THERMAL INSTABILITY OF COMPRESSIBLE FLUIDS WITH HALL CURRENTS THROUGH POROUS MEDIUM

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In this paper it has been found the effect of compressibility to be stabilizing and the Hall currents having a destabilizing effect. The case of overstability has been also considered and the sufficient conditions for the nonexistence of overstability have been investigated.

The thermal instability of compressible fluids in the presence of Hall currents through porous medium is considered. The effect of compressibility is found to be stabilizing and the Hall currents have a destabilizing effect. The system is stable for $\frac{c_p}{g}\beta < 1$; c_p, g and β denoting the specific heat at constant pressure, the acceleration due to gravity and the uniform temperature gradient respectively. In contrast to the nonoscillatory modes in the absence of magnetic field, the presence of magnetic field introduces oscillatory modes for $\frac{c_p}{g}\beta > 1$. The case of overstability is also considered and the sufficient conditions for the nonexistence of overstability are investigated.

1. Introduction. The problem of convection in a horizontal layer of fluid heated from below (referred to as thermal instability problem), under varying assumptions of hydrodynamics and hydromagnetics, has been discussed in detail by Chandrasekhar (1961). The effect of Hall currents on the thermal instability of a horizontal layer of conducting fluid has been studied by Gupta (1967).

Lapwood (1948) has investigated the stability of convective flow in hydrodynamics in a porous medium using Rayleigh's procedure and has shown that the criterion for convective flow is $R_c = 4\pi^2$, where R_c is the critical Rayleigh number. The problem to study the breakdown of stability of a layer of fluid subject to a vertical temperature gradient in a porous medium and to study the possibility of convective flow has been of considerable interest in recent years particularly among geophysical fluid dynamicists.

Just as in hydrodynamics, when a conducting fluid permeates a porous material in the presence of a magnetic field the actual path of an individual particle of fluid can not be followed analytically. The gross effect, as the fluid slowly percolates through the pores of the rock, must be represented by a macroscopic law applying to masses of fluid which is the usual Darcy's law. The usual viscous term in the equations of fluid motion will be replaced by the resistance term $\left(\frac{\mu}{k_1}\right)\vec{q}$, where μ is the viscosity of the fluid, k_1 the permeability of the medium (which has the dimension of length squared), and $\vec{q}$ the velocity of the fluid, calculated from Darcy's law.

In all the above studies, the Boussinesq approximation has been used which means that density variations are disregarded in all the terms in the equations of motion except the one in the external force. The equations governing the system become quite complicated when the fluids are compressible. To simplify them, Boussinesq tried to justify the approximation for compressible fluids when the density variations arise principally from thermal effects.

Spiegel and Veronis (1960) have simplified the set of equations governing the flow of compressible fluids under the following assumptions :

(*i*) The fluctuations in density, pressure and temperature, introduced due to motion, do not exceed their total static variations and

(*ii*) the depth of fluid layer is much less than the scale height as defined by them.

Under the above approximations, Spiegel and Veronis (1960) have found the flow equations to be the same as for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic.

The Hall currents are likely to be important and should not be ignored in cases like earth's molten core. The compressibility effects are more realistic in such regions. The present paper attempts to study the thermal instability of compressible fluids in the presence of Hall currents through porous medium.

2. Formulation of the Problem and Perturbation Equations. Consider an infinite horizontal finitely conducting and compressible fluid of depth d flowing through a porous medium in which a uniform temparature gradient $\beta\left(=\left|\frac{dz}{dT}\right|\right)$ is maintained. Consider the cartesian coordinates (x,y,z) and the origin to be on the lower boundary $z = 0$ with the axis of z perpendicular to it along the vertical. The uniform vertical magnetic field $\vec{H}(0, 0, H)$ and gravity force $\vec{g}(0, 0, -g)$ pervade the system. In addition to the resistance term as derived from Darcy's law, the viscous term is also considered in the equations of motion.

Spiegel and Veronis (1960) expressed any space variable, say X, in the form

$$X = X_m + X_0(z) + X'(x, y, z, t) , \tag{1}$$

where X_m stands for the constant space distribution of X, X_0 is the variation in X in the absence of motion and $X'(x, y, z, t)$ stands for the fluctuations in X due to the motion of the fluid.

The initial state is one in which the velocity, temperature, pressure and density at any point in the fluid are respectively given by

$$\vec{q} = 0,\ T = T(z),\ p = p(z)\, ,\ \rho = \rho(z)\ , \tag{2}$$

where according to Spiegel and Veronis (1960)

$$\begin{aligned} T(z) &= -\beta z + T_0\ , \\ p(z) &= p_m - g \int_0^z (\rho_m + \rho_0)\, dz\ , \\ \rho(z) &= \rho_m [1 - \alpha_m (T - T_m) + K_m (p - p_m)], \\ \alpha_m &= -\left(\frac{1}{\rho}\,\frac{\partial \rho}{\partial T}\right)_m , \\ K_m &= \left(\frac{1}{\rho}\,\frac{\partial \rho}{\partial p}\right)_m . \end{aligned} \tag{3}$$

Let μ_e, μ, $\nu\left(= \dfrac{\mu}{\rho_m}\right)$, K', $K\left(= \dfrac{K}{\rho_m c_p}\right)$ and $\dfrac{g}{c_p}$ denote the magnetic permeability, the viscosity, the kinematic viscosity, the thermal conductivity, the thermal diffusivity and the adiabatic gradient respectively. Then the flow of a conducting fluid $\vec{q}$ and the temperature distribution T through a porous medium are governed by the following system of equations :

$$\rho \frac{d\vec{q}}{dt} = -\nabla p + \rho \vec{g} + \frac{\mu_e}{4\pi} (\nabla \times \vec{H}) \times \vec{H} - \frac{\rho\nu}{k_1} \vec{q} + \rho\nu \nabla^2 \vec{q}\, , \tag{4}$$

$$\frac{d\rho}{dt} + \rho\, (\nabla \cdot \vec{q}) = 0\, , \tag{5}$$

$$\frac{dT}{dt} = \left(\beta - \frac{g}{c_p}\right) w + K \nabla^2 T . \tag{6}$$

The Maxwell's equations and the modified Ohm's law taking Hall currents into account give

$$\frac{\partial \vec{\mathbf{H}}}{\partial t} = \nabla \times (\vec{q} \times \vec{H}) + \eta \nabla^2 \vec{H} - \frac{1}{4\pi Ne} \nabla \times [(\nabla \times \vec{H}) \times \vec{H}], \tag{7}$$

and

$$\nabla \cdot \vec{\mathbf{H}} = 0, \tag{8}$$

μ, N and e denote the resistivity, the electron number density and charge of an electron respectively.

Let θ, $\delta\rho$, δp, $\vec{q}(u, v, w)$ and $\vec{h}(h_x, h_y, h_z)$ denote respectively the perturbations in temperature T, density ρ, pressure p, velocity and magnetic field $\vec{H}$. Then the linearized perturbation forms of Eqs. (1)-(5) are

$$\rho \frac{\partial \vec{q}}{\partial t} = -\nabla \cdot \delta p - \vec{g}\, \alpha' \rho \theta + \frac{\mu_e}{4\pi} (\nabla \times \vec{h}) \times \vec{H} - \frac{\rho\nu}{k_1} \vec{q} + \rho\nu\nabla^2 \vec{q}, \tag{9}$$

$$\nabla \cdot \vec{q} = 0, \tag{10}$$

$$\frac{\partial \vec{h}}{\partial t} = \nabla \times (\vec{q} \times \vec{H}) + \eta\nabla^2 \vec{h} - \frac{1}{4\pi Ne} \nabla \times [(\nabla \times \vec{h}) \times \vec{H}], \tag{11}$$

$$\nabla \cdot \vec{h} = 0, \tag{12}$$

$$\frac{\partial \theta}{\partial t} = \left(\beta - \frac{g}{c_p}\right) w + K \nabla^2 \theta, \tag{13}$$

α_m ($= \alpha'$, say) being the coefficient of thermal expansion. In writing Eq. (7), use has been made of the Boussinesq equation of state

$$\delta\rho = -\alpha' \rho_m \theta. \tag{14}$$

Equations (7)-(11) give

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2 + \frac{\nu}{k_1}\right) \nabla^2 w + g k^2 \alpha' \theta - \frac{\mu_e H}{4\pi\rho} \nabla^2 \frac{\partial h_z}{\partial z} = 0, \tag{15}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2 + \frac{\nu}{k_1}\right) \zeta = \frac{\mu_e H}{4\pi\rho} \frac{\partial \xi}{\partial z}, \tag{16}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) \xi = H \frac{\partial \xi}{\partial z} + \frac{H}{4\pi Ne} \nabla^2 \frac{\partial h_z}{\partial z}, \tag{17}$$

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right) h_z = H \frac{\partial w}{\partial z} - \frac{H}{4\pi Ne} \frac{\partial \xi}{\partial z}, \tag{18}$$

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right) \theta = \left(\beta - \frac{g}{c_p}\right) w, \tag{19}$$

where $\xi = \frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}$ and $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ denote the z-components of current density and vorticity respectively.

3. The Dispersion Relation. Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, h_z, \zeta, \xi] = [\,W(z), \Theta(z), K(z), Z(z), X(z)] \exp(ik_x x + ik_y y + nt), \tag{20}$$

where k_x, k_y $(k^2 = k_x^2 + k_y^2)$ are the horizontal wave numbers of the harmonic disturbance and n is, in general, a complex constant.

Putting $a = kd$, $\sigma = \frac{nd^2}{\nu}$, $p_1 = \frac{\nu}{K}$, $p_2 = \frac{\nu}{\eta}$, $x' = \frac{x}{d}$, $y' = \frac{y}{d}$, $z' = \frac{z}{d}$, $G = \frac{c_p}{g}\beta$ and $D = \frac{d}{dz'}$, *Eqs.* (15)-(19) in the dimensionless form, with the help of expression (20), become

$$(D^2 - a^2)\left(D^2 - a^2 - \sigma - \frac{d^2}{k_1}\right) w - \frac{g\alpha' d^2}{\nu} a^2 \Theta + \frac{\mu_e Hd}{4\pi \rho_m \nu}(D^2 - a^2) DK = 0, \tag{21}$$

$$\left(D^2 - a^2 - \sigma - \frac{d^2}{k_1}\right) Z = -\left(\frac{\mu_e Hd}{4\pi \rho_m \nu}\right) DX, \tag{22}$$

$$(D^2 - a^2 - p_2\sigma) X = -\left(\frac{Hd}{\eta}\right) DZ - \left(\frac{H}{4\pi Ne\,\eta d}\right)(D^2 - a^2) DK, \tag{23}$$

$$(D^2 - a^2 - p_2\sigma) K = -\left(\frac{Hd}{\eta}\right) DW + \left(\frac{Hd}{4\pi Ne\,\eta}\right) DX, \tag{24}$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\frac{d^2}{K}\left(\beta - \frac{g}{c_p}\right) W. \tag{25}$$

Here both the boundaries are considered to be free and the medium adjoining the fluid is nonconducting. The boundary conditions appropriate for the problem to the first order are [Chandrasekhar (1961), Lapwood (1948)] :

$$\left.\begin{array}{l} W = \Theta = D^2 W = DZ = 0 \\ X = 0 \text{ and } h_x, h_y, h_z \text{ are continuous} \end{array}\right\} \text{at } z' = 0 \text{ and } 1. \tag{26}$$

In the absence of any surface current, tangential components of magnetic field are continuous. Hence the boundary conditions in addition to (26) are

$$DK = 0 \quad \text{on the boundaries.} \tag{27}$$

Eliminating Θ, Z, K and X between Eqs. (21)-(25), we obtain

$$\left[(D^2-a^2)\left(D^2-a^2-\sigma-\frac{d^2}{k_1}\right)(D^2-a^2-p_1\sigma)+R\left(\frac{G-1}{G}\right)\right]\times$$

$$\times\left[(D^2-a^2-p_2\sigma)\left\{\left(D^2-a^2-\sigma-\frac{d^2}{k_1}\right)(D^2-a^2-p_2\sigma)-Q\,D^2\right\}+\right.$$

$$\left.+M_1{}^2(D^2-a^2)\left(D^2-a^2-\sigma-\frac{d^2}{k_1}\right)-D^2\right]W-Q\,(D^2-a^2)\,(D^2-a^2-p_1\sigma)\times$$

$$\times\left\{\left(D^2-a^2-\sigma-\frac{d^2}{k_1}\right)(D^2-a^2-p_2\sigma)-QD^2\right\}D^2\,W=0\,, \tag{28}$$

where $Q=\dfrac{\mu_e H^2 d^2}{4\pi\rho_m \nu\eta}$ is the Chandrasekhar number, $R=\dfrac{g\alpha\beta\,d^4}{\nu K}$ is the Rayleigh number, $M_1=\dfrac{H}{4\pi\,Ne\,\eta}$ is a nondimensional number accounting for Hall currents and $G=\dfrac{c_p}{g}\beta$.

Dropping the dashes for convenience and using the boundary conditions (26), it can be shown with the help of Eqs. (21)-(25) that all the even order derivatives of W vanish at the boundaries. Hence the proper solution of Eq. (28) characterizing the lowest mode is

$$W=W_0\sin\pi z\,, \tag{29}$$

where W_0 is a constant.

Substituting (29) in Eq. (28) and letting $a^2=\pi^2 x$, $Q_1=\dfrac{Q}{\pi^2}$, $R_1=\dfrac{R}{\pi^4}$, $M_1{}^2=M$ and $P=\dfrac{d^2}{\pi^2 k_1}$, we obtain the dispersion relation

$$R_1 x=\left(\frac{G}{G-1}\right)\left[(1+x)\left(1+x+\frac{\sigma}{\pi^2}+P\right)\left(1+x+p_1\frac{\sigma}{\pi^2}\right)+\right. \tag{30}$$

$$\left.+\frac{Q\,(1+x)\left(1+x+p_1\frac{\sigma}{\pi^2}\right)\left\{\left(1+x+\frac{\sigma}{\pi^2}+P\right)\left(1+x+p_2\frac{\sigma}{\pi^2}\right)+Q_1\right\}}{\left(1+x+p_2\frac{\sigma}{\pi^2}\right)\left\{\left(1+x+\frac{\sigma}{\pi^2}+P\right)\left(1+x+p_2\frac{\sigma}{\pi^2}\right)+Q_1\right\}+M_1(1+x)\left(1+x+\frac{\sigma}{\pi^2}+P\right)}\right].$$

4. Stability of the System and Oscillatory Modes. Multiplying Eq. (21) by W^*, the complex conjugate of W, and using Eqs. (22)-(25) together with the boundary conditions (26) and (27), we obtain

$$I_1 + \left(\sigma + \frac{d^2}{k_1}\right) I_2 + \frac{\mu_e \eta}{4\pi \rho_m \nu} (I_5 + p_2 \sigma^* I_6) + \frac{\mu_e \eta}{4\pi \rho_m \nu} d^2 (I_7 + p_2 \sigma I_8) +$$

$$+ d^2 \left[I_9 + \left(\sigma^* + \frac{d^2}{k_1}\right) I_{10} \right] = \frac{c_p \alpha' K a^2}{\nu (G-1)} (I_3 + p_1 \sigma^* I_4), \quad (31)$$

where

$$\left.\begin{aligned}
I_1 &= \int_0^1 (|DW|^2 + 2a^2 |DW|^2 + a^4 |W|^2)\, dz, \\
I_2 &= \int_0^1 (|DW|^2 + a^2 |W|^2)\, dz, \\
I_3 &= \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, dz, \\
I_4 &= \int_0^1 |\Theta|^2\, dz, \\
I_5 &= \int_0^1 (|D^2 K|^2 + 2a^2 |DK|^2 + a^4 |K|^2)\, dz, \\
I_6 &= \int_0^1 (|DK|^2 + a^2 |K|^2)\, dz, \\
I_7 &= \int_0^1 (|DX|^2 + a^2 |X|^2)\, dz, \\
I_8 &= \int_0^1 |X|^2\, dz, \\
I_9 &= \int_0^1 (|DZ|^2 + a^2 |Z|^2)\, dz, \\
I_{10} &= \int_0^1 |Z|^2\, dz,
\end{aligned}\right\} \quad (32)$$

which are all positive definite.

Putting $\sigma = \sigma_r + i\sigma_i$ and equating real and imaginary parts of Eq. (31) we obtain

$$\left[I_1 + \frac{d^2}{k_1} I_2 + \frac{c_p \alpha' K a^2}{\nu (G-1)} I_3 + \frac{\mu_e \eta}{4\pi \rho_m \nu} (I_5 + d^2 I_7) + d^2 \left(I_9 + \frac{d^2}{k_1} I_{10}\right)\right] +$$

$$+ \sigma_r \left[I_2 + \frac{c_p \alpha' K a^2}{\nu (G-1)} p_1 I_4 + \frac{\mu_e \eta}{4\pi \rho_m \nu} p_2 (I_6 + d^2 I_8) + d^2 I_{10}\right] = 0, \quad (33)$$

and

$$\sigma_i \left[I_2 + \frac{c_p \alpha' K a^2}{\nu (G-1)} p_1 I_4 - \frac{\mu_e \eta}{4\pi \rho_m \nu} p_2 (I_6 - d^2 I_8) - d^2 I_{10}\right] = 0 . \quad (34)$$

Theorem 1. If $G < 1$, the system is stable.

It is evident from Eq. (33) that, if $G < 1, \sigma_r$ is negative. This means that the system is stable.

Theorem 2. In contrast to the nonoscillatory modes for $G > 1$ in the absence of magnetic field, the presence of magnetic field (and hence the presence of Hall effect) introduces oscillatory modes in the system.

In the absence of magnetic field, Eq. (34) reduces to

$$\sigma_i \left[I_2 + \frac{c_p \alpha' K a^2}{\nu (G-1)} p_1 I_4\right] = 0 , \quad (35)$$

which implies that, for $G > 1$, the term in the bracket is positive and so σ_i is zero. Thus in the absence of magnetic field (and hence in the absence of Hall currents) the oscillatory modes do not exist. The presence of magnetic field introduces oscillatory modes as σ_i may be nonzero, as is evident from Eq. (34).

5. The Stationary Convection. For stationary convection, putting $\sigma = 0$ in Eq. (30) reduces it to

$$R_1 = \left(\frac{1+x}{x}\right)\left(\frac{G}{G-1}\right)\left[\frac{\{(1+x+P)(1+x)+Q_1\}^2 + M(1+x)(1+x+P)^2}{(1+x+P)(1+x) + Q_1 + M_1(1+x+P)}\right]. \quad (36)$$

For fixed values of Q_1, M and P, let the nondimensional number G accounting for the compressibility effects be also kept as fixed, then we find that

$$\bar{R}_c = \left(\frac{G}{G-1}\right) R_c , \quad (37)$$

where $\bar{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility. The effect of compressibility is, thus, to postpone the onset of thermal instability. Hence we obtain stabilizing effect of compressibility.

To find the effect of Hall currents on thermal instability, we examine the nature of dR_1/dM.

From Eq. (36), it follows that

$$\frac{dR_1}{dM} = -\frac{\left(\frac{1+x}{x}\right)\left(\frac{G}{G-1}\right) Q_1 (1+x+P)\,[Q_1 + (1+x)(\mathrm{i}+x+P)]}{[(1+x)(1+x+P) + Q_1 + M(1+x+P)]^2},$$

which is always negative. This shows that Hall currents have a destabilizing effect on the system.

6. The case of Overstability. Put $\frac{\sigma}{\pi^2} = i\sigma_1$, where σ may be complex. Equation (30) becomes

$$R_1 x = \left(\frac{G}{G-1}\right)\Bigg[(1+x)(1+x+i\sigma_1+P)(1+x+ip_1\sigma_1) + \tag{38}$$

$$+\frac{Q_1(1+x)(1+x+ip_1\sigma_1)\{(1+x+i\sigma_1+P)(1+x+ip_2\sigma_1)+Q_1\}}{[(1+x+ip_2\sigma_1)\{(1+x+i\sigma_1+P)(\mathrm{i}+x+ip_2\sigma_1)+Q_1\} + M(1+x)(1+x+i\sigma_1+P)]}\Bigg].$$

Since for overstability we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (38) will admit of solutions with σ_1 real.

Equating real and imaginary parts of Eq. (38) and eliminating R_1 between them, after a little algebra, we obtain

$$A_1\beta'^3 + B_1\beta'^2 + C_1\beta' + D_1 = 0, \tag{39}$$

where

$$\begin{aligned} \beta' &= \sigma_1^2, \quad 1+x = \alpha, \\ A_1 &= p_2^4\,[\alpha(1+p_1) + p_1 P], \end{aligned} \tag{40}$$

and

$$\begin{aligned} D_1 = {} & (1+p_1)\alpha^7 + [P(2+3p_1)+2M(1+p_1)]\,\alpha^6 + [(2p_1-p_2+2)\,Q_1 + MP(4+5p_1) + \\ & + P^2(1+Q_1) + M^2(1+p_1) + Q_1 MP]\alpha^5 + [MQ_1(2p_1+p_2+2) + Q_1 P(5p_1-2p_2+2) + \\ & + p_1(Q_1+P^3) + MP(2P+2M+3Mp_1)]\alpha^4 + [Q_1^2(2p_1-2p_2+1) + 3Q_1^2P^2(p_1-p_2) + \\ & + 2p_2\,Q_1 P + M(2p_1P^3 + 5p_1Q_1P + 2p_2Q_1P + p_1Q_1 + 2Q_1P) + \\ & + M^2(2p_1P^2 + P + p_1P)]\alpha^3 + [MQ_1^2(p_1-1) + PQ_1^2(3p_1-2p_2) + \\ & + M^2p_1P^3 + p_1Q_1^2 + M_1Q_1P(p_1+p_2P+3p_1P)]\alpha^2 + \alpha Q_1^2[Q_1(p_1-p_2) + p_1\,MP]. \end{aligned}$$

We have not calculated and written the values of B_1 and C_1 as only A_1 and D_1 are needed for our discussion.

Since σ_1 is real for overstability, the three values of β' are positive. Now the product of the roots of (39) is $-\frac{D_1}{A_1}$ and if this is to be positive then $D_1 < 0$ since from (40), $A_1 > 0$. Equation (40) shows this is impossible if

$$p_1 > 1 \quad \text{and} \quad p_1 > p_2 , \tag{41}$$

which implies that

$$\nu > K \quad \text{and} \quad \eta > K . \tag{42}$$

Thus for $K<\nu$ and $K<\eta$, overstability is not possible and the principle of exchange of stabilities is valid. (42) are therefore sufficient conditions for nonexistence of overstability.

REFERENCES

[1] CHANDRASEKHAR, S. : Hydrodynamic and Hydromagnetic Stability, Oxford University Press (1961), Chaps. 2-5.

[2] GUPTA, A.S. : Rev. Roumaine Math. Pures Appl. 12 (1967), 665.

[3] LAPWOOD, E.R. : Proc. Camb. Phil. Soc. 44 (1948), 508.

[4] SPIEGEL, E.A. and VERONIS, G. : Astrophys. J. 131 (1960), 442.

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ÖZET

Bu çalışmada sıkıştırılabilirlik etkisinin kararlı ve Hail akımlarının kararsız bir etkisinin olacağı saptanmaktadır. Aynı zamanda aşırı kararlılık durumu gözönüne alınmakta ve aşırı kararlılığın ortaya çıkmaması için yeterli koşullar incelenmektedir.