CONFORMAL MOTION IN A SYMMETRIC FINSLER SPACE

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In this paper a conformal motion in a symmetric Finsler space defined by the infinitesimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt$$

is studied, where $dt\, v^i(x)$ is any vector field and dt is an infinitesimal constant.

1. Introduction. Let F_n be an n-dimensional Finsler space equipped with the line element $(x, \dot{x})$ and a fundamental metric function $f(x^i, \dot{x}^i)$ which satisfies the requisite conditions [1][1]. The metric tensor of F_n is given by

$$g_{ij}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2}\, \dot{\partial}^2_{ij} f^2(x, \dot{x}) \text{ where } \dot{\partial}^2_{ij} = \partial/\partial\dot{x}^i\, \partial\dot{x}^j \text{ and } g^{ij} g_{jk} = \delta^i_k.$$

The contravariant and covariant components of the metric tensor are symmetric in their indices and positively homogeneous of degree zero in the $\dot{x}^i$ s. Let $X^i(x, \dot{x})$ be a vector field depending on position as well as on directional argument. The covariant derivative of $X^i(x, \dot{x})$ in the sense of Berwald is given by

$$X^i_{(k)} = \partial_k X^i - \dot{\partial}_m X^i G^m_{k\gamma} \dot{x}^\gamma + x^m G^i_{mk}\,, \tag{1.1}$$

where $G^i_{hk}(x, \dot{x}) \overset{\text{def.}}{=} \dot{\partial}^2_{hk} G^i$ [2] is Berwald's connection parameter and $G^i(x, \dot{x})$ is homogeneous function of degree two in $\dot{x}^i$.

The commutation formula for Berwald's covariant derivative of a tensor field T^i_j is expressed by

$$2\, T^i_{j[(h)(k)]}{}^{3)} = T_j^\gamma H^i_{hk\gamma} - T_\gamma^i H^\gamma_{hkj} - (\dot{\partial}_\gamma T_j^i)\, H^\gamma_{hk}\,, \tag{1.2}$$

where

$H^i_{jhk}(x, \dot{x})$ is the Berwald's curvature tensor. It is given by

[1] Numbers in brackets refer to the references given at the end of the paper.

[2] $\partial_i = \partial/\partial x^i$ and $\dot{x}^i = dx^i/dt$

[3] $2X_{[hk]} = X_{hk} - X_{kh}$.

$$H^i{}_{hjk} = 2\,\{\partial_{[k}G^i{}_{j]h} + G^\gamma{}_{h[j}G^i{}_{k]\gamma} + G^i{}_{\gamma h[k}\dot{\partial}_{j]}\,G^\gamma\}\ . \tag{1.3}$$

These functions satisfy the following identities :

$$H^i{}_{hjk} + H^i{}_{khj} + H^i{}_{jkh} = 0 \tag{1.4a}$$

$$H^i{}_{hk} = -\,H^i{}_{kh}\,,\ H^i{}_{jkh}\,\dot{x}^j = H^i{}_{kh}\ \text{and}\ H^i{}_{hj}\,\dot{x}^h = H^i{}_j\,. \tag{1.4b}$$

For the contraction of $H^i{}_{jhk}(x, \dot{x})$ we have

$$H_{kh}(x, \dot{x}) \stackrel{\text{def.}}{=} H^i{}_{khi}\,,\ H^i{}_{ki} = H_k \tag{1.5a}$$

$$H(x, \dot{x}) \stackrel{\text{def.}}{=} \frac{1}{n-1}\,H^i{}_i\ \text{and} \tag{1.5b}$$

the skew-symmetric properties of $H^i{}_{jhk}$, reduces to

$$H_{kh} - H_{hk} = H^i{}_{ihk}\,. \tag{1.6}$$

We consider the infinitesimal point transformation

$$\overline{x}^i = x^i + \mathrm{v}^i(x)\,dt\,, \tag{1.7}$$

where $dt\ \mathrm{v}^i(x)$ is any vector field and dt is an infinitesimal constant. The Lie-derivatives of a tensor field $T^i{}_j(x, \dot{x})$ and the connection parameter $G^i{}_{jk}(x, \dot{x})$ are respectively given by

$$\pounds\,T^i{}_j(x, \dot{x}) = T^i{}_{j(h)}\,\mathrm{v}^h - T^h{}_j\,\mathrm{v}^i{}_{(h)} + T^i{}_h\,\mathrm{v}^h{}_{(j)} + (\dot{\partial}_h\,T^i{}_j)\,\mathrm{v}^h{}_{(s)}\,\dot{x}^s \tag{1.8}$$

and

$$\pounds\,G^i{}_{jk}(x, \dot{x}) = \mathrm{v}^i{}_{(j)(k)} + \mathrm{v}^h\,H^i{}_{hjk} + G^i{}_{jkh}\,\mathrm{v}^h{}_{(s)}\,\dot{x}^s\,. \tag{1.9}$$

The commutation formulae for the operator $\pounds$ and (k) of the tensor field $T_j{}^i$ is given by

$$\pounds\,(T_j{}^i{}_{(k)}) - (\pounds\,T_j{}^i)_{(k)} = T_j{}^h\,\pounds\,G^i{}_{kh} - T_h{}^i\,\pounds\,G^h{}_{kj} - (\dot{\partial}_h\,T_j{}^i)\,\pounds\,G^h{}_{ks}\,\dot{x}^s\,. \tag{1.10}$$

In particular, we have

$$2\,\pounds\,G^i{}_{h\,[k(j)]} = \pounds\,H^i{}_{jkh} + 2\,(\pounds\,G^s{}_{m[j})\,\dot{x}^m\,G^i{}_{k]\,sh}\,. \tag{1.11}$$

The conformal transformation in F_n is characterized by

$$\overline{g}_{ij}(x, \dot{x}) = e^{2\sigma}\,g_{ij}(x, \dot{x})\,, \tag{1.12}$$

where g_{ij} and $\overline{g}_{ij}$ are two metric tensors obtained respectively from two distinct metric functions $f(x, \dot{x})$ and $\overline{f}(x, \dot{x})$, and σ is the function of x only. We obtain the following entities of the conformal Finsler space :

$$\overline{G}^i = G^i - \sigma_m\,B^{im}\,,\ \sigma_m = \partial_m\,\sigma \tag{1.13}$$

$$\overline{G}^i{}_{hk} = G^i{}_{hk} - \sigma_m\,B^{im}{}_{hk}\,(B^{im}{}_{hk} \stackrel{\text{def.}}{=} \dot{\partial}^2{}_{hk}\,B^{im})\,, \tag{1.14}$$

where

$B^{hk}(x, \dot{x}) \overset{\text{def.}}{=} \frac{1}{2} f^2 g^{hk} - \dot{x}^h \dot{x}^k$ is positively homogeneous of degree two in its directional argument.

2. Conformal Symmetric Finsler Space. If the Berwald's curvature tensor field $H^i_{hjk}(x, \dot{x})$ satisfies the relation

$$H^i_{jkh(m)} = 0\,, \tag{2.1}$$

then such a Finsler space is called a symmetric Finsler space [4]. The following relations have been obtained [4] :

$$\text{a) } H^i_{jk(m)} = 0\,, \quad \text{b) } H^i_{k(m)} = 0\,, \quad \text{c) } H_{(m)} = 0\,. \tag{2.2}$$

If the infinitesimal point change (1.7) implies that the magnitudes of vector defined in the same tangent space are proportional and the angle between two directions is also the same with respect to the metrices then it is called a conformal motion in F_n. The variation of $G^i_{jk}(x, \dot{x})$ under the infinitesimal point transformation is $\pounds\, G^i_{jk}$ and that under the conformal change is $\overline{G}^i_{jk}$. The two transformations will coincide if the corresponding variations are the same. Thus we have :

Theorem (2.1). A necessary and sufficient condition that the infinitesimal change (1.7) be a conformal motion is that

$$\pounds\, G^i_{jk}(x, \dot{x}) = -\,\sigma_m\, B^{im}{}_{jk}\,. \tag{2.3}$$

Thus, for a conformal motion we have the following relations :

$$\pounds\, G^i(x, \dot{x}) = -\,\sigma_m\, B^i{}_m \tag{2.4a}$$

$$\pounds\, G^i_{hk\gamma}(x, \dot{x}) = -\,\sigma_m\, B^{im}{}_{hk\gamma}\,, \tag{2.4a}$$

where $B^{im}(x, \dot{x})$ is a symmetric function with respect to its indices.

Definition (2.1). A finsler space F_n admitting (2.1) and (2.3) is called a conformal symmetric Finsler space. If we apply formula (1.10) to the deviation tensor field $H_j^i(x, \dot{x})$ we obtain in view of (2.2b) and (2.3)

$$(\pounds\, H_j^i)_{(k)} = -\,\sigma_m\, \{H^i_h\, B^{hm}{}_{kj} - H_j^h\, B^{im}{}_{kh} + (\dot{\partial}_h H_j^i)\, B^{hm}{}_{ks}\, \dot{x}^s\}\,. \tag{2.5}$$

On contracting indices i and j of (2.5) and using the equation (1.4b) we get

$$(\pounds H)_{(k)} + (\dot{\partial}_h H)\, B^{hm}{}_{ks}\, \dot{x}^s\, \sigma_m = 0\,. \tag{2.6}$$

Thus we have the following theorem :

Theorem (2.2). If an infinitesimal point transformation (1.7) defines a conformal motion in symmetric Finsler space then (2.6) holds. From (2.3) it is clear that for vanishing of function $B^{ih}(x, \dot{x})$ is the necessary and sufficient condition that the conformal motion becomes an affine motion. Hence from (2.6) we have :

Corollary (2.2). In a symmetric Finsler space if the conformal motion becomes affine motion in the same space then it is necessary and sufficient condition that $(\pounds H)_{(k)}$ must vanish. Taking the Lie-derivative of $H^i_{hjk}(x, \dot{x})$ and using equations (1.4b), (1.5b), (2.3) and (2.4b) we get [6]

$$\begin{aligned}(\pounds H)_{(k)} = \frac{1}{(n-1)} [\sigma_m \{B_i{}^{im}{}_{(j)(k)} \dot{x}^j - 2 (B^{im}{}_{(i)(k)} - B^{i\gamma}{}_{(k)} G^m{}_{\gamma i} - \\ - G^m{}_{\gamma i (k)} B^{i\gamma}) - B^{i\gamma}{}_{i(k)} G^m{}_\gamma - G^m{}_{\gamma (k)} B_i{}^{i\gamma}\} + \sigma_{m(k)} \{B^{im}{}_{i(j)} \dot{x}^j - \\ - 2 (B^{im}{}_{(i)} - B^{i\gamma} G^m{}_{\gamma i}) - B_i{}^{i\gamma} G^m{}_\gamma\} - \{2 (B^{im}{}_{(k)} \sigma_{mi} + \sigma_{mi(k)} B^{im}) + \\ + B^{im}{}_{i(k)} \sigma_{mj} \dot{x}^j + B_i{}^{im} \sigma_{mj(k)} \dot{x}^j\}] .\end{aligned} \tag{2.7}$$

From (2.6) and (2.7) we obtain

$$\begin{aligned}(\dot{\partial}_h H) B^{hm}{}_{ks} \dot{x}^s \sigma_m + \frac{1}{n-1} [\sigma_m \{B_i{}^{im}{}_{(j)(k)} \dot{x}^j - 2 (B^{im}{}_{(i)(k)} - B^{i\gamma}{}_{(k)} G^m{}_{\gamma i} - \\ - G^m{}_{\gamma i (k)} B^{i\gamma}) - B^{i\gamma}{}_{i(k)} G_\gamma{}^m - G^m{}_{\gamma (k)} B_i{}^{i\gamma}\} + \sigma_{m(k)} \{B^{im}{}_{i(j)} \dot{x}^j - \\ - 2 (B^{im}{}_{(i)} - B^{i\gamma} G^m{}_{\gamma i}) - B_i{}^{i\gamma} G_\gamma{}^m\} - \{2 (B^{im}{}_{(k)} \sigma_{mi} + \sigma_{mi(k)} B^{im}) \\ + B^{im}{}_{i(k)} \sigma_{mj} \dot{x}^j + B_i{}^{im} \sigma_{mj(k)} \dot{x}^j\}] = 0 .\end{aligned} \tag{2.8}$$

Thus we have the following theorem :

Theorem (2.3). In a symmetric Finsler space when conformal motion is given by (1.7) then (2.8) holds.

Theorem (2.2) can be proved in a different way also as follows :

Using the equations (1.10), (2.1) and (2.2a) Meher [2] has obtained the following relation :

$$(\pounds H^i_{jk})_{(m)} = 2H^i{}_{s[k} v^s{}_{<(m)>(j)]} - H^s{}_{jk} v^i{}_{(m)(s)} + H^i{}_{jkh} v^h{}_{(m)(s)} \dot{x}^s \text{ }^{4)} . \tag{2.9}$$

Substituting the value of $v^i_{(j)(k)}$ from the equation (1.9) and using equation (2.3) we have after arranging the terms

$$\begin{aligned}(\pounds H^i_{jk})_{(m)} = \sigma_p \{H^i{}_{sj} B^{sp}{}_{mk} - H^i{}_{sk} B^{sp}{}_{mj} + H^s{}_{jk} B^{ip}{}_{ms} - H^i{}_{jkh} B^{hp}{}_{ms} \dot{x}^s\} + \\ + v^h \{H^s{}_{jk} H^i{}_{hms} - 2H^s{}_{hm[k} H^i{}_{j]s} - H^i{}_{jks} H^s{}_{km}\} + \\ + v^h{}_{(\gamma)} \dot{x}^\gamma \{H^s{}_{jk} G^i{}_{msh} - H^i{}_{sk} G^s{}_{mjh} + H^i{}_{sj} G^s{}_{mkh}\} .\end{aligned} \tag{2.10}$$

4) The indices in brackets < > are free from symmetric and skew symmetric parts.

The following commutation formulae for $H^i_{jk}(x, \dot{x})$ are given by

$$2\, H^i_{jk[(h)(m)]} = \{H^s_{jk}\, H^i_{hms} - 2H^s_{hm[k}\, H^i_{j]s} - H^i_{jks}\, H^s_{hm}\} \tag{2.11}$$

and

$$\dot{\partial}_h (H^i_{jk(m)}) - (\dot{\partial}_h H^i_{jk})_{(m)} = H^i_{jk}\, G^i_{hms} - 2H^i_{s[k}\, G^s_{j]\,hm}\,. \tag{2.12}$$

In a symmetric Finsler space F_n characterized by $H^i_{jhk(m)}=0$ and $H^i_{jh(m)}=0$ the above equations reduce to respectively

$$H^s_{jk}\, H^i_{hms} - 2H^s_{hm[k}\, H^i_{j]s} - H^i_{jks}\, H^s_{hm} = 0 \tag{2.13}$$

and

$$H^s_{jk}\, G^i_{msh} - H^i_{sk}\, G^s_{mjh} + H^i_{sj}\, G^s_{mkh} = 0. \tag{2.14}$$

Substituting equations (2.13) and (2.14) in (2.10) we obtain

$$(\pounds\, H^i_{jk})_{(m)} = \{H^i_{sj}\, B^{sp}_{mk} - H^i_{sk}\, B^{sp}_{mj} + H^s_{jk}\, B^{ip}_{ms} - H^i_{jkh}\, B^{hp}_{ms}\, \dot{x}^s\}\, \sigma_p\,. \tag{2.15}$$

Transvecting equation (2.15) by $\dot{x}^k$ and then contracting the resulting equation in the indices i and j we get the required result (2.6).

The Lie-derivative of $H^i_{jkh}(x, \dot{x})$ can be obtained with the help of equations (1.11) and after transvection by $\dot{x}^h$ we get

$$\pounds H^i_{jk} = (\pounds G^i_{hj(k)} - \pounds G^i_{hk(j)})\, \dot{x}^h\,. \tag{2.16}$$

By using equation (2.3) and by homogenity property of $G^i(x, \dot{x})$ we have

$$\pounds H^i_{jk} = 2\, \{B^{ip}_{[j(k)]}\, \sigma_p + \sigma_{p[(k)}\, B^{ip}_{j]}\}\,, \tag{2.17}$$

where we have used $\dot{x}^i_{(k)} = 0$. By taking the Lie-derivative of equation (2.14) and substituting $\pounds\, G^i_{kh\gamma}(x, \dot{x})$ and $\pounds\, H^i_{jk}(x, \dot{x})$ from equations (2.4b) and (2.16) respectively, we obtain

$$\begin{aligned} &2\,\{G^i_{hms}\,(B^{sp}_{[j(k)]}\,\sigma_p + \sigma_{p[(k)}\, B^{sp}_{j]}) - G^s_{jhm}\,(B^{ip}_{[s(k)]}\,\sigma_p + \\ &+ \sigma_{p[(k)}\, B^{ip}_{s]} + G^s_{khm}\,(B^{ip}_{[s(j)]}\,\sigma_p + \sigma_{p[(j)}\, B^{ip}_{s]})\} - \\ &- (H^s_{jk}\, B^{ip}_{hms} - 2H^i_{s[k}\, B^{sp}_{j]hm})\,\sigma_p = 0. \end{aligned} \tag{2.18}$$

On contracting with respect to the indices i and j in the above equation and using (1.5a) we get

$$H_s\, B^{sp}_{khm}\, \sigma_p = 2\,\{B^{ip}_{[s(i)]}\,\sigma_p + \sigma_{p[(i)}\, B^{ip}_{s]}\}\, G^s_{khm}\,. \tag{2.19}$$

Thus we have the following theorem:

Theorem (2.4). The infinitesimal transformation (1.7) defining a conformal motion in a symmetric F_n satisfies (2.19).

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ÖZET

Bu çalışmada $dt\, v^i(x)$ herhangi bir vektör alanı ve dt bir infinitezimal sabit olmak üzere

$$\bar{x}^i = x^i + v^i(x)\, dt$$

nokta dönüşümü ile tanımlanan bir simetrik Finsier uzaymdaki bir konform hareket incelenmektedir.