THERMAL - CONVECTIVE INSTABILITY WITH FINITE LARMOR RADIUS

R.C. SHARMA - KIRTI PRAKASH

Thermal-convective instability of a stellar atmosphere is considered to include finite Larmor radius effect in the presence of a uniform vertical magnetic field. The effect of a uniform rotation is also included. It is found that the criterion for monotonic instability is the same as in the absence or presence of these two effects. The growth rates of instability are discussed.

1. Introduction. Defouw [1] has termed the thermal-convective instability as the convective instability of a thermally unstable atmosphere and has generalized the Schwarzschild criterion for convection to include departures from adiabatic motion.

Defouw [1] has given a criterion that a stellar atmosphere will be unstable if

$$D = \frac{1}{C_p}(L_T - \rho\alpha L_\rho) + \mathrm{K}\, k^2 < 0, \qquad (1)$$

where L is the energy lost minus the energy gained per gram per second and $\alpha, \rho, \mathrm{K}, k, L_T, L_\rho$ denote respectively the coefficient of thermal expansion, the density, the coefficient of thermometric conductivity, the wave number of the perturbation, the partial derivative of L with respect to T and the partial derivative of L with respect to ρ, both evaluated in the equilibrium state. C_p is the specific heat at constant pressure. In general the instability due to inequality (1) may be either oscillatory or monotonic.

The criterion for instability (1) has been found to be unchanged by the presence of a uniform rotation and a uniform magnetic field, separately by Defouw [1] and simultaneously by Bhatia [2].

In the above studies the Larmor radii of the charged particles (electrons and protons) are assumed zero. In many astrophysical situations such as the solar corona, interplanetary and interstellar plasmas, it is known that the approximation (zero Larmor radius) is not valid. The effect of the finiteness of the ion Larmor radius, which exhibits itself in the form of a magnetic viscosity in the fluid equations, on plasma instabilities have been studied by Rosenbluth et al [5] and Roberts and Taylor [4].

It is, therefore, interesting to study the modification, if any, in the criterion for instability when the effects due to rotation and finite Larmor radius are included in the thermal-convective instability of a stellar atmosphere.

2. Formulation of the problem. Let us consider an infinite horizontal layer which is in a state of uniform rotation $\vec{\Omega}(o, o, \Omega)$, acted on by a vertical magnetic field $\vec{H}(0, 0, H)$ and gravity force $\vec{g}(o, o, -g)$. This layer is heated from below such that a steady temperature gradient $\beta\left(=\frac{dT}{dz}\right)$ is maintained.

Let

$$\delta \overleftrightarrow{P}, \delta\rho, \vec{q}\,(u, v, w) \text{ and } \vec{h}\,(h_x, h_y, h_z)$$

denote the perturbations in stress tensor $\overleftrightarrow{P}$, density ρ, velocity and magnetic field H respectively; g, ν and η denote respectively, the gravitational acceleration, the kinematic viscosity and the resistivity. Then the linearized hydromagnetic perturbation equations are :

$$\rho \frac{\partial \vec{q}}{\partial t} = -\nabla\delta\overleftrightarrow{P} + \rho\nu\nabla^2 \vec{q} + \frac{1}{4\pi}(\nabla\times\vec{h})\times\vec{H} + 2\rho\,(\vec{q}\times\vec{\Omega}) + \vec{g}\delta\rho, \tag{2}$$

$$\nabla\,.\,\vec{q} = 0, \quad \nabla\,.\,\vec{h} = 0, \tag{3}$$

$$\frac{\partial \vec{h}}{\partial t} = (\vec{H}\,.\,\nabla)\,\vec{q} + \eta\nabla^2\vec{h}. \tag{4}$$

The first law of thermodynamics can be written as

$$C_v \frac{dT}{dt} = -L + \frac{K}{\rho}\nabla^2 T + \frac{p}{\rho^2}\frac{d\rho}{dt}, \tag{5}$$

where K, C_v, T, t and p denote the thermal conductivity, the specific heat at constant volume, the temperature, the time and the pressure respectively.

The linearized perturbation form of equation (5), following Defouw [1], is

$$\frac{\partial\theta}{\partial t} + \frac{1}{C_p}(L_T - \rho\alpha L_\rho)\,\theta - K\nabla^2\theta = -\left(\beta + \frac{g}{C_p}\right) w\,, \tag{6}$$

where 0 is the perturbation in temperature. In obtaining (6), use has been made of the Boussinesq equation of state

$$\delta\rho = -\alpha\rho 0. \tag{7}$$

For the vertical magnetic field $\vec{H}(0,0,H)$, the stress tensor components $\delta \overleftrightarrow{P}$, taking into account finite ion gyration, have the components (Sharma [6]) :

$$\delta P_{xx} = \delta p - \rho\nu_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \delta P_{xy} = \delta P_{yx} = \rho\nu_0\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right),$$

$$\delta P_{xz} = \delta P_{zx} = -2\rho\nu_0\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \delta P_{yy} = \delta p + \rho\nu_0\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{8}$$

$$\delta P_{yz} = \delta P_{zy} = 2\rho\nu_0\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \delta P_{zz} = \delta p .$$

In equations (8), δp is the perturbation in scalar part of the pressure and

$$\rho\nu_0 = \frac{NT}{4\omega_H},$$

where ω_H is the ion-gyration frequency, while N and T denote respectively, the number density and temperature of the ions. We consider the case in which both the boundaries are free and the medium adjoining the fluid is nonconducting. The boundary conditions appropriate for the problem are (Chandrasekhar [3]) :

$$\omega = 0 \quad , \quad \theta = 0 ,$$
$$\frac{\partial^2 w}{\partial z^2} = 0 \quad , \quad \frac{\partial \rho}{\partial z} = 0 , \tag{9}$$

$\xi = 0$ and $\vec{h}$ is continuous with an external vacuum field. Here p and ξ denote, respectively, the z-components of vorticity and current density.

3. Dispersion relation and discussion. Analyzing in terms of normal modes, we seek solutions whose dependence on space and time coordinates is of the form

$$\exp\left[ik_x x + ik_y y + nt\right] \sin k_z z , \tag{10}$$

where k_z is an integral multiple of π divided by thickness of the fluid layer, $k\left(=\sqrt{k_x{}^2 + k_y{}^2 + k_z{}^2}\right)$ is the wave number of the perturbation and n is the growth rate.

From equations (2) - (4), (6) and (8), we have :

$$\frac{\partial}{\partial t}(\nabla^2\omega) = g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \nu\nabla^4\omega - 2\Omega\frac{\partial\rho}{\partial z} + \frac{H}{4\pi h}\frac{\partial}{\partial z}\nabla^2 h_z + $$
$$+ \nu_0\left(\nabla^2 - 3\frac{\partial^2}{\partial z^2}\right)\frac{\partial\rho}{\partial z}, \tag{11}$$

$$\frac{\partial\rho}{\partial t} = \nu\nabla^2\rho + 2\Omega\frac{\partial\omega}{\partial z} + \frac{H}{4\pi\rho}\frac{\partial\xi}{\partial z} - \nu_0\left(\nabla^2 - 3\frac{\partial^2}{\partial z^2}\right)\frac{\partial\omega}{\partial z}, \tag{12}$$

$$\frac{\partial h_z}{\partial t} = H \frac{\partial \omega}{\partial z} + \eta \nabla^2 h_z \,, \tag{13}$$

$$\frac{\partial \xi}{\partial t} = H \frac{\partial \rho}{\partial z} + \eta \nabla^2 \xi \,, \tag{14}$$

$$n\theta + \left[\frac{1}{C_p} (L_T - \rho \alpha L_\rho) + \mathrm{K} k^2 \right] \theta = - \left(\beta + \frac{g}{C_p} \right) \omega \,. \tag{15}$$

Eliminating 0, ξ, h_z and ρ from equations (11)-(15) and using (10), we obtain the dispersion relation

$$n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0 = 0 \,, \tag{16}$$

where

$$A_4 = D + 2k^2 (\nu + \eta) \,,$$

$$A_3 = 2k^2 D(\nu + \eta) + k^4 (\nu^2 + 4\nu\eta + \eta^2) + 2k_z^2 V^2 + \Gamma \left(\beta + \frac{g}{C_p} \right) + \frac{k_z^2}{k^2} A^2 \,,$$

$$A_2 = k^2 (\nu + 2\eta) \Gamma \left(\beta + \frac{g}{C_p} \right) + k^4 D(\nu^2 + 4\nu\eta + \eta^2) + 2k_z^2 V^2 D + 2k^6 \nu\eta (\nu + \eta) + $$
$$+ 2k_z^2 k^2 V^2 (\nu + \eta) + \frac{k_z^2}{k^2} A^2 (D + 2\eta k^2) \,, \tag{17}$$

$$A_1 = 2k_z^2 k^2 V^2 D (\nu + \eta) + 2k^6 D \nu\eta (\nu + \eta) + \Gamma \left(\beta + \frac{g}{C_p} \right) \eta k^4 (\eta + 2\nu) + \nu^2 \eta^2 k^8 +$$
$$+ \Gamma \left(\beta + \frac{g}{C_p} \right) k_z^2 V^2 + 2\nu\eta k^4 k_z^2 V^2 + k_z^4 V^4 + \eta k_z^2 A^2 (\eta k^2 + 2D) \,,$$

$$A_0 = (\nu\eta k^4 + k_z^2 V^2) \left\{ k_z^2 V^2 D + \nu\eta k^4 D + \Gamma \left(\beta + \frac{g}{C_p} \right) \eta k^2 \right\} + \eta^2 k^2 k_z^2 A^2 D \,,$$

$$\Gamma = \frac{g\alpha (k_x^2 + k_y^2)}{k^2} \,, \quad V^2 = \frac{H^2}{4\pi\rho} \text{ and } A = 2\Omega + \nu_0 (k^2 - 3k_z^2) \,.$$

Setting $\nu = \eta = 0$ in equation (16), as the effects of viscosity and resistivity are negligible in many cases of astrophysical interest, the dispersion relation reduces to :

$$n^5 + Dn^4 + \left[2k_z^2 V^2 + \Gamma \left(\beta + \frac{g}{C_p} \right) + \frac{k_z^2}{k^2} (2\Omega + \nu_0 \overline{k^2 - 3k_z^2})^2 \right] n^3 +$$
$$+ D \left[2k_z^2 V^2 + \frac{k_z^2}{k^2} (2\Omega + \nu_0 \overline{k^2 - 3k_z^2})^2 \right] n^2 + k_z^2 V^2 \left[\Gamma \left(\beta + \frac{g}{C_p} \right) + k_z^2 V^2 \right] n +$$
$$+ k_z^4 V^4 D = 0 \,. \tag{18}$$

When $D < 0$ i.e., when inequality (1) is satisfied, the constant term in equation (18) is negative. This means that the equation (18) has positive real root, leading to monotonie instability. The criterion for instability, therefore, is the same even if the rotation and finite Larmor radius effects are included on thermal-convective instability of a stellar atmosphere.

We shall now discuss the dispersion relation (18) in some detail. Let us regard D to be small and solve the dispersion relation in order to study the nature of instability and growth rates in case of instability. Putting

$$n = n_0 + n_1 D$$

in Eq. (18) and neglecting terms involving powers of D higher than the first, we get the following equations determining n_0 and n_1 :

$$n_0 \left[n_0^4 + PA^2 \left(2f + 1 + \frac{\cos^2\theta}{P} \right) n_0^2 + P^2 A^4 f (1 + f) \right] = 0, \tag{19}$$

$$n_1 = - \frac{n_0^4 + P A^2 \left(2f + \frac{\cos^2\theta}{P} \right) n_0^2 + P^2 A^4 f^2}{5 n_0^4 + 3 PA^2 \left(2f + 1 + \frac{\cos^2\theta}{P} \right) n_0^2 + P^2 A^4 f (1 + f)} . \tag{20}$$

From Eq. (19), we have

$$n_0 = 0 , \tag{21}$$

$$2 \left(\frac{n_0}{A} \right)^2 = - [(1 + 2f) P + \cos^2\theta \mp \sqrt{P^2 + 2P(1 + 2f)\cos^2\theta + \cos^4\theta}], \tag{22}$$

where

$$P = \frac{\Gamma\left(\beta + \frac{g}{C_p}\right)}{A^2}, f = \frac{k_z^2 V^2}{PA^2}, \frac{k_z}{k} = \cos\theta , \tag{23}$$

P, f and A^2 are all positive.

Corresponding to $n_0 = 0$,

$$n_1 = - \frac{f}{1+f} . \tag{24}$$

The modes (21) and (24) correspond to the growing mode with growth rate given by

$$n = - \frac{f}{1+f} D = - D \left[\frac{1}{1 + \frac{\Gamma\left(\beta + \frac{g}{C_p}\right)}{k_z^2 V^2}} \right] .$$

Thus, when $D<0$, as k_z and H increase, growth rate increases whereas as temperature gradient increases growth rate decreases.

In the equation (22), the expression within the square root sign is positive. Therefore all the four values of n_0 are purely imaginary occurring in conjugate pairs so that we have the oscillatory instability when $D = 0$. The frequencies of these oscillations depend on k_z, k, A, F, P, θ i.e. magnetic field, temperature gradient, rotation, finite Larmor radius and wave numbers.

And

$$\left.\begin{array}{l} n_{1,2} = (-)\dfrac{1}{2} \cdot \dfrac{E}{F} \qquad , \qquad (25,26) \\ E = P\left[P + \cos^2\theta \mp \sqrt{P^2 + 2P(1+2f)\cos^2\theta + \cos^4\theta}\right] \\ F = \left[(1-f-2f^2)P^2 + \cos^2\theta(2+3f)P + \cos^4\theta - [\cos^2\theta + P(f+1)\}\sqrt{P^2 + 2P(1+2f)\cos^2\theta + \cos^4\theta}\right] \end{array}\right\}$$

the frequency n_1 with the negative sign corresponding to the value of n_0^2 given by upper sign in Eq, (22) and the frequency n_2 with the positive sign corresponding to the values of n_0 given by lower sign in Eq. (22).

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DEPARTMENT OF MATHEMATICS
HIMACHAL PRADESH UNIVERSITY
SIMLA - 171005

ÖZET

Bu çalışmada bir yıldız atmosferinin termik konvektivitesi dengesizliği, bir düzgün düşey manyetik alandaki sonlu Larmor yarıçapı etkisi gözönüne alınarak incelenmektedir. Aynı zamanda düzgün bir dönme etkisi de gözönüne alınmaktadır. Monoton dengesizlik kriterinin, adı geçen iki etkinin varlığında veya yokluğunda aynı olduğu saptanmakta ve dengesizliğin büyüme oranları incelenmektedir.