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AN $R-\oplus$ RECURRENT FINSLER SPACE WITH NON-SYMMETRIC CONNECTION

H.D. PANDE - K.K. GUPTA

In the present paper it has been obtained necessary and sufficient conditions for first order recurrency of curvature tensors $R^h{}_{ijk}$ and $\tilde{R}^h{}_{ijk}$ in a Finsler space with non-symmetric connection $\Gamma^i{}_{jk}$.

1. Introduction. Let F_n^* be an n-dimensional Finsler space having $2n$-line elements $(x^i, \dot{x}^i)$ $(i,j,k,\dots$ etc. $= 1,2,3,\dots,n)$ equipped with non-symmetric connection $\Gamma^i{}_{jk}(x, \dot{x})$ based on non-symmetric metric tensor $g_{ij}(x, \dot{x})$.

Let us write $\Gamma^i{}_{jk}$ as given below [2][1)]

$$\Gamma^i{}_{jk} = M^i{}_{jk} + \frac{1}{2} N^i{}_{jk}\,, \tag{1.1}$$

where $M^i{}_{jk}$ and $\frac{1}{2} N^i{}_{jk}$ are the symmetric and skew-symmetric parts of $\Gamma^i{}_{jk}$ respectively. Let us introduce another connection $\tilde{\Gamma}^i{}_{jk}(x, \dot{x}) \equiv \Gamma^i{}_{kj}(x, \dot{x})$ and define two types of co-variant derivatives :

$$x^{\overset{i}{+}}_{|j} = \partial_j x^i - (\dot{\partial}_m x^i)\,\Gamma^m{}_{pj}\,\dot{x}^p + x^m\,\Gamma^i{}_{mj}{}^{2)}\,, \tag{1.2}$$

$$x^{\overset{i}{-}}_{|j} = \partial_j x^i - (\dot{\partial}_m x^i)\,\tilde{\Gamma}^m{}_{pj}\,\dot{x}^p + x^m\,\tilde{\Gamma}^i{}_{mj}\,. \tag{1.2}'$$

The duality in the nature of co-variant derivatives introduces two curvature tensors:

$$\begin{aligned} R^i{}_{jkl} &= \partial_l\,\Gamma^i{}_{jk} - \partial_k \Gamma^i{}_{jl} - (\dot{\partial}_m\,\Gamma^i{}_{jk})\,\Gamma^m{}_{pl}\,\dot{x}^p + (\dot{\partial}_m\,\Gamma^i{}_{jl})\,\Gamma^m{}_{pk}\,\dot{x}^p + \\ &\quad + \Gamma^p{}_{jk}\,\Gamma^i{}_{pl} - \Gamma^p{}_{jl}\,\Gamma^i{}_{pk}\,. \end{aligned} \tag{1.3}$$

$$\begin{aligned} \tilde{R}^i{}_{jkl} &= \partial_l\,\tilde{\Gamma}^i{}_{jk} - \partial_k\,\tilde{\Gamma}^i{}_{jl} - (\dot{\partial}_m\,\tilde{\Gamma}^i{}_{jk})\,\tilde{\Gamma}^m{}_{pl}\,\dot{x}^p + (\dot{\partial}_m\,\tilde{\Gamma}^i{}_{jl})\,\tilde{\Gamma}^m{}_{pk}\,\dot{x}^p + \\ &\quad + \tilde{\Gamma}^p{}_{jk}\,\tilde{\Gamma}^i{}_{pl} - \tilde{\Gamma}^p{}_{jl}\,\tilde{\Gamma}^i{}_{pk}\,. \end{aligned} \tag{1.3}'$$

[1)] The numbers in square brackets refer to the references given at the end of the paper.

[2)] $\partial_i \equiv \partial/\partial x^i$, $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$

It can be easily verified that both the co-variant derivatives of $\dot{x}^l$ vanish, i.e.

$$\dot{x}^{i}_{\overset{+}{}|k} = 0 = \dot{x}^{i}_{\overset{-}{}|k} . \tag{1.4}$$

The following notations and abreviations will be extensively used in the sequel.

$$R^i_{jk} \equiv \dot{x}^h R^i_{hjk} \tag{1.5a}$$

$$R^i_j \equiv \dot{x}^h R^i_{hj} \tag{1.5b}$$

$$R \equiv R^i_i \tag{1.5c}$$

$$R^i_{hjk} = - R^i_{hkj} , \; R^i_{jk} = - R^i_{kj} , \; N^i_{jk} = - N^i_{kj} . \tag{1.5d}$$

Commutation formulae are as follows :

$$\dot{\partial}_k (T^{\overset{i}{+}}_{\underset{+}{j} | h}) - (\dot{\partial}_{\underset{+}{k}} T^{\overset{i}{+}}_{\underset{+}{j}})_{|h} = T_j^m \dot{\partial}_k \Gamma^i_{mh} - T^i_m \dot{\partial}_k \Gamma^m_{jh} - (\dot{\partial}_m T_j^i) (\dot{\partial}_k \Gamma^m_{ph}) \dot{x}^p . \tag{1.6}$$

$$T^{\overset{i}{+}}_{\underset{+}{j} | hk} - T^{\overset{i}{+}}_{\underset{+}{j} | kh} = - (\dot{\partial}_m T_j^i) R^m_{hk} + T_j^m R^i_{mhk} - T_m^i R^m_{jhk} + (T^{\overset{i}{+}}_{\underset{+}{j} | m}) N^m_{kh} , \tag{1.7}$$

where N^i_{jk} is defined in (1.1).

2. $R - \oplus$ Recurrent Finsler space.

Definition 2.1. F_n^* will be called $R - \oplus$ recurrent F_n^* if its first curvature tensor R^h_{ijk} satisfies the following condition

$$R^{\overset{h}{+}}_{\underset{+++}{i\, j\, k} | l} = \lambda_l R^h_{ijk} \quad (\lambda_l \neq 0) , \tag{2.1}$$

where λ_l is known as recurrence vector field.

Transvecting (2.1) with $\dot{x}^i$ and using (1.4) and (1.5a), we find

$$R^{\overset{h}{+}}_{\underset{++}{j\, k} | l} = \lambda_l R^h_{jk} . \tag{2.2}$$

Again transvecting (2.2) with $\dot{x}^j$ and using (1.4), (1.5b) we have

$$R^{\overset{h}{+}}_{\underset{+}{k} | l} = \lambda_l R_k^h . \tag{2.3}$$

Contracting $R^{\overset{h}{+}}_{\underset{+}{k} | l}$ with respect to the indices h and k, and using (1.5c) we get

$$R_{|l} = \lambda_l R\,. \tag{2.4}$$

From (2.2), (2.3), (2.4), we conclude that $R^h{}_{jk}$, $R_k{}^h$ and R are also $\oplus$ recurrent of first order in an $R-\oplus$ recurrent F_n^*. The converse of this statement is not necessarily true:

Theorem 2.1. An $R^i{}_{jk}-\oplus$ recurrent F_n^* will be $R-\oplus$ recurrent F_n^* if and only if the recurrence vector field satisfies

$$\dot x^p (\dot\partial_{\underset{+}{h}} \overset{i}{R^{+}}_{\underset{+\,+\,+}{p\,j\,k}})_{|l} = (\dot\partial_h \lambda_l)\, R^i{}_{jk} + \lambda_l\, \dot x^p\, \dot\partial_h R^i{}_{pjk} - R^m{}_{jk}\, \dot\partial_h \Gamma^i{}_{ml} + R^i{}_{mk}\, \dot\partial_h \Gamma^m{}_{jl} + \\ + R^i{}_{jm}\, \dot\partial_h \Gamma^m{}_{kl} + (\dot\partial_m R^i{}_{jk})\,(\dot\partial_h \Gamma^m{}_{pl})\, \dot x^p\,. \tag{2.5}$$

Proof. Let F_n^* be $R^i{}_{jk}-\oplus$ recurrent,
viz.

$$\overset{i}{R^{+}}_{\underset{+\,+}{j\,k}|l} = \lambda_l R^i{}_{jk}\,.$$

Differentiating the above equation, partially with respect to $\dot x^h$ and applying the commutation formula (1.6) together with relations (1.4), (1.5a), we get

$$\overset{i}{R^{+}}_{\underset{+\,+\,+}{h\,j\,k}|l} - \lambda_l R^i{}_{hjk} = (\dot\partial_h \lambda_l)\, R^i{}_{jk} + \lambda_l\, \dot x^p\, \dot\partial_h R^i{}_{pjk} - \dot x^p (\dot\partial_{\underset{+}{h}} \overset{i}{R^{+}}_{\underset{+\,+\,+}{p\,j\,k}}) - \\ - R^m{}_{jk}\, \dot\partial_h \Gamma^i{}_{ml} + R^i{}_{mk}\, \dot\partial_h \Gamma^m{}_{jl} + R^i{}_{jm}\, \dot\partial_h \Gamma^m{}_{kl} + \\ + (\dot\partial_m R^i{}_{jk})\,(\dot\partial_h \Gamma^m{}_{pl})\, \dot x^p\,. \tag{2.6}$$

Now in $R-\oplus$ recurrent F_n^*, left hand side of (2.6) vanishes and hence (2.5) holds good.

When the connection coefficients $\Gamma^i{}_{jk}$ are homogenous of degree zero in their directional arguments, then:

Theorem 2.2. In an $R^i{}_{jk}-\oplus$ recurrent F_n^*, the following identity is satisfied:

$$\dot x^p\, \dot x^h (\dot\partial_{\underset{+}{h}} \overset{i}{R^{+}}_{\underset{+\,+\,+}{p\,j\,k}|l}) = (\dot\partial_h \lambda_l)\, R^i{}_{jk}\, \dot x^h + \lambda_l\, \dot x^p\, \dot x^h\, \dot\partial_h R^i{}_{pjk}\,. \tag{2.7}$$

Proof. Transvecting (2.6) with $\dot x^h$ and using (1.4), (1.5c) we get

$$\overset{i}{R^{+}}_{\underset{+\,+}{j\,k}|l} - \lambda_l R^i{}_{jk} = (\dot\partial_h \lambda_l)\, R^i{}_{jk}\, \dot x^h + \lambda_l\, \dot x^p\, \dot x^h\, \dot\partial_h R^i{}_{pjk} - \dot x^p\, \dot x^h (\dot\partial_{\underset{+}{h}} \overset{i}{R^{+}}_{\underset{+\,+\,+}{p\,j\,k}|l}) - \\ - R^m{}_{jk}\, (\dot\partial_h \Gamma^i{}_{ml})\, \dot x^h + R^i{}_{mk}\, (\dot\partial_h \Gamma^m{}_{jl})\, \dot x^h + R^i{}_{jm}\, (\dot\partial_h \Gamma^m{}_{kl})\, \dot x^h + \\ + (\dot\partial_m R^i{}_{jk})\,(\dot\partial_h \Gamma^m{}_{pl})\, \dot x^p\, \dot x^h\,.$$

Using homogeneity property of Γ^i_{jk} and noting the fact that left hand side of above identity vanishes in $R^i_{jk} - \oplus$ recurrent F_n^*, we get (2.7).

Theorem 2.3. The necessary and sufficient condition that an $R_j^i - \oplus$ recurrent F_n^* will be an $R^i_{jk} - \oplus$ recurrent F_n^*, is that

$$\dot{x}^h (\dot{\partial}_{\underset{+}{k}} R^{\overset{i}{+}}_{\underset{++}{h\,j|l}}) = (\dot{\partial}_k \lambda_l) R_j^i + \lambda_l \dot{x}^h (\dot{\partial}_k R^i_{hj}) - R_j^m \dot{\partial}_k \Gamma^i_{ml} + R_m^i \dot{\partial}_k \Gamma^m_{jl} + (\dot{\partial}_m R_j^i)(\dot{\partial}_k \Gamma^m_{pl}) \dot{x}^p . \tag{2.8}$$

Proof. Let F_n^* be $R_j^i - \oplus$ recurrent space, viz.

$$R^{\overset{i}{+}}_{\underset{+}{j}|l} = \lambda_l R_j^i . \tag{A}$$

Differentiating it partially with respect to $\dot{x}^k$ and using (1.6), (1.5b), we have, after a rearrangement of its members,

$$R^{\overset{i}{+}}_{\underset{++}{k\,j|l}} - \lambda_l R^i_{kj} = - \dot{x}^h (\dot{\partial}_{\underset{+}{k}} R^{\overset{i}{+}}_{\underset{++}{h\,j|l}}) - R_j^m \dot{\partial}_k \Gamma^i_{ml} + R_m^i \dot{\partial}_k \Gamma^m_{jl} + (\dot{\partial}_m R_j^i)(\dot{\partial}_k \Gamma^m_{pl}) \dot{x}^p + (\dot{\partial}_k \lambda_l) R_j^i + \lambda_l \dot{x}^h \dot{\partial}_k R^i_{hj} . \tag{2.9}$$

If F_n^* becomes $R^i_{jk} - \oplus$ recurrent, first member of (2.9) vanishes identically and we have the result (2.8).

When the connection coefficients Γ^i_{jk} are homogeneous of degree zero in their directional arguments, then :

Theorem 2.4. In an $R_j^i - \oplus$ recurrent F_n^*, following identity is true :

$$\dot{x}^k \dot{x}^h (\dot{\partial}_{\underset{+}{k}} R^{\overset{i}{+}}_{\underset{++}{h\,j|l}}) = (\dot{\partial}_k \lambda_l) R_j^i \dot{x}^k + \lambda_l \dot{x}^h \dot{x}^k \dot{\partial}_k R^i_{hl} . \tag{2.10}$$

Proof. Transvecting (2.9) with $\dot{x}^k$ and after using (1.4) and (1.5b), we have

$$R^{\overset{i}{+}}_{\underset{+}{j}|l} - \lambda_l R_j^i = - \dot{x}^k \dot{x}^h (\dot{\partial}_{\underset{+}{k}} R^{\overset{i}{+}}_{\underset{++}{h\,j|l}}) - R_j^m (\dot{\partial}_k \Gamma^i_{ml}) \dot{x}^k + R_m^i (\dot{\partial}_k \Gamma^m_{jl}) \dot{x}^k + (\dot{\partial}_m R_j^i)(\dot{\partial}_k \Gamma^m_{pl}) \dot{x}^p \dot{x}^k + (\dot{\partial}_k \lambda_l) R_j^i \dot{x}^k + \lambda_l \dot{x}^h \dot{x}^k \dot{\partial}_k R^i_{hj} . \tag{2.11}$$

Now, in an $R_j^i - \oplus$ recurrent F_n^* first member of (2.11) vanishes, hence using the homogeneity property of Γ^i_{jk}, (2.11) takes the form of (2.10).

Theorem 2.5. In an $R_j^i-\oplus$ recurrent F_n^*, the relation

$$(\lambda_{\underset{+}{l|m}}-\lambda_{\underset{+}{m|l}}+\lambda_\gamma N^\gamma{}_{lm})R_j^i=-(\dot\partial_\gamma R_j^i)R^\gamma{}_{lm}+R_j^\gamma R^i{}_{\gamma lm}-R_\gamma{}^iR^\gamma{}_{jlm} \tag{2.12}$$

is always true.

Proof. Differentiating the relation given in (A), $\oplus$ co-variantly with respect to x^m, we have after using (A),

$$\underset{+}{R}{}^{\,i}_{j\,|lm}=(\lambda_{\underset{+}{l|m}}+\lambda_l\lambda_m)R_j^i. \tag{2.13}$$

Subtracting the result obtained after interchanging the indices l and m in (2.13) from (2.13), we have after using commutation formula (1.7) :

$$-(\dot\partial_\gamma R_j^i)R^\gamma{}_{lm}+R_j^\gamma R^i{}_{\gamma lm}-R_\gamma{}^iR^\gamma{}_{jlm}+(\underset{+}{R}{}^{\,i}_{j\,|\gamma})N^\gamma{}_{ml}=$$
$$=(\lambda_{\underset{+}{l|m}}-\lambda_{\underset{+}{m}})R_j^i. \tag{2.14}$$

From (2.14) and (A), we get (2.12), after a rearrangement of terms.

Theorem 2.6. In $r-\oplus$ recurrent F_n^*, the Bianchi's Identity takes the following form :

$$3\lambda_{[l}R^h{}_{jk]}{}^{3)}+3R^h{}_{p[k}N^p{}_{jl]}+\dot x^iE^h{}_{iljk}=0, \tag{2.15}$$

where

$$E^h{}_{iljk}\overset{\text{def.}}{=\!=}3(\dot\partial_m\Gamma^h{}_{i[l})R^m{}_{jk]}.$$

Proof. From Theorem 2.1 in [2], we are given with the following Bianchi Identity :

$$\underset{+}{R}{}^{\,h}_{i\,jk|l}+\underset{+}{R}{}^{\,h}_{i\,kl|j}+\underset{+}{R}{}^{\,h}_{i\,lj|k}+E^h{}_{iljk}=0. \tag{2.16}$$

But as we know,

$$\underset{+}{R}{}^{\,h}_{i\,jk|l}=\underset{+++}{R}{}^{\,h}_{i\,j\,k|l}+R^h{}_{ipk}\Gamma^p{}_{jl}+R^h{}_{ijp}\Gamma^p{}_{kl}. \tag{2.17}$$

Applying (2.1), (2.17) can be rewritten as

$$\underset{+}{R}{}^{\,h}_{i\,jk|l}=\lambda_lR^h{}_{ijk}+R^h{}_{ipk}\Gamma^p{}_{jl}+R^h{}_{ijp}\Gamma^p{}_{kl}. \tag{2.17$'$}$$

After changing the indices j, k, l cyclically in (2.17)′ we shall get two other similar results, substitute all the results thus obtained in (2.16) and after using (1.5d), we will get (2.15).

[3] $A_{[ijk]}\overset{\text{def.}}{=\!=}\frac{1}{3}(A_{ijk}+A_{jki}+A_{kij})$.

It is noteworthy that similar theorems as given above can be obtained when we define $\widetilde{R} - \Theta$ recurrent F_n^* as follows:

$$\widetilde{R}^h_{\underline{i}\,\underline{j}\,\underline{k}|l} = \mu_l \widetilde{R}^h_{ijk} \quad (\mu_l \neq 0) .$$

REFERENCES

[1] NITESCU, C. : *Bianchi's Identities in a non-symmetric connection space*, Bul. Inst. Politehn. Iaşi (N.S.) **20 (24)** (1974), Fasc. 1-2, Sect. I, 69-72.

[2] PANDE, H.D. and GUPTA, K.K. : *Bianchi's Identities in a Finsler Space with non-symmetric connection*, Bui. Classe des Sciences Mathématiques et Naturelles, Knez (Belgrade) (communicated).

[3] PANDE, H.D. and SINGH, B. : *On Projective Recurrent Finsler spaces of the first order*, Rendiconti Accademia Nazionale dei XL, Serie IV, **XXXIV-XXV** (1974), 1-8.

[4] RUND, H. : The differential Geometry of Finsler Spaces, Springer Verlag (1959).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
GORAKHPUR (U.P.) INDIA
AND
DEPARTMENT OF MATHEMATICS
U.N.P.G.COLLEGE, PADRAUNA (DEORIA)
(U.P.) INDIA

ÖZET

Bu çalışmada, simetrik olmayan T^i_{jk} bağlacım haiz bir Finsler uzaymdaki R^h_{ijk} ve $\widetilde{R}^h_{ijk}$ eğrilik tensörlerinin birinci mertebeden tekrarlılığma dair gerek ve yeter koşullar elde edilmektedir.