ON WAVE SOLUTIONS OF COUPLED ELECTROMAGNETIC AND ZERO-REST-MASS SCALAR FIELDS IN A GENERALIZED PERES SPACE-TIME

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In this paper it has been proposed to find the plane wave-like solutions of the general relativistic field equations for regions of space containing electromagnetic fields and a scalar zero-rest-mass-meson field.

Recently Lal and Pandey [1][1)] have investigated the wave solutions of the field equations of general relativity containing electromagnetic fields in a generalized Peres space-time.

1. Introduction. Many authors have tried to find the solutions of relativistic field equations in the presence of scalar meson field in the hope that they may be useful in future. This problem has been mentioned briefly by Brahmachary [2]. Scalar fields associated with a meson of zero-rest-mass have also been considered by Bergmann and Leipnik [3] and Buchdahl [4]. Janis et al [5] have presented a solution of the coupled zero-rest-mass scalar field and gravitational field in the spherically symmetric case. Penney [6] and Gautreau [7] have solved the same field equations in axially symmetric static case and following Erez and Rozen [8] Penney [6] reduced the axially symmetric solutions to spherical symmetry. Lal and Singh [9] have found cylindrical wave solutions of the coupled zero-rest-mass scalar field and gravitational field. Recently Singh [10] and Patel [11] have obtained plane symmetric solutions of the field equations corresponding to zero-rest-mass scalar fields.

Lai and Pandey [1] considered a space time whose metric is given by

$$ds^2 = - A\, dx^2 - B\, dy^2 - (1 - E)\, dz^2 - 2E\, dz\, dt + (1 + E)\, dt^2 \qquad (1.1)$$

as a generalization of Peres metric by assuming that A and B are functions of z, t and E is any function of x, y, z, t and obtained the wave like solutions of the field equations of general relativity. In the present paper, taking the line element (1.1) we find the wave like solutions of the field equations of general relativity for regions of space containing the electromagnetic field and a scalar zero-rest-mass-meson field. These equations according to G.Stephenson [12] are

1) Numbers in brackets refer to the references at the end of the paper.

$$G_{ij} \equiv K_{ij} - \frac{1}{2} g_{ij} K = - 8\pi (E_{ij} + M_{ij}) , \qquad (1.2)$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0 , \qquad (1.3)$$

$$F^{ij}_{;j} = 0 \quad (F^{ij} \sqrt{-g})_{,j} = 0 , \qquad (1.4)$$

$$g^{ij} V_{;ij} = 0 , \qquad (1.5)$$

where K_{ij} is the Ricci tensor, F_{ij} is the electromagnetic field tensor, V is a zero-rest-mass-scalar field. E_{ij} and M_{ij} are the electromagnetic energy-momentum tensor and energy momentum tensor of the zero-rest-mass-scalar-meson field respectively defined by

$$E_{ij} = - F_{is} F^s_j + \frac{1}{4} g_{ij} F_{sp} F^{sp} , \qquad (1.6)$$

$$M_{ij} = V_{,i} V_{,j} - \frac{1}{2} g_{ij} (V_{,l} V_{,m} g^{lm}) . \qquad (1.7)$$

A comma and a semi-colon followed by an index denote partial and covariant differentiations respectively.

2. Calculations of G_{ij} and M_{ij}. The non-vanishing components of electromagnetic energy tensor E_{ij} as obtained in [1] are

$$E_{33} = - E_{34} = E_{44} = p^2/B + \sigma^2/A . \qquad (2.1)$$

The non-vanishing components of G_{ij} for the metric (1.1) are given by

$$\left.\begin{aligned}
G_{11} &= -(A/2) [(B_{33} - B_{44}) / B - (B_3^2 - B_4^2) / 2B^2 + R + \\
&\quad + (E_3 + E_4)(B_3 + B_4) / B + EH / B - E(B_3 + B_4)^2 / 2B^2] . \\
G_{13} &= - G_{14} = (1/2)(E_{13} + E_{14}) - (1/4) E_1 P , \\
G_{22} &= -(B/2) [(A_{33} - A_{44}) / A - (A_3^2 - A_4^2) / 2A^2 + R + \\
&\quad + (E_3 + E_4)(A_3 + A_4) / A + EN / A - E(A_3 + A_4)^2 / 2A^2] , \\
G_{23} &= - G_{24} = (1/2)(E_{23} + E_{24}) + (1/4) E_2 P , \\
G_{34} &= A_{34} / 2A + B_{34} / 2B + U - (A_3 A_4 / 4A^2 + B_3 B_4 / 4B^2) - \\
&\quad - E(E_3 + E_4) Q/4 - E_3 T + E_4 S - E(A_{33} - A_{44}) / 2A - \\
&\quad - E(B_{33} - B_{44})/2B + E(A_3^2 - A_4^2)/4A^2 + E(B_3^2 - B_4^2) / 4B^2 - \\
&\quad - E(A_3 B_3 - A_4 B_4)/4AB - E^2 (A_3 + A_4)(B_3 + B_4) / 4AB - (E^2/2) W ,
\end{aligned}\right\} \qquad (2.2)$$

$$\left.\begin{aligned}G_{33} &= (A_{44}/2A + B_{44}/2B) + E(A_{33} - A_{44})/2A + E(B_{33} - B_{44})/2B - \\ &- U - (A_4^2/4A^2 + B_4^2/4B^2) - E(A_3^2 - A_4^2)/4A^2 - E(B_3^2 - B_4^2)/4B^2 - \\ &- (E_3 + E_4)(2 - E)Q/4 + E_3 S + (E_4 + 2E_3)T - (1 - E)(A_3B_3 - A_4B_4)/4AB - \\ &- E(1 - E)(A_3 + A_4)(B_3 + B_4)/4AB - E(1 - E)W/2. \\ G_{44} &= (A_{33}/2A + B_{33}/2B) + E(A_{33} - A_{44})/2A + E(B_{33} - B_{44})/2B - U - \\ &- (A_3^2/4A^2 + B_3^2/4B^2) - E(A_3^2 - A_4^2)/4A^2 - E(B_3^2 - B_4^2)/4B^2 + \\ &+ (E_3 + E_4)(2 + E)Q/4 - (E_3 + 2E_4)S - E_4 T + (1 + E)(A_3B_3 - A_4B_4)/4AB + \\ &+ E(1 + E)(A_3 + A_4)(B_3 + B_4)/4AB + E(1 + E)W/2,\end{aligned}\right|$$

where

$$P = (A_3 + A_4)/A - (B_3 + B_4)/B, \qquad Q = (A_3 + A_4)/A + (B_3 + B_4)/B,$$

$$R = E_{33} + 2E_{34} + E_{44}, \qquad S = A_3/4A + B_3/4B, \qquad T = A_4/4A + B_4/4B,$$

$$U = (1/2)(E_{11}/A + E_{22}/B), \quad N = (A_{33} + 2A_{34} + A_{44}), \quad H = (B_{33} + 2B_{34} + B_{44})$$

and

$$W = N/A + H/B - (A_3 + A_4)^2/2A^2 - (B_3 + B_4)^2/2B^2.$$

The non-vanishing components of M_{ij} are

$$\left.\begin{aligned}M_{11} &= V_{,1}^2/2 - (A/2B)V_{,2}^2 - AE\,V_{,3}V_{,4} - A(V_{,3}^2 - V_{,4}^2)/2 - \\ &- AE(V_{,3}^2 + V_{,4}^2)/2, \\ M_{12} &= V_{,1}V_{,2}, \qquad M_{13} = V_{,1}V_{,3}, \qquad M_{14} = V_{,1}V_{,4}, \\ M_{22} &= V_{,2}^2/2 - (B/2A)V_{,1}^2 - BE\,V_{,3}V_{,4} - B(V_{,3}^2 - V_{,4}^2)/2 - \\ &- BE(V_{,3}^2 + V_{,4}^2)/2, \\ M_{23} &= V_{,2}V_{,3}, \qquad M_{24} = V_{,2}V_{,4}, \\ M_{33} &= (1 - E)(V_{,1}^2/A + V_{,2}^2/B)/2 + (1 + E^2)(V_{,3}^2 + V_{,4}^2)/2 - \\ &- E(1 - E)V_{,3}V_{,4} - EV_{,4}^2, \\ M_{34} &= -E(V_{,1}^2/A + V_{,2}^2/B)/2 + (1 - E^2)V_{,3}V_{,4} - E(V_{,3}^2 - V_{,4}^2)/2 - \\ &- E(V_{,3}^2 + V_{,4}^2)/2, \\ M_{44} &= (1 + E)(V_{,1}^2/A + V_{,2}^2/B)/2 + (1 + E^2)(V_{,3}^2 + V_{,4}^2)/2 + \\ &+ E(1 + E)V_{,3}V_{,4} + EV_{,3}^2.\end{aligned}\right\} \quad (2.3)$$

3. Solutions of equations (1.2) and (1.5). The field equations (1.3) and (1.4) have already been solved by Lai and Pandey [1]. Using (1.1) in the field equation (1.5), we get

$$\begin{aligned}&[\{V_{,11} + V_{,3}((1 + E)A_3/2 + EA_4/2) + V_{,4}(EA_3/2 - (1 - E)A_4/2)\}/A + \\ &+ \{V_{,22} + V_{,3}((1 + E)B_3/2 + EB_4/2) + V_{,4}(EB_3/2 - (1 - E)B_4/2)\}/B + \qquad (3.1) \\ &+ (V_{,33} - V_{,44}) + (E_3 + E_4)(V_{,3} + V_{,4}) + E(V_{,33} + 2V_{,34} + V_{,44})] = 0.\end{aligned}$$

Using (2.2), (2.1) and (2.3) in (1.2) we find that for $i = 1$ and $j = 2$, equation (1.2) gives the condition

$$V_{,1} V_{,2} = 0 . \tag{3.2}$$

There are three possibilities namely (3.2) (*i*) $V_{,1} = 0$, $V_{,2} \neq 0$, (3.2) (*ii*) $V_{,1} \neq 0$ and $V_{,2} = 0$ and (3.2) (*iii*) $V_{,1} = 0$, $V_{,2} = 0$, which show that V is function of Z only.

In [1] we see that for (1.4) $F^{1j}_{;j} = 0$ and $F^{2j}_{;j} = 0$ give the condition

$$(A_3 + A_4)/A = (B_3 + B_4)/B \tag{3.3}$$

and $F^{3j}_{;j} = 0$ and $F^{4j}_{;j} = 0$ give the condition

$$A\rho_2 - B\sigma_1 = 0 , \tag{3.4}$$

where the suffixes 1,2,3,4 denote partial derivatives of A, B, E, ρ and σ with respect to x, y, z and t.

Case I. Let quantities on either side of equation (3.3) vanish separately. This gives on integration

$$\text{a)}\quad A = A(Z) \qquad \text{and} \qquad \text{b)}\quad B = B(Z) . \tag{3.5}$$

Using (2.1), (2.2), (3.2) (*iii*) and (3.3) in equation (1.2) we find

$$E_{33} + 2E_{34} + E_{44} = 0 , \tag{3.6}$$

$$E_{13} + E_{14} = 0 , \tag{3.7}$$

$$E_{23} + E_{24} = 0 , \tag{3.8}$$

$$- (\overline{\overline{A}} / A + \overline{\overline{B}} / B)/2 + U + (\overline{A}^2 / A + \overline{B}^2 / B)/4 + (E_3 + E_4)(\overline{A} / A + \overline{B} / B)/4 =$$

$$= 8\pi \, [(\rho^2/B + \sigma^2/A) + \overline{V}^2] . \tag{3.9}$$

Integrating (3.6), (3.7) and (3.8) we find that the form of E satisfying (3.6), (3.7) and (3.8) is given by

$$\text{a)}\quad E = (x + y + z) f_1(z - t) + f_2(x, y) f_3(z - t) \tag{3.10}$$

or

$$\text{b)}\quad E = F(x, y, z - t) + z f(z - t) .$$

Thus the plane wave like solutions of the field equations (1.2) are composed of g_{ij} given by (1.1) under the condition (3.4), (3.9) and (3.10).

Using (3.2) (*iii*) in (3.1), it is seen that under condition (3.5) it is identically satisfied.

Case II. Let $(A_3 + A_4)/A = (B_3 + B_4)/B \neq 0$. Differentiating it partially with respect to z and t, we get

$$\text{a)} \quad (A_{33} + A_{34})/A - (B_{33} + B_{34})/B = (A_3 + A_4)(A_3/A - B_3/B)/A\,, \quad (3.11)$$

$$\text{b)} \quad (A_{34} + A_{44})/A - (B_{34} + B_{44})/B = (A_3 + A_4)(A_4/A - B_4/B)/A\,.$$

Using (3.3) and (3.11) in $G_{ii} = -8\pi(E_{ii} + M_{ii})$ $(i = 1,2)$ we have

$$(A_3 - A_4)/A - (B_3 - B_4)/B = 0\,. \quad (3.12)$$

From (3.3) and (3.12) we have $A_3/A = B_3/B$ and $A_4/A = B_4/B$ on integration which yields $A/B =$ constant and by certain transformation it can be reduced to

$$A = B\,. \quad (3.13)$$

Thus using (2.1), (2.2), (3.13) and (3.2) (*iii*) in (1.2) we get

$$\left.\begin{aligned}
&\text{a)} \quad A_{33} - A_{44} + EN + (A_3 + A_4)(E_3 + E_4) + AR - (A_3^2 - A_4^2)/2A - \\
&\qquad - E(A_3 + A_4)^2/2A = 0\,, \\
&\text{b)} \quad E_{13} + E_{14} = 0\,, \\
&\text{c)} \quad E_{23} + E_{24} = 0\,, \\
&\text{d)} \quad -\mu + 2A_{44} - A_4^2/A - (A_3^2 - A_4^2)/2A - 2(E_3 + E_4)(A_3 + A_4) + \\
&\qquad + E_3 A_3 + E_4 A_4 + 2E_3 A_4 + E(A_3 + A_4)^2/2A - 2EN = \\
&\qquad = -16\pi[(\rho^2 + \sigma^2) + A\overline{V}^2]\,, \\
&\text{e)} \quad \mu + 2A_{34} - E_3 A_4 + E_4 A_3 - A_3 A_4/A = 16\pi[(\rho^2 + \sigma^2) + A\overline{V}^2]\,, \\
&\text{f)} \quad \mu + 2A_{33} - A_3^2/A + (A_3^2 - A_4^2)/2A + 2(E_3 + E_4)(A_3 + A_4) - \\
&\qquad - E_3 A_3 - E_4 A_4 - 2E_4 A_3 - E(A_3 + A_4)^2/2A + 2EN = \\
&\qquad = -16\pi[(\rho^2 + \sigma^2) + A\overline{V}^2]\,,
\end{aligned}\right\} \quad (3.14)$$

where

$$\begin{aligned}\mu = E_{11} + E_{22} - E(E_3 + E_4)(A_3 + A_4) + E^2(A_3 + A_4)^2/2A - \\ - 2E(A_{33} - A_{44}) - 2E^2 N\,.\end{aligned}$$

Solving (3.14) d), e) and f) we find

$$2N - (1/A)(A_3 + A_4)^2 = 0\,. \quad (3.15)$$

In order to find a solution of (1.2) in this case it is necessary to solve (3.15) and then (3.14) a), b) and c). Unfortunately at this stage, it is not possible to get a solution directly by integrating (3.15).

Hence it should be remarked that if we put $A = e^{mz} e^{nt}$ in (3.15), we get $m + n = 0$ and we can arrive at a solution satisfying $A = B = ke^{m(z-t)}$, where k and m are constants. However this is a special case of (3.5) and belongs to Case I.

Lastly we add that in case we take (3.10) b) as the form of E, and take the function f, to be zero; E will be then Peres function and we shall get the plane wave like solutions of the general relativistic field equations for regions of space-time containing electromagnetic fields and scalar zero-rest-mass-meson field.

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ÖZET

Bu çalışmada, elektromanyetik alanları ve bir kütlesiz mezon alanını (scalar zero-rest-mass-meson field) içeren uzay bölgelerine ait genel rölativistik alan denklemlerinin düzlem dalgasal çözümlerinin bulunması problemi ele alınmaktadır.