İstanbul Üniv. Fen Fak. Mec. Seri A, 43 (1978), 121-130

FLOW AND HEAT TRANSFER IN AN ELASTICO-VISCOUS LIQUID OVER AN OSCILLATING PLATE IN A ROTATING FRAME

NANIGOPAL DATTA - RABINDRA NATH JANA

An exact solution is given for the flow of an elasticoviscous fluid induced by an oscillating flat plate in a rotating frame. It is found that the phase difference of the primary shear stress decreases while that of the secondary shear stress increases with increase in the elastic parameter. The problem of heat transfer from the plate, taking viscous dissipation into account, has also been considered when the temperature of the plate oscillates with the same frequency as that of the plate.

1. Introduction. In technological fields, the flow of non-Newtonian fluids are being extensively used. The unsteady flow of a visco-elastic fluid of Walters [1] model B' over a plate has been studied by Gulati [2], Kaloni [3], Soundalgekar and Puri [4], Missiha [5] and Puri [6]. Recently Puri [6] has studied the flow induced by the oscillations of a plate in a rotating frame of reference.

The purpose of the present study is two-fold. Firstly, to re-investigate the problem studied by Puri [7], because we feel that there is some error in his solution. Secondly, to study the fluctuations in heat transfer from the plate when the plate temperature fluctuates with the same frequency as that of the plate.

2. Mathematical Analysis. The constitutive equations for Walters [liquid (Model B') are]

$$p_{ik} = -p\, g_{ik} + p'_{ik}\,, \tag{1}$$

$$p'_{ik} = 2\int_{-\infty}^{t} \psi(t-t')\,\frac{\partial x^i}{\partial x'^m}\cdot\frac{\partial x^k}{\partial x'^r}\, e^{(1)mr}\, dt'\,, \tag{2}$$

where p_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} the metric tensor, $e^{(1)ik}$ the rate of strain tensor and

$$\psi(t-t') = \int_{0}^{\infty} \frac{N(\tau)}{\tau}\, \exp\left[-(t-t')/\tau\right] d\tau\,, \tag{3}$$

where $N(\tau)$ is the distribution function of relaxation times. For liquids with short memories (i.e. short relaxation times), the above equations give

$$p'^{ik} = 2\eta e^{(1)ik} - 2k_0 \frac{\delta}{\delta t} e^{(1)ik}, \tag{4}$$

where $\eta = \int_0^\infty N(\tau)\, d\tau$ is the limiting viscosity at small rates of shear, $k_0 = \int_0^\infty \tau N(\tau)\, d\tau$ and $\delta/\delta t$ denotes the convected differentiation of a tensor.

In a rotating frame of reference, the equation of continuity and the equation of motion are

$$\frac{\partial u_i}{\partial x^i} = 0, \tag{5}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x^j} + 2\varepsilon_{ijk} V_j \Omega_k^* = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^i} + \frac{1}{\rho}\frac{\partial p'_{ik}}{\partial x^k}, \tag{6}$$

where u_i is the velocity vector, Ω_k^* the angular velocity vector and p^* is the modified fluid pressure which includes the centrifugal force.

Consider the flow of an elastico-viscous liquid cccupying the space $z > 0$ induced by simple harmonic oscillations $U_0 \cos \omega^* t$ in the x-direction of an infinite plate $z = 0$ in a rotating frame of reference. The plate is rotating in uniform with the liquid with an angular velocity Ω^* about the z-axis. Since the plate is infinite all physical quantities will be functions of z and t only. The equation of continuity together with the no-slip condition at the plate then shows that the z-component of the velocity vanishes everywhere.

Using (1) and (4), the non-dimensional equations of momentum (6) can be written as

$$\left[\left(1 - k\frac{\partial}{\partial T}\right)\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial T}\right] u_1 + 2\Omega u_2 = 0, \tag{7}$$

$$\left[\left(1 - k\frac{\partial}{\partial T}\right)\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial T}\right] u_2 - 2\Omega u_1 = 0, \tag{8}$$

where

$$\begin{aligned} &\xi = U_0 z/\nu, \ T = U_0^2 t/\nu, \ u_1 = u_x/U_0, \ u_2 = u_y/U_0, \\ &\Omega = \nu \Omega^* U_0^2, \ k = k^* U_0^2/\nu^2, \ \nu = \eta/\rho, \ k^* = k/\rho, \end{aligned} \tag{9}$$

and u_x, u_y denote the velocity components along x and y-directions.

The boundary conditions are

$$u_1 = \exp(i\omega T),\ u_2 = 0 \quad \text{at} \quad \xi = 0\ , \qquad (10)$$
$$u_1 \to 0\ ,\ u_2 \to 0 \qquad \text{as} \quad \xi \to \infty\ .$$

Assuming

$$u_1 = f_1 e^{i\omega T} \quad \text{and} \quad u_2 = f_2 e^{i\omega T}\ , \qquad (11)$$

equations (5) and (6) become

$$(1 - ik\omega)\frac{d^2 f_1}{d\xi^2} - i\omega f_1 + 2\Omega f_2 = 0\ , \qquad (12)$$

$$(1 - ik\omega)\frac{d^2 f_2}{d\xi^2} - i\omega f_2 - 2\Omega f_1 = 0\ . \qquad (13)$$

The boundary conditions (8) give

$$f_1(0) = 1 \quad \text{and} \quad f_1(\infty) = 0\ , \qquad (14)$$

$$f_2(0) = 0 \quad \text{and} \quad f_2(\infty) = 0\ . \qquad (15)$$

Eliminating f_2 from equations (12) and (13), we have

$$(1 - ik\omega)^2 \frac{d^4 f_1}{d\xi^4} - 2i\omega(1 - ik\omega)\frac{d^2 f_1}{d\xi^2} - (\omega^2 - 4\Omega^2) f_1 = 0. \qquad (16)$$

From equation (16), the solution of $f_1(\xi)$ satisfying the boundary conditions (14) can be easily obtained. Having found $f_1(\xi)$, one can determine $f_2(\xi)$ from equation (12) using the boundary conditions (15). These solutions are

$$u_1 = \frac{1}{2}\left[\exp\{-A\xi + i(\omega T - B\xi)\} + \exp\{-A_1\xi + i(\omega T - B_1\xi)\}\right] \qquad (17)$$
$$\text{for} \quad \omega > 2\Omega,$$

$$u_2 = -\frac{i}{2}\left[\exp\{-A\xi + i(\omega T - B\xi)\} - \exp\{-A_1\xi + i(\omega T - B_1\xi)\}\right] \qquad (18)$$

and

$$u_1 = \frac{1}{2}\left[\exp\{-A\xi + i(\omega T - B\xi)\} + \exp\{-A_2\xi + i(\omega T + B_2\xi)\}\right] \qquad (19)$$
$$\text{for} \quad \omega < 2\Omega,$$

$$u_2 = -\frac{i}{2}\left[\exp\{-A\xi + i(\omega T - B\xi)\} - \exp\{-A_2\xi + i(\omega T + B_2\xi)\}\right], \qquad (20)$$

where

$$A = \frac{1}{\sqrt{2}}\left(\frac{\omega + 2\,\Omega}{1 + k^2\omega^2}\right)^{1/2}(\alpha - \beta) \ , \quad B = \frac{1}{\sqrt{2}}\left(\frac{\omega + 2\,\Omega}{1 + k^2\,\omega^2}\right)^{1/2}(\alpha + \beta) \, , \tag{21}$$

$$A_1 = \frac{1}{\sqrt{2}}\left(\frac{\omega - 2\,\Omega}{1 + k^2\omega^2}\right)^{1/2}(\alpha - \beta) \ , \quad B_1 = \frac{1}{\sqrt{2}}\left(\frac{\omega - 2\,\Omega}{1 + k^2\omega^2}\right)^{1/2}(\alpha + \beta) \, , \tag{22}$$

$$\alpha = \frac{1}{\sqrt{2}}\,\{(1 + k^2\omega^2)^{1/2} + 1\}^{1/2} \, , \quad \beta = \frac{1}{\sqrt{2}}\,\{(1 + k^2\,\omega^2)^{1/} - 1\}^{1/2} \, , \tag{23}$$

and

$$A_2 = \frac{1}{\sqrt{2}}\left(\frac{2\,\Omega - \omega}{1 + k^2\omega^2}\right)^{1/2}\{(1 + k^2\,\omega^2)^{1/2} + k\omega\}^{1/2} \, , \tag{24}$$

$$B_2 = \frac{1}{\sqrt{2}}\left(\frac{2\,\Omega - \omega}{1 + k^2\omega^2}\right)^{1/2}\{(1 + k^2\,\omega^2)^{1/2} - k\omega\}^{1/2} \, . \tag{25}$$

The solution (17) - (20) agree with the solution for Newtonian fluid studied by Puri [8] by making $k = 0$ except for the term 1 and the minus sign in the oscillatory part and that the suction has to be zero. Because of some mathematical error in Puri [7], the solution for the Newtonian fluid (Puri [8]) cannot be deduced from those of former one.

3. Results and discussion. For $\omega > 2\,\Omega$, the velocity consists of damped harmonic oscillations with amplitudes $U_0\, e^{-A\xi}$ and $U_0\, e^{-A_1\xi}$ and having phase lags $B\xi$ and $B_1\xi$ respectively relative to the wall. The depths of penetration or wave lengths of the two layers are respectively $2\pi\,\nu/U_0 B$ and $2\pi\,\nu/U_0\, B_1$. Because of the presence of elasticity of the fluid, A and B decrease while A and A_1 increase for $0 < k\omega \leqslant 0.58$ and decrease for $0.59 \leqslant k\omega$. This implies that the depths of penetration $2\pi\,\nu/U_0 B$ and $2\pi\,\nu/U_0\, B_1$ decrease for $0 < k\omega \leqslant 0.58$ and increase for $0.59 \leqslant k\omega$.

For $\omega < 2\,\Omega$, the velocity consists of two damped oscillations of which one is the same as that for the case $\omega > 2\,\Omega$ while the second one has an amplitude $U_0\, e^{-A_2\xi}$ and has a phase advance $B_2\,\xi$ with respect to the plate. The depth of penetration of the second layer is $2\pi\,\nu/U_0\,|\,B_2\,|$ which decreases with $k\omega$.

For $\omega = 2\,\Omega$, we have a phenomenon of resonance similar to the result obtained by Thornley [9] in her study of non-torsional oscillations of an infinite non-porous plate rotating in accord with a viscous fluid. This resonance implies that the whole liquid is affected by the motion of the plate and the oscillation is not confined to a well-defined Ekman layer near the plate.

In the absence of rotation ($\Omega = 0$), $u_2 = 0$ and u_1 is given by

$$u_1 = \exp\{-A^*\xi + i(\omega T - B^*\xi)\}\ , \tag{26}$$

where A^* and B^* are obtained from equations (21) - (23) by putting $\Omega = 0$. It is seen from (26) that the two layers merge into one, which oscillates with amplitude $U_0\, e^{-A^*\xi}$ and a phase lag $B^*\xi$.

The non-dimensional shear stress components at the plate $\xi = 0$ are

$$\tau_{zx} = \left[\left(1 - k\frac{\partial}{\partial T}\right)\frac{\partial u_1}{\partial \xi}\right]_{\xi=0} \quad \text{and} \quad \tau_{zy} = \left[\left(1 - k\frac{\partial}{\partial T}\right)\frac{\partial u_2}{\partial \xi}\right]_{\xi=0} . \tag{27}$$

Using equations (17) - (20) in equation (27) we get

$$\tau_{zx} = -\frac{1}{2} R_1 \exp\{i(\omega T + \theta_1)\}\ , \tag{28}$$

$$\tau_{zy} = -\frac{1}{2} R_2 \exp\{i(\omega T - \theta_2)\}\ , \tag{29}$$

$$\left.\begin{aligned} R_1 &= \sqrt{2}\,[(1 + k^2\omega^2)^{1/2}\,\{\omega + (\omega^2 - 4\Omega^2)^{1/2}\}\,]^{1/2}\ , \\ R_2 &= \sqrt{2}\,[(1 + k^2\omega^2)^{1/2}\,\{\omega - (\omega^2 - 4\Omega^2)^{1/2}\}\,]^{1/2}\ , \\ \tan\theta_1 &= \frac{(\alpha + \beta) - k\omega(\alpha - \beta)}{(\alpha - \beta) + k\omega(\alpha + \beta)}\ ,\ \tan\theta_2 = \frac{(\alpha - \beta) + k\omega(\alpha + \beta)}{(\alpha + \beta) - k\omega(\alpha - \beta)} \end{aligned}\right\}\ \omega > 2 \tag{30}$$

and

$$\left.\begin{aligned} R_1 &= (1 + k^2\omega^2)^{1/2}\,[\,(A + A_2)^2 + (B - B_2)^2]^{1/2}\ , \\ R_2 &= (1 + k^2\omega^2)^{1/2}\,[\,(A - A_2)^2 + (B + B_2)^2]^{1/2}\ , \\ \tan\theta_1 &= \frac{(B - B_2) - k\omega(A + A_2)}{(A + A_2) + k\omega(B - B_2)}\ ,\ \tan\theta_2 = \frac{(A - A_2) + k\omega(B + B_2)}{(B + B_2) - k\omega(A - A_2)} . \end{aligned}\right\}\ \omega < 2 \tag{31}$$

It is seen from equation (30) that for $\omega > 2\Omega$, both R_1 and R_2 increase with increase in k when Ω and ω are fixed while for fixed k and Ω, R_1 increases and R_2 decreases with increase in ω. For $\omega < 2\Omega$, the values of R_1 and R_2 are given in Tables 1 and 2 respectively. It is observed that both R_1 and R_2 increase with increase in either k or ω. It is interesting to note that ω for Newtonian fluid ($k = 0$) both R_1 and R_2 depend on when $\omega > 2\Omega$ while for $\omega < 2\Omega$, the effect of frequency on R_1 and R_2 is prominent only when the elasticity of fluid ($k \neq 0$) is taken into account.

TABLE - 1

Values of R_1 for $\Omega = 5.0$

ω / k	0.0	0.05	0.1
1.0	4.4721357	4.4749283	4.4832745
2.0	4.4721357	4.4832744	4.5162011
3.0	4.4721357	4.4970817	4.5695304
4.0	4.4721357	4.5162012	4.6411909

TABLE - 2

Values of R_2 for $\Omega = 5.0$

ω / k	0.0	0.05	0.10
1.0	4.4721357	4.4749272	4.4832730
2.0	4.4721357	4.4832729	4.5162009
3.0	4.4721357	4.4970812	4.5695299
4.0	4.4721357	4.5162010	4.6411904

The values of $\tan\theta_1$ and $\tan\theta_2$ have been plotted against k for various values of ω and Ω in Fig. 1. It is observed that for fixed ω, $\tan\theta_1$ decreases with increase in k. It is also observed that $\tan\theta_2$ increases with increase in either k or ω for both the cases $\omega > 2\Omega$ and $\omega < 2\Omega$. It is seen from equation (30) and from the definition of α and β given by equation (23) that for $\omega > 2\Omega$ both $\tan\theta_1$ and $\tan\theta_2$ are independent of rotation parameter Ω.

Further for the Newtonian fluid ($k = 0$), both $\tan\theta_1$ and $\tan\theta_2$ are independent of frequency ω when $\omega > 2\Omega$. In the presence of elasticity of the fluid ($k \neq 0$), the frequency ω influences the values of $\tan\theta_1$ and $\tan\theta_2$. However, for large rotation ($\omega < 2\Omega$), the effect of ω on $\tan\theta_1$ and $\tan\theta_2$ is prominent even in the case of Newtonian fluid ($k = 0$).

4. Heat Transfer. We consider now the heat transfer problem when the plate temperature oscillates with the same frequency as that of the oscillating plate. The energy equation is

$$\frac{\partial T^*}{\partial t} = \alpha^* \frac{\partial^2 T^*}{\partial z^2} + \frac{\nu}{c_p}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2\right], \tag{32}$$

where α^* is the thermal diffusivity, ν the kinematic coefficient of viscosity, c_p the specific heat at constant pressure and T^* the temperature of the fluid. In writing equation (32) we have neglected elastic dissipation of the fluid.

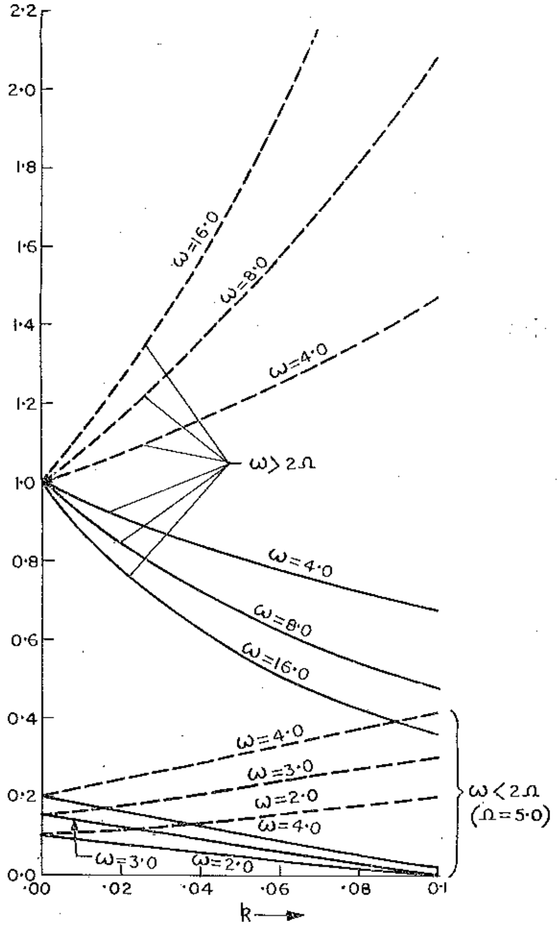

Fig. 1. Graphs of $\tan\theta_1$ and $\tan\theta_2$ against k

——— $\tan\theta_1$ and — — — — $\tan\theta_2$

Boundary conditions are

$$T^* - T_\infty = (T_\omega - T_\infty)\cos\omega^* t \quad at \quad z = 0\,,$$
$$T^* \to T_\infty \quad \text{as} \quad z \to \infty\,. \tag{33}$$

Introducing

$$\theta = (T^* - T_\infty) / (T_\omega - T_\infty), \ \sigma = \alpha^*/\nu, \ E = U_0^2/c_p (T_\omega - T_\infty), \tag{34}$$

and using (9), the equation (32) becomes

$$\sigma \frac{\partial \theta}{\partial T} = \frac{\partial^2 \theta}{\partial \xi^2} + \sigma E \left[\left(\frac{\partial u_1}{\partial \xi} \right)^2 + \left(\frac{\partial u_2}{\partial \xi} \right)^2 \right]. \tag{35}$$

Boundary conditions (33) can be written as

$$\theta = \cos \omega T \text{ at } \xi = 0 \quad \text{and} \quad \theta \to 0 \quad \text{as} \quad \xi \to \infty . \tag{36}$$

Assuming

$$\theta (\xi , T) = \theta_1 (\xi) \, e^{i\omega T} + \theta_2 (\xi) \, e^{2i\omega T} , \tag{37}$$

and using equations (17) - (20), equation (35) has been integrated under the boundary conditions (26) to give

$$\theta = \exp \{-(1 + i) (\sigma\omega/2)^{1/2} \xi\} \, e^{i\omega T} + \sigma E (m_1 - in_1) [\exp \{-(1 + i) (\sigma\omega)^{1/2} \xi\} - \\ - \exp \{- [(A + A_1) + i (B + B_1)] \xi\}] \, e^{2i\omega T} ; \text{ for } \omega > 2\Omega , \tag{38}$$

$$\theta = \exp \{- (1 + i) (\sigma\omega/2)^{1/2} \xi\} \, e^{i\omega T} + \sigma E (m_4 + in_4) [\exp \{- (1 + i) (\sigma\omega)^{1/2} \xi\} - \\ - \exp \{- [(A + A_2) + i (B - B_2)] \xi\}] \, e^{2i\omega T} ; \text{ for } \omega < 2\Omega , \tag{39}$$

where

$$
\begin{aligned}
m_1 &= (\omega^2 - 4\Omega^2)^{1/2} \{\omega (1 - \sigma) + (\omega^2 - 4\Omega^2)^{1/2}\} /2R, \\
n_1 &= (\omega^2 - 4\Omega^2)^{1/2} \sigma k \omega^2 /2R , \\
R &= \{(1 - \sigma) \omega + (\omega^2 - 4\Omega^2)^{1/2}\}^2 + \sigma^2 k^2 \omega^4 , \\
m_2 &= AA_2 - BB_2 , \quad n_2 = AB_2 - BA_2 , \\
m_3 &= m_2 (1 + k^2 \omega^2) - k\omega^2 , \quad n_3 = (n_2 + \sigma\omega) (1 + k^2 \omega^2) - \omega , \\
m_4 &= (m_2 m_3 + n_2 n_3)/2 (m_3^2 + n_3^2) , \quad n_4 = (m_2 n_3 - n_2 m_3)/2 (m_3^2 + n_3^2).
\end{aligned}
\tag{40}
$$

The rate of heat transfer at the plate $\xi = 0$ is

$$\left(\frac{d\theta}{d\xi} \right)_{\xi=0} = - [(\sigma\omega/2)^{1/2} (\cos \omega T - \sin \omega T) + \sigma E \{(m_1 m_5 + n_1 n_5) \cos 2\omega T - \\ - (m_1 n_5 - n_1 m_5) \sin 2\omega T\}] ; \text{ for } \omega > 2\Omega , \tag{41}$$

$$\left(\frac{d\theta}{d\xi} \right)_{\xi=0} = - [(\sigma\omega/2)^{1/2} (\cos \omega T - \sin \omega T) + \sigma E \{(m_4 m_6 - n_4 n_6) \cos 2\omega T - \\ - (m_4 n_6 + n_4 m_6) \sin 2\omega T\}] ; \text{ for } \omega < 2\Omega , \tag{42}$$

where

$$
\begin{aligned}
m_5 &= (\sigma\omega)^{1/2} - (A + A_1) , \ n_5 = (\sigma\omega)^{1/2} - (B + B_1) , \\
m_6 &= (\sigma\omega)^{1/2} - (A + A_2) , \ n_6 = (\sigma\omega)^{1/2} - (B - B_2) .
\end{aligned}
$$

The numerical values of $(d\theta/d\xi)_{\xi=0}$ for $\omega T = \pi/2$, $E = 0.02$, $\sigma = 0.72$, $\Omega = 2{,}5$ and for different values of ω and k are given in Table - 3. It is observed that for fixed ω, the rate of heat transfer decreases with increase in k while for fixed k, the rate of heat transfer increases with increase in ω. It is also observed that the rate of heat transfer is less in the case of visco-elastic fluid ($k \neq 0$) than those of Newtonian fluid ($k = 0$).

TABLE - 3

Rate of heat transfer $(d\theta/d\xi)_{\xi=0}$ for $\omega T = \pi/2$ and $\sigma = 0.72$

		k				
ω	Ω	0.0	0.01	0.02	0.03	0.04
1.00		0.529809	0.529088	0.528384	0.527699	0.527043
2.00	5.0	0.794259	0.792789	0.791375	0.790023	0.788737
3.00		0.997871	0.995655	0.993541	0.991544	0.989677
5.0		1.368080	1.363918	1.359828	1.355886	1.352161
10.0	2.0	1.936287	1.924225	1.912909	1.902893	1.894534
15.0		2.371471	2.349536	2.330378	2.315631	2.305731

Acknowledgement

One of the authors (R.N.J.) wishes to express his thanks to C.S.I.R. for financial provision for this research.

REFERENCES

[1] WALTERS, K. : *The solution of flow problems in the case of materials with memory*, J. Mech. 1 (1962), 474.

[2] GULATI, S.P. : Linear oscillations of a sphere surrounded by an Infinite Mass of an elastico-viscous fluid, 5 (1967), 250.

[3] KALONI, P.N. : Fluctuating flow of an elastico-viscous fluid past a porous flat plate, Phys. Fluids **10** (1967), 1344.

[4] SOUNDALGEKAR, V.M. and PURI, P. : *On fluctuating flow of an elastico-viscous fluid past an infinite plate with variable suction*, J. Fluid Mech. 35 (1969), 561.

[5] MESSIHA, S.A.S. : *Laminar boundary layers in oscillatory flow along an infinite flat plate with variable suction*, Proc. Camb. Phil. Soc. 62 (1966), 329.

[6] PURI, P. : *On unsteady flow of an elastico-viscous fluid past an infinite plate with variable suction*, Appl. Sei. Res. 28 (1973), 111.

[7] PURI, P. : *Rotating flow of an elastico-viscous fluid on an oscillating plate*, ZAMP **54** (1974), 743.

[8] PURI, P. : Fluctuating flow of a viscous fluid on a porous plate in a rotating medium, Acta Mechanica **21** (1975), 153.

[9] THORNBY, C. : On stokes and Rayleigh layers in a rotating system, Quart. J. Mech. Appl. Math. **21** (1968), 451.

ÖZET

Bu çalışmada bir dönel çerçeveye yerleştirilmiş, titreşen düzgün bir levhanın oluşturduğu elastik viskoziteli bir sıvının akışına ait kesin bir çözüm verilmektedir. İkinci kesme geriliminin faz farkı elastik parametreye bağlı olarak artarken, birinci kesme gerilimininkinin azaldığı saptanmaktadır. Viskozite dağılımı gözönüne alınmak suretiyle levhanın ısı transferi problemi, levhanın ısısının, levhanın frekansına eşit bir frekansla titreşmesi halinde incelenmektedir.

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY
GORAKHPUR, INDIA