + \frac{2}{\pi}\int_0^{\infty} \frac{K_1(ay)}{I_1(ay)} I_0(ry) \cosh y\, dy\right] +$$

$$+ \int_0^1 \frac{g'(t)}{(r^2 - t^2)^{1/2}}\, dt + \frac{2}{\pi}\int_0^1 g'(t)\, dt \int_0^{\infty} \frac{K_1(ay)}{I_1(ay)} I_0(ry) \cosh(ty)\, dy$$

$$= -\int_0^1 \frac{tg(t)}{(r^2 - t^2)^{3/2}}\, dt - \frac{2}{\pi} g(1) \int_0^{\infty} \frac{K_1(ay)}{I_1(ay)} I_0(ry) \cosh y\, dy$$

$$+ \frac{2}{\pi}\int_0^1 g'(t)\, dt \int_0^{\infty} \frac{K_1(ay)}{I_1(ay)} I_0(ry) \cosh(ty)\, dy\,.$$

Therefore, the stress intensity factor N is given by

$$N = \lim_{r \to 1+} \left[(r-1)^{1/2} (\sigma_{zz})_{\substack{z=0\\ r>1}}\right] = \sqrt{2}\, \mu\, g(1)\,.$$

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ÖZET

Bu çalışmada yalıtılmış bir uzun silindirdeki penny şeklinde bir çatlak civarındaki ısı streslerini belirleme problemi ele alınmış olup, çözüm Dini serisi cinsinden elde edilmiştir.