ON THE GEOMETRIC MEANS OF PRODUCTS OF ENTIRE FUNCTIONS OF SLOW GROWTH

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In this paper it has been considered the geometric means of the products of two entire functions and obtained some of their properties. The results can easily be extended to any finite number of entire functions. Some results of Jain and Chugh can be obtained as particular cases of the results obtained here.

1. Introduction. Throughout this paper we shall assume $f_1(z), f_2(z), \dots, f_m(z)$ are m entire functions, other than polynomials, of orders zero. Let $f_1(z), f_2(z), \dots, f_m(z)$ be of logarithmic orders $\rho_1^*, \rho_2^*, \dots, \rho_m^*$ and lower logarithmic orders $\lambda_1^*, \lambda_2^*, \dots, \lambda_m^*$ respectively. For an entire function, say $f_1(z)$, of this nature, the logarithmic order ρ_1^* and lower logarithmic order λ_1^* are defined as [5]:

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r,f_1)}{\log \log r} = \begin{matrix} \rho_1^* \\ \lambda_1^* \end{matrix}, \quad 1 \leqslant \lambda_1^* \leqslant \rho_1^* \leqslant \infty , \tag{1.1}$$

where $M(r,f_1) = \max\limits_{|z|=r} |f_1(z)|$.

The geometric mean of $f_1(z) f_2(z) \dots f_m(z)$, for $|z| = r$, has been defined as [4, p.144]:

$$G(r, f_1 f_2 \dots f_m) = \\ = \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log |f_1(re^{i\theta}) f_2(re^{i\theta}) \dots f_m(re^{i\theta})| \, d\theta \right]. \tag{1.2}$$

Let us introduce the following mean values of $f_1(z) f_2(z) \dots f_m(z)$:

$$g_\delta(r, f_1 f_2 \dots f_m) = \exp\left[\frac{\delta+1}{r^{\delta+1}} \int_0^r x^\delta \log G(x, f_1 f_2 \dots f_m) \, dx\right] \tag{1.3}$$

and

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$$g_\delta^* (r, f_1 f_2 \dots f_m) = \tag{1.4}$$
$$= \exp \left[\frac{\delta + 1}{(\log r)^{\delta+1}} \int_1^r x^{-1} (\log x)^\delta \log G(x, f_1 f_2 \dots f_m)\, dx \right],$$

where $0 < \delta < \infty$.

Using Jensen's formula [1, p.2] in (1.2), we have

$$\log G(r, f_1 f_2 \dots f_m) = \log |f_1(0) f_2(0) \dots f_m(0)| + \tag{1.5}$$
$$+ \int_0^r \left[n(x, f_1) + n(x, f_2) + \dots + n(x, f_m) \right] \frac{dx}{x},$$

where $n(r, f_1)$, $n(r, f_2), \dots, n(r, f_m)$ represent the number of zeros of $f_1(z), f_2(z), \dots, f_m(z)$, respectively in $|z| \leqslant r$ and $f_1(0) \neq 0$, $f_2(0) \neq 0, \dots, f_m(0) \neq 0$.

In this paper, we have considered the geometric means of the products of two entire functions only and have obtained some of their properties. The results can easily be extended to any finite number of entire functions. Some results of Jain and Chugh ([2, p.98], [3, p.22]) can be obtained as particular cases of the results obtained here.

2. We prove :

Theorem 1. If

$$\limsup_{r\to\infty} \frac{\log \log G(r, f_1 f_2)}{\log \log r} = A, \tag{2.1}$$

$$\limsup_{r\to\infty} \frac{\log \log g_\delta (r, f_1 f_2)}{\log \log r} = B \tag{2.2}$$

and

$$\limsup_{r\to\infty} \frac{\log \log g_\delta^* (r, f_1 f_2)}{\log \log r} = C, \tag{2.3}$$

then

$$A = B = C = \max (\rho_1^*, \rho_2^*).$$

Proof. Let $M(r, f_1)$ and $M(r, f_2)$ denote, respectively the maximum modulii of $f_1(z)$ and $f_2(z)$ for $|z| = r$, then in view of lemma on p. 311 [6], (1.2) gives

$$\log G(r, f_1 f_2) \leqslant \log \left[\frac{1}{2\pi} \int_0^{2\pi} |f_1(re^{i\theta}) f_2(re^{i\theta})|\, d\theta \right] \tag{2.4}$$
$$\leqslant \log [M(r, f_1)\, M(r, f_2)].$$

Again, let $f(z)$ be regular in $|z| \leqslant R$ and let $z = re^{i\theta}$, $\theta < r < R$, then Poisson-Jensen formula gives

$$\log |f(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) \log |f(Re^{i\varphi})|}{R^2 - 2Rr \cos (\theta - \varphi) + r^2} d\varphi - \sum_{\mu} \log \left| \frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)} \right|,$$

where a_μ are the zeros of $f(z)$ inside the circle $|z| \leqslant R$. Since each term in Σ is positive, for $f(z) = f_1(z) f_2(z)$, this yields

$$\log |f_1(z) f_2(z)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) \log |f_1(Re^{i\varphi}) f_2(Re^{i\varphi})|}{R^2 - 2R r \cos (\theta - \varphi) + r^2} d\varphi.$$

Choosing z in such a manner that

$$\begin{matrix} \log [M(r, f_1) |f_2(re^{i\theta})|] \\ \log [|f_1(re^{i\theta})| M(r, f_2)] \end{matrix} \leqslant \frac{R + r}{R - r} \log G(R, f_1 f_2),$$

according as $\rho_1^* \geqslant \rho_2^*$ or $\rho_1^* \leqslant \rho_2^*$. Taking $R = 2r$, this gives

$$\log G(2r, f_1 f_2) \geqslant \begin{matrix} \frac{1}{3} \log [M(r, f_1) |f_2(re^{i\theta})|) \\ \\ \frac{1}{3} \log [|f_1(re^{i\theta})| M(r, f_2)] \end{matrix} . \tag{2.5}$$

Taking logarithms on both sides of (2.4) and (2.5), proceeding to limits as $r \to \infty$ and combining the results thus obtained, we get

$$A = \max (\rho_1^*, \rho_2^*) .$$

Further, since $\log G(r, f_1 f_2)$ is an increasing function of r we have

$$\log g_\delta (r, f_1 f_2) = \frac{\delta + 1}{r^{\delta+1}} \int_0^r x^\delta \log G(x, f_1 f_2) dx \leqslant \log G(r, f_1 f_2),$$

which leads to $B \leqslant A$. Also,

$$\log g_\delta (2r, f_1 f_2) = \frac{\delta + 1}{(2r)^{\delta+1}} \int_0^{2r} x^\delta \log G(x, f_1 f_2) dx$$

$$> \frac{\delta + 1}{(2r)^{\delta+1}} \int_r^{2r} x^\delta \log G(x, f_1 f_2) dx$$

$$\geqslant \frac{2^{\delta+1} - 1}{2^{\delta+1}} \log G(r, f_1 f_2),$$

which leads to $B \geqslant A$. Thus, we obtain

$$A = B = \max(\rho_1^*, \rho_2^*) . \tag{2.6}$$

Again, we have

$$\log g_\delta^*(r, f_1 f_2) = \frac{\delta + 1}{(\log r)^{\delta+1}} \int_1^r x^{-1} (\log x)^\delta \log G(x, f_1 f_2)\, dx$$

$$\leqslant \log G(r, f_1 f_2).$$

This gives $C \leqslant A$. Also

$$\log g_\delta^*(r^2, f_1 f_2) > \frac{\delta + 1}{(\log r)^{\delta+1}} \int_r^{r^2} x^{-1} (\log x)^\delta \log G(x, f_1 f_2)\, dx$$

$$\geqslant (2^{\delta+1} - 1) \log G(r, f_1 f_2) .$$

This gives $C \geqslant A$. Hence

$$A = C = \max(\rho_1^*, \rho_2^*) . \tag{2.7}$$

(2.6) and (2.7) complete the proof of theorem 1.

Theorem 2. Let $n(r, f_1)$ and $n(r, f_2)$ denote the number of zeros of $f_1(z)$ and $f_2(z)$, respectively in $|z| \leqslant r$ and $f_1(0) \neq 0$, $f_2(0) \neq 0$; then

$$\lim_{r\to\infty} \sup \frac{\log [n(r, f_1) + n(r, f_2)]}{\log \log r} = A - 1 ,$$

where A is defined by (2.1).

This theorem can easily be proved with the help of (1.5) and (2.1).

Corollary 1. If $f_2(z) \equiv 1$, we get the following results due to Jain and Chugh ([2, p. 98], [3, p. 22]) as particular cases of the theorems 1 and 2 :

$$\lim_{r\to\infty} \sup \frac{\log \log G(r, f_1)}{\log \log r} = \rho_1^*$$

and

$$\lim_{r\to\infty} \sup \frac{\log n(r, f_1)}{\log \log r} = \rho_1^* - 1 .$$

Theorem 3. If

$$\lim_{r\to\infty} \inf \frac{n(r, f_1) + n(r, f_2)}{(\log r)^u \log \log r} > 1 , \tag{2.8}$$

then

$$\lim_{r\to\infty} \inf \frac{\log g_\delta^* (r, f_1 f_2)}{(\log r)^{u+1} \log\log r} \geqslant \frac{\delta+1}{(u+1)(\delta+u+2)}, \tag{2.9}$$

and if

$$\lim_{r\to\infty} \sup \frac{n(r, f_1) + n(r, f_2)}{(\log r)^u \log\log r} < 1, \tag{2.10}$$

then

$$\lim_{r\to\infty} \sup \frac{\log g_\delta^* (r, f_1 f_2)}{(\log r)^{u+1} \log\log r} \leqslant \frac{\delta+1}{(u+1)(\delta+u+2)}, \tag{2.11}$$

where $u \geqslant 0$.

Proof. From (2.8), we have for any $\varepsilon > 0$ and $r > r_0$,

$$n(r, f_1) + n(r, f_2) > (1-\varepsilon)(\log r)^u \log\log r.$$

Substituting this in (1.5), we get

$$\log G(r, f_1 f_2) > \frac{(1-\varepsilon)(\log r)^{u+1} \log\log r}{u+1} (1 + o(1)). \tag{2.12}$$

Substituting for $\log G(r, f_1 f_2)$ from (2.12) in (1.4), we get

$$\log g_\delta^* (r, f_1 f_2) > \frac{(1-\varepsilon)(\delta+1)(\log r)^{u+1} \log\log r}{(u+1)(\delta+u+2)} (1 + o(1)). \tag{2.13}$$

Dividing both sides of (2.13) by $(\log r)^{u+1} \log\log r$ and taking limits as $r \to \infty$, (2.9) follows.

Again, from (2.10), we have for any $\varepsilon > 0$ and $r > r_0$,

$$n(r, f_1) + n(r, f_2) < (1+\varepsilon)(\log r)^u \log\log r,$$

which together with (1.5) gives

$$\log G(r, f_1 f_2) < \frac{(1+\varepsilon)(\log r)^u \log\log r}{u+1} (1 + o(1)) + 0(1). \tag{2.14}$$

Substituting this in (1.4), we get

$$\log g_\delta^* (r, f_1 f_2) < \frac{(1+\varepsilon)(\delta+1)(\log r)^{u+1} \log\log r}{(u+1)(\delta+u+2)} (1 + o(1)) + 0(1),$$

from which (2.11) follows immediately.

This completes the proof of theorem 3.

Corollary 2. We find

$$\liminf_{r\to\infty} \frac{\log G(r, f_1 f_2)}{(\log r)^{u+1} \log\log r} \geqslant \frac{1}{u+1},$$

provided (2.8) holds, and

$$\limsup_{r\to\infty} \frac{\log G(r, f_1 f_2)}{(\log r)^{u+1} \log\log r} \leqslant \frac{1}{u+1},$$

provided (2.10) holds.

These results are immediate consequence of (2.12) and (2.14), respectively.

Theorem 4. For a class of entire functions for which

$$\liminf_{r\to\infty} \frac{\log\log g_\delta^* (r, f_1 f_2)}{\log\log r} = \infty,$$

and $\log\log G(r, f_1 f_2)$ is an increasing convex function of $\log r$, we have

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_3 g_\delta^* (r, f_1 f_2)}{\log\log r} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left[\dfrac{\log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)}\right]}{\log\log r},$$

where $\log_3 x = \log\log\log x$.

The proof of this theorem is based on the following lemma :

Lemma 1. Under hypothesis of the theorem, $(\log r)^{\delta+1} \log G(r, f_1 f_2)$ is an increasing convex function of $(\log r)^{\delta+1} \log g_\delta^* (r, f_1 f_2)$.

Proof. We have

$$\frac{d\,[(\log r)^{\delta+1} \log G(r, f_1 f_2)]}{d\,[(\log r)^{\delta+1} \log g_\delta^*(r, f_1 f_2)]} = \frac{\dfrac{d}{dr}\,[(\log r)^{\delta+1} \log G(r, f_1 f_2)]}{\dfrac{d}{dr}\,[(\log r)^{\delta+1} \log g_\delta^* (r, f_1 f_2)]}$$

$$= \frac{(\delta+1)(\log r)^{\delta} \log G(r, f_1 f_2) + r(\log r)^{\delta+1}\,[G^{(1)}(r, f_1 f_2)/G(r, f_1 f_2)]}{(\delta+1)(\log r)^{\delta} \log G(r, f_1 f_2)}$$

$$= 1 + \frac{r(\log r)\, G^{(1)}(r, f_1 f_2)}{(\delta+1)\, G(r, f_1 f_2) \log G(r, f_1 f_2)},$$

which increases with r, for large values of r, under the hypothesis that $\log\log G(r, f_1 f_2)$ is an increasing convex function of $\log r$.

Proof of theorem 4. We have

$$\log [(\log r)^{\delta+1} \log g_\delta^* (r, f_1 f_2)] = (\delta + 1) \int_1^r \frac{\log G(x, f_1 f_2)}{x (\log x) \log g_\delta^* (x, f_1 f_2)} dx ,$$

since numerator on the right-hand side is the differential coefficient of the denominator. From lemma 1, $\log G(r, f_1 f_2) / \log g_\delta^* (r, f_1 f_2)$ is an increasing function of r and hence,

$$\log [(\log r)^{\delta+1} \log g_\delta^* (r, f_1 f_2)] <$$

$$< \frac{(\delta + 1) \log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)} \int_{r_0}^r \frac{dx}{x \log x} + 0(1) , \quad r > r_0$$

$$= \frac{(\delta + 1) \log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)} (\log \log r) (1 - o(1)) + 0(1) .$$

Taking logarithms on both sides, dividing by $\log \log r$ and taking limits as $r \to \infty$, we get

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_3 g_\delta^* (r, f_1 f_2)}{\log \log r} \leqslant \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left[\frac{\log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)}\right]}{\log \log r} ,$$

since $\lim_{r\to\infty} \inf \frac{\log \log g_\delta^* (r, f_1 f_2)}{\log \log r} = \infty$.

Again, we have

$$\log [(\log r^2)^{\delta+1} \log g_\delta^* (r^2 , f_1 f_2)] > (\delta + 1) \int_r^{r^2} \frac{\log G(x, f_1 f_2)}{x (\log x) \log g_\delta^* (x, f_1 f_2)} dx$$

$$\geqslant (\delta + 1) \frac{\log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)} \log 2.$$

Consequently,

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log_3 g_\delta^* (r, f_1 f_2)}{\log \log r} \geqslant \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left[\frac{\log G(r, f_1 f_2)}{\log g_\delta^* (r, f_1 f_2)}\right]}{\log \log r} .$$

This proves theorem 4.

3. Let us set :

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n (r, f_1) + n (r, f_2)}{(\log r)^{\rho^*-1}} = \begin{matrix}\alpha^*\\ \beta^*\end{matrix} , \tag{3.1}$$

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log G(r, f_1 f_2)}{(\log r)^{\rho^*}} = \begin{matrix} a^* \\ b^* \end{matrix}, \tag{3.2}$$

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log g_\delta(r, f_1 f_2)}{(\log r)^{\rho^*}} = \begin{matrix} c^* \\ d^* \end{matrix}, \tag{3.3}$$

and

$$\lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log g_\delta^*(r, f_1 f_2)}{(\log r)^{\rho^*}} = \begin{matrix} p^* \\ q^* \end{matrix}, \tag{3.4}$$

where $\rho^* = \max(\rho_1^*, \rho_2^*)$. We prove here the following results :

Theorem 5. We find

$$(\rho^* - 1)\,\alpha^* \leqslant \rho^{*\rho^*} a^* - b^* \leqslant (\rho^* - 1)\,\rho^{*\rho^*-1}\,\alpha^*, \tag{i}$$

$$(\rho^* - 1)\,\beta^* \leqslant \rho^{*\rho^*} b^* - a^* \leqslant (\rho^* - 1)\,\rho^{*\rho^*-1}\,\beta^*, \tag{ii}$$

$$a^* = 2\,c^* - d^*, \tag{iii}$$

$$b^* = 2\,d^* - c^*, \tag{iv}$$

$$a^* \leqslant 2^{(\delta+\rho^*+1)/(\delta+1)}\,p^* - q^* \leqslant 2^{\rho^*/(\delta+1)}\,a^*, \tag{v}$$

and

$$b^* \leqslant 2^{(\delta+\rho^*+1)/(\delta+1)}\,q^* - p^* \leqslant 2^{\rho^*/(\delta+1)}\,b^*. \tag{vi}$$

Proof. We know that

$$[n(r^{\rho^*}, f_1) + n(r^{\rho^*}, f_2)]\log r \geqslant \frac{1}{\rho^*-1}\int_r^{r^{\rho^*}} [n(x, f_1) + n(x, f_2)]\,\frac{dx}{x}.$$

Adding $\frac{1}{\rho^*-1}\log G(r, f_1 f_2)$ on both sides, we get

$$\frac{1}{\rho^*-1}\log G(r, f_1 f_2) + [n(r^{\rho^*}, f_1) + n(r^{\rho^*}, f_2)]\log r \geqslant$$
$$\geqslant \frac{1}{\rho^*-1}\log G(r^{\rho^*}, f_1 f_2). \tag{3.5}$$

Similarly, we get

$$\frac{1}{\rho^*-1}\log G(r, f_1 f_2) + [n(r, f_1) + n(r, f_2)]\log r \leqslant \frac{1}{\rho^*-1}\log G(r^{\rho^*}, f_1 f_2). \tag{3.6}$$

Dividing (3.5) and (3.6) by $(\log r)^{\rho^*}$, taking limits as $r \to \infty$ and using (3.1) and (3.2), the results (i) and (ii) follow. Further, we have

$$\log G(2^{1/(\delta+1)} r, f_1 f_2) \geqslant \frac{\delta+1}{r^{\delta+1}} \int_r^{2^{1/(\delta+1)} r} x^\delta \log G(x, f_1 f_2)\, dx.$$

Adding $\log g_\delta(r, f_1 f_2)$ on both sides, this gives

$$\log g_\delta(r, f_1 f_2) + \log G(2^{1/(\delta+1)} r, f_1 f_2) \geqslant 2 \log g_\delta(2^{1/(\delta+1)} r, f_1 f_2). \quad (3.7)$$

In a similar manner, we get

$$\log g_\delta(r, f_1 f_2) + \log G(r, f_1 f_2) \leqslant 2 \log g_\delta(2^{1/(\delta+1)} r, f_1 f_2). \quad (3.8)$$

Dividing (3.7) and (3.8) by $(\log r)^{\rho^*}$, taking limits as $r \to \infty$ and using (3.2) and (3.3), the results (iii) and (iv) follow.

Also, we have

$$\log G(r^{2^{1/(\delta+1)}}, f_1 f_2) \geqslant \frac{\delta+1}{(\log r)^{\delta+1}} \int_r^{r^{2^{1/(\delta+1)}}} x^{-1} (\log x)^\delta \log G(x, f_1 f_2)\, dx,$$

or

$$\log g_\delta^*(r, f_1 f_2) + \log G(r^{2^{1/(\delta+1)}}, f_1 f_2) \geqslant 2 \log g_\delta^*(r^{2^{1/(\delta+1)}}, f_1 f_2). \quad (3.9)$$

Similarly, we get

$$\log g_\delta^*(r, f_1 f_2) + \log G(r, f_1 f_2) \leqslant 2 \log g_\delta^*(r^{2^{1/(\delta+1)}}, f_1 f_2). \quad (3.10)$$

The results (v) and (vi) follow from (3.9) and (3.10), on proceeding to limits as $r \to \infty$, in view of (3.2) and (3.4).

Theorem 6. We find

$$\frac{\beta^*(\delta+1)}{\rho^*(\delta+\rho^*+1)} \leqslant \liminf_{r\to\infty} \frac{\log g_\delta^*(r, f_1 f_2)}{(\log r)^{\rho^*}} \leqslant$$

$$\leqslant \limsup_{r\to\infty} \frac{\log g_\delta^*(r, f_1 f_2)}{(\log r)^{\rho^*}} \leqslant \frac{\alpha^*(\delta+1)}{\rho^*(\delta+\rho^*+1)},$$

where $\rho^* = \max(\rho_1^*, \rho_2^*)$.

Proof. From (3.1), we have for any $\varepsilon > 0$ and $r > r_0$,

$$(\beta^* - \varepsilon)(\log r)^{\rho^*-1} < n(r, f_1) + n(r, f_2) < (\alpha^* + \varepsilon)(\log r)^{\rho^*-1}. \quad (3.11)$$

Therefore, from (1.5), we get for $r > r_0$

$$\log G(r, f_1 f_2) > \frac{(\beta^* - \varepsilon)(\log r)^{\rho^*}}{\rho^*}(1 - o(1)) \quad (3.12)$$

and

$$\log G(r, f_1 f_2) < \frac{(\alpha^* + \varepsilon)(\log r)^{\rho^*}}{\rho^*}(1 - o(1)) + 0(1). \qquad (3.13)$$

Further, from (1.4), we have

$$\log g_\delta^*(r, f_1 f_2) = \frac{\delta + 1}{(\log r)^{\delta+1}} \int_{r_0}^{r} x^{-1} (\log x)^\delta \log G(x, f_1 f_2)\, dx + 0(1). \qquad (3.14)$$

Substituting for $\log G(r, f_1 f_2)$ from (3.12) and (3.13) in (3.14), we get for $r > r_0$,

$$\log g_\delta^*(r, f_1 f_2) > \frac{(\beta^* - \varepsilon)(\delta + 1)(\log r)^{\rho^*}}{\rho^*(\delta + \rho^* + 1)}(1 - o(1)) \qquad (3.15)$$

and

$$\log g_\delta^*(r, f_1 f_2) < \frac{(\alpha^* + \varepsilon)(\delta + 1)(\log r)^{\rho^*}}{\rho^*(\delta + \rho^* + 1)}(1 - o(1)) + 0(1). \qquad (3.16)$$

Dividing (3.15) and (3.16) by $(\log r)^{\rho^*}$ and taking limits as $r \to \infty$, the result follows.

Corollary 3. We have

$$\frac{\beta^*}{\rho^*} \leqslant \lim_{r \to \infty} \inf \frac{\log G(r, f_1 f_2)}{(\log r)^{\rho^*}} \leqslant \lim_{r \to \infty} \sup \frac{\log G(r, f_1 f_2)}{(\log r)^{\rho^*}} \leqslant \frac{\alpha^*}{\rho^*}.$$

This easily follows from (3.12) and (3.13).

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ÖZET

Bu çalışmada iki tam fonksiyon çarpımının geometrik ortalaması gözönüne alınmakta ve bunların bazı özelikleri elde edilmektedir. Elde edilen bu sonuçlar, herhangi sonlu sayıda tam fonksiyona kolayca teşmil edilebilir. Jain ve Chugh'un bazı sonuçları, yukarıdaki sonuçların özel halleri olarak elde edilebilir.