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ON WAVE SOLUTIONS OF GAUGE INVARIANT GENERALIZATION OF FIELD THEORIES WITH ASYMMETRIC FUNDAMENTAL TENSOR

K.B. LAL - MUSTAQEEM

In this paper wave solutions of gauge-invariant generalizations of field theories with asymmetric fundamental tensor have been obtained.

1. Introduction. Combining the unifield field theories of Weyl [8] and Einstein [3], Buchdahl ([1], [2]) has developed a gauge-invariant theory based on an asymmetric covariant tensor g_{ij} and a covariant vector K_i relating these to an asymmetric linear connection $\Gamma^k{}_{ij}$ in such a way that this theory can be regarded equivalently as 'gauge-invariant generalization of Einstein's non-symmetric theory' or 'the asymmetric generalization of Weyl's theory'.

In Buchdahl's theory [2] the asymmetric tensor $g_{ij} = h_{ij} + f_{ij}$ (where h_{ij} is the symmetric part of g_{ij} which coincides with the fundamental tensor of the metric of the space-time and f_{ij} is the skew-symmetric part of g_{ij}), the vector K_i and the linear connection $\Gamma^k{}_{ij}$ of gauge-invariant theory are defined by

$$L^s{}_{i1}\, g_{sj} + L^s{}_{1j}\, g_{is} - g_{ij,1} = 0\,, \tag{1.1}$$

$$\gamma^s{}_{i1}\, g_{sj} + \gamma^s{}_{1j}\, g_{is} - g_{ij}\, k_1 = 0\,, \tag{1.2}$$

$$\Gamma^i{}_{jk} = L^i{}_{jk} - \gamma^i{}_{jk}\,, \tag{1.3}$$

where $L^i{}_{jk}$ is the linear connection of Einstein's non-symmetric theory [3]. The indices $i, j, k,\ldots$ take the values 1,2,3,4 and a comma (,) before an index i denotes partial differentiation with respect to x^i. The simplest field equations as given by Buchdahl [2] are

$$\Gamma^i \equiv \Gamma^s{}_{\underset{\vee}{is}} = 0\,, \tag{1.4}$$

$$G_{ij} = P_{ij} - (k_{\underset{+}{i};\, j} + k_{\underset{+}{j};\, i}) + \overset{s}{\overset{0}{\gamma}}{}_{\underset{+-}{ij};\, s} + \gamma^s{}_{it}\, \gamma^t{}_{sj} - 2\gamma^s{}_{ij}\, k_s = 0\,, \tag{1.5}$$

where P_{ij} is defined by

$$P_{ij} = -\, L^s{}_{ij,s} + \frac{1}{2}\, \{L^s{}_{\underline{is},\, j} + L^s{}_{\underline{sj},\, i}\} + L^s{}_{it}\, L^t{}_{sj} - L^s{}_{ij}\, L^t{}_{st}\,. \tag{1.6}$$

A semi-colon (;) denotes covariant differentiation with respect to L^k_{ij}, the sign '+' , '—' or 'o' below an index fixes the position of covariant index i in the connection as $L^{\cdot}_{\cdot i}$, $L^{\cdot}_{i\cdot}$ or $L^{\cdot}_{\underset{o}{\cdot} i}$ and a bar (—) and a hook ($\smile$) below the indices denote the symmetry and skew-symmetry respectively between those indices. Tensors G_{ij} and P_{ij} are the hermitian tensors of gauge-invariant and Einstein's usual asymmetric theory respectively.

Further Buchdahl [2] has considered the field equations (1.4) and (1.5) in a static spherically symmetric space-time. Recently Lal and Singh [5] have obtained the solutions of these field equations in a cylindrically symmetric space-time. In the present paper we consider these field equations in a space-time represented by the metric ([4] , [7])

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 + 2A\,dzdt + 2B\,dxdy. \tag{1.7}$$

The non-symmetric tensor g_{ij} [7] is given by

$$(g_{ij}) = \begin{bmatrix} -1 & B & \rho & -\rho \\ B & -1 & \sigma & -\sigma \\ -\rho & -\sigma & -1 & A \\ \rho & \sigma & A & 1 \end{bmatrix}, \tag{1.8}$$

where $A \equiv A\,(z, t)$ and $B \equiv B\,(x, y)$ and ρ, σ are functions of $(z-t)$.

Equation (1.1) is the first equation of Einstein's non-symmetric unified field theory and it has been solved in [7], whose solution gives various components of the non-symmetric connection coefficients of Einstein's unified field theory which together with γ^k_{ij} gives connection coefficients in gauge-invariant theory.

2. Solution of Equation (1.2). Mishra [6] has proved that γ^k_{ij} can be put in the form

$$\gamma^k_{ij} = H^k_{ij} + S^k_{ij} + U^k_{ij}, \tag{2.1}$$

where

$$U^k_{ij} = 2h^{km}\,S^1_{m(i}f_{j)1}, \tag{2.2}$$

$$H^k_{ij} = h^{k1}\,(k_j\,h_{i1} + k_i\,h_{1j} - k_1\,h_{ij})/2, \tag{2.3}$$

$$S^k_{ij} = \gamma^k_{[ij]} = h^{k1}\,(K'_{ij1} + 2U^m_{\ 1[i}f_{j]m}), \tag{2.4}$$

$$\left.\begin{aligned} &\text{a)}\quad K'_{ijk} = K_{ijk} + 2f_{n[i}\,H^n_{\ j]k}, \\ &\text{b)}\quad K_{ijk} = (k_j\,f_{ik} + k_k f_{ij} - k_i\,f_{jk})/2, \end{aligned}\right\} \tag{2.5}$$

where h^{ij}, the reciprocal of h_{ij}, are used to raise the indices of a tensor. Eliminating $U^k{}_{ij}$ between (2.2) and (2.4), as given in [6], we get

$$S_{ijk} = K'_{ijk} + 2f_{[i}{}^{1} f_{j]}{}^{m} S_{k1m} + 2f_{[i}{}^{1} S_{j]1m} f_k{}^m , \tag{2.6}$$

where

$$f_i{}^j = f_{ik} h^{kj} . \tag{2.7}$$

From (1.8) and (2.3) the non-vanishing components of $H^k{}_{ij} (= H^k{}_{ji})$ are

$$H^1{}_{11} = \{k_1 - B(2k_1 B + k_2)\} / 2(1 - B^2) ,$$

$$H^2{}_{22} = \{k_2 - B(2k_2 B + k_1)\} / 2(1 - B^2) ,$$

$$H^3{}_{33} = \{k_3 + A(2k_3 A + k_4)\} / 2(1 + A^2) ,$$

$$H^4{}_{44} = \{k_4 + A(2k_4 A - k_3)\} / 2(1 + A^2) ,$$

$$H^2{}_{12} = - H^2{}_{11} = - H^1{}_{33} = H^2{}_{44} = \frac{1}{A} H^2{}_{34} = (k_1 B + k_2) / 2(1 - B^2) ,$$

$$H^2{}_{12} = - H^1{}_{22} = - H^1{}_{33} = H^1{}_{44} = \frac{1}{A} H^1{}_{34} = (k_2 B + k_1) / 2(1 - B^2) , \tag{2.8}$$

$$H^3{}_{11} = H^2{}_{22} = - H^1{}_{44} = - H^4{}_{34} = - \frac{1}{B} H^2{}_{12} = (k_4 A - k_3) / 2(1 + A^2) ,$$

$$H^4{}_{11} = H^4{}_{22} = H^4{}_{33} = H^3{}_{34} = - \frac{1}{B} H^4{}_{12} = (k_3 A + k_4) / 2(1 + A^2) ,$$

$$H^1{}_{13} = H^2{}_{23} = k_3/2 , \qquad H^3{}_{13} = H^4{}_{14} = k_1/2 ,$$

$$H^1{}_{14} = H^2{}_{24} = k_4/2 , \qquad H^3{}_{23} = H^4{}_{24} = k_2/2 ,$$

$$H^2{}_{13} = H^4{}_{13} = H^2{}_{14} = H^3{}_{14} = H^1{}_{23} = H^4{}_{23} = H^1{}_{24} = H^3{}_{24} = 0 .$$

From (1.8) and (2.5b) the non-vanishing components of $K_{ijk} (= - K_{jik})$ are

$$K_{131} = - K_{141} = \rho k_1 , \; K_{232} = - K_{242} = \sigma k_2 , \; K_{342} = - \sigma(k_3 + k_4) / 2 ,$$

$$K_{132} = - K_{142} = K_{231} = - K_{241} = (\sigma k_1 + \rho k_2) / 2 , \; K_{133} = \rho k_3 ,$$

$$K_{341} = - \rho(k_3 + k_4) / 2 , \; K_{134} = K_{143} = \rho(k_4 - k_3) / 2 , \; K_{233} = \sigma k_3 , \tag{2.9}$$

$$K_{234} = K_{243} = \sigma(k_4 - k_3) / 2 , \; K_{123} = - K_{124} = (\rho k_2 - \sigma k_1) / 2 ,$$

$$K_{144} = - \rho k_4 , \; K_{244} = - \sigma k_4 , \; K_{ijk} = 0 \quad \text{for all} \quad i = j .$$

Using (1.8), (2.8) and (2.9) in (2.5a) the non-vanishing components of $K'_{ijk} (= - K'_{jik})$ are

$$
\begin{aligned}
K'_{131} &= - K'_{141} = (B\rho + \sigma)(k_1 B + k_2) / 2(1 - B^2) , \\
K'_{232} &= - K'_{242} = (B\sigma + \rho)(k_2 B + k_1) / 2(1 - B^2) , \\
K'_{132} &= - K'_{142} = - (B\sigma + \rho)(k_1 B + k_2) / 2(1 - B^2) , \\
K'_{231} &= - K_{241} = - (B\rho + \sigma)(k_2 B + k_1) / 2(1 - B^2) , \\
K'_{121} &= - (B\rho + \sigma) \{ (k_3 + k_4) + A(k_3 - k_4) \} / 2(1 + A^2) , \\
K'_{122} &= (B\sigma + \rho) \{ (k_3 + k_4) + A(k_3 - k_4) \} / 2(1 + A^2) , \\
K'_{133} &= \rho(1 - A)(k_3 A + k_4) / 2(1 + A^2) , \\
K'_{134} &= - \rho(1 + A)(k_3 A + k_4) / 2(1 + A^2) , \\
K'_{143} &= - \rho(1 - A)(k_4 A - k_3) / 2(1 + A^2) , \\
K'_{144} &= \rho(1 + A)(k_4 A - k_3) / 2(1 + A^2) , \\
K'_{233} &= \sigma(1 - A)(k_3 A + k_4) / 2(1 + A^2) , \\
K'_{234} &= - \sigma(1 + A)(k_3 A + k_4) / 2(1 + A^2) , \\
K'_{243} &= - \sigma(1 - A)(k_4 A - k_3) / 2(1 + A^2) , \\
K'_{244} &= \sigma(1 + A)(k_4 A - k_3) / 2(1 + A^2) , \\
K'_{343} &= - (1 - A) \{ \rho (k_2 B + k_1) + \sigma(k_1 B + k_2) \} / 2(1 - B^2) , \\
K'_{344} &= (1 + A) \{ \rho (k_2 B + k_1) + \sigma(k_1 B + k_2) \} / 2(1 - B^2) .
\end{aligned}
\tag{2.10}
$$

From (1.8) and (2.7) the components of f_i^j are given by

$$
(f_i^j) = \begin{bmatrix}
0 & 0 & -\dfrac{\rho(1 + A)}{(1 + A^2)} & -\dfrac{\rho(1 - A)}{(1 + A^2)} \\
0 & 0 & -\dfrac{\sigma(1 + A)}{(1 + A^2)} & -\dfrac{\sigma(1 - A)}{(1 + A^2)} \\
\dfrac{(B\sigma + \rho)}{(1 - B^2)} & \dfrac{(B\rho + \sigma)}{(1 - B^2)} & 0 & 0 \\
-\dfrac{(B\sigma + \rho)}{(1 - B^2)} & -\dfrac{(B\rho + \sigma)}{(1 - B^2)} & 0 & 0
\end{bmatrix} . \tag{2.11}
$$

Using (2.10) and (2.11) in (2.6) we get 24 simultaneous equations. Solving these equations the 24 independent components of S_{ijk} will be given by

$$
\begin{aligned}
&S_{131} = - S_{141} = (B\rho + \sigma)(k_1 B + k_2) / 2(1 - B^2)\,,\\
&S_{232} = - S_{242} = (B\sigma + \rho)(k_2 B + k_1) / 2(1 - B^2)\,,\\
&S_{132} = - S_{142} = - (B\sigma + \rho)(k_1 B + k_2) / 2(1 - B^2)\,,\\
&S_{231} = - S_{241} = - (B\rho + \sigma)(k_2 B + k_1) / 2(1 - B^2)\,,\\
&S_{121} = - (B\rho + \sigma)\{(k_3 + k_4) + A(k_3 - k_4)\} / 2(1 + A^2)\,,\\
&S_{122} = (B\sigma + \rho)\{(k_3 + k_4) + A(k_3 - k_4)\} / 2(1 + A^2)\,,\\
&S_{133} = \rho(1 - A)(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S_{134} = - \rho(1 + A)(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S_{233} = \sigma(1 - A)(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S_{234} = - \sigma(1 + A)(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S_{143} = - \rho(1 - A)(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S_{144} = \rho(1 + A)(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S_{244} = \sigma(1 + A)(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S_{243} = - \sigma(1 - A)(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S_{343} = - (1 - A)\{\rho(k_2 B + k_1) + \sigma(k_1 B + k_2)\} / 2(1 - B^2)\,,\\
&S_{344} = (1 + A)\{\rho(k_2 B + k_1) + \sigma(k_1 B + k_2)\} / 2(1 - B^2)\,,\\
&S_{123} = S_{124} = S_{341} = S_{342} = 0\,.
\end{aligned}
\tag{2.12}
$$

Therefore the components of $S^k{}_{ij} = h^{k1} S_{ij1}$ are

$$
\begin{aligned}
&S^1{}_{12} = \sigma\{(k_3 + k_4) + A(k_3 - k_4)\} / 2(1 + A^2)\,,\\
&S^2{}_{12} = - \rho\{(k_3 + k_4) + A(k_3 - k_4)\} / 2(1 + A^2)\,,\\
&S^1{}_{13} = - S^1{}_{14} = - \sigma(k_1 B + k_2) / 2(1 - B^2)\,,\\
&S^2{}_{13} = - S^2{}_{14} = \rho(k_1 B + k_2) / 2(1 - B^2)\,,\\
&S^3{}_{13} = S^4{}_{13} = - \rho(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S^3{}_{14} = S^4{}_{14} = \rho(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S^1{}_{23} = - S^1{}_{24} = \sigma(k_2 B + k_1) / 2(1 - B^2)\,,\\
&S^2{}_{23} = - S^2{}_{24} = - \rho(k_2 B + k_1) / 2(1 - B^2)\,,\\
&S^3{}_{23} = S^4{}_{23} = - \sigma(k_3 A + k_4) / 2(1 + A^2)\,,\\
&S^3{}_{24} = S^4{}_{24} = \sigma(k_4 A - k_3) / 2(1 + A^2)\,,\\
&S^3{}_{34} = S^4{}_{34} = \{\rho(k_2 B + k_1) + \sigma(k_1 B + k_2)\} / 2(1 - B^2)\,,\\
&S^1{}_{34} = S^2{}_{34} = S^3{}_{12} = S^4{}_{12} = 0\,.
\end{aligned}
\tag{2.13}
$$

Using (1.8) and (2.13) in (2.2) we find that all the components of $U^k{}_{ij}$ are zero. Therefore using (2.8) and (2.13) in (2.1) the components of $\gamma^k{}_{ij}$ are given by

$$
\begin{aligned}
\gamma^k_{11} &= [\ \{k_1 - B(2k_1 B + k_2)\}/2(1-B^2), \quad -(k_1 B + k_2)/2(1-B^2), \\
&\quad (k_4 A - k_3)/2(1+A^2), \quad (k_3 A + k_4)/2(1+A^2)\ ], \\
\gamma^k_{22} &= [-(k_2 B + k_1)/2(1-B^2), \quad \{k_2 - B(2k_2 B + k_1)\}/2(1-B^2), \\
&\quad (k_4 A - k_3)/2(1+A^2), \quad (k_3 A + k_4)/2(1+A^2)\ ], \\
\gamma^k_{33} &= [-(k_2 B + k_1)/2(1-B^2), \quad -(k_1 B + k_2)/2(1-B^2), \\
&\quad \{k_3 + A(2k_3 A + k_4)\}/2(1+A^2), \quad (k_3 A + k_4)/2(1+A^2)\ ], \\
\gamma^k_{44} &= [(k_2 B + k_1)/2(1-B^2), \quad (k_1 B + k_2)/2(1-B^2), \\
&\quad -(k_4 A - k_3)/2(1+A^2), \quad \{k_4 + A(2k_4 A - k_3)\}/2(1+A^2)\ ], \\
\gamma^k_{\substack{12\\21}} &= [(k_1 B + k_2)/2(1-B^2) \pm \sigma\{(k_3+k_4) + A(k_3-k_4)\}/2(1+A^2), \\
&\quad (k_2 B + k_1)/2(1-B^2) \mp \rho\{(k_3+k_4) + A(k_3-k_4)\}/2(1+A^2), \\
&\quad -B(k_4 A - k_3)/2(1+A^2), \quad -B(k_3 A + k_4)/2(1+A^2)\ ],
\end{aligned}
\tag{2.14}
$$

similar expressions for $\gamma^k_{\substack{13\\31}}$, $\gamma^k_{\substack{14\\41}}$, $\gamma^k_{\substack{23\\32}}$, $\gamma^k_{\substack{24\\42}}$, $\gamma^k_{\substack{34\\43}}$ are omitted for brevity's sake.

Thus the solutions of equation (1.2) in the space-time (1.7) are given by (2.14).

3. Solution of Equation (1.4). Using the components of $L^k{}_{ij}$ from [7] and $\gamma^k{}_{ij}$ from (2.14) in equation (1.4) we find that when $i = 1$ and 2, equations $\underset{\vee}{\Gamma^s_{is}} = 0$ give

$$(k_3 + k_4) + A(k_3 - k_4) = 0, \tag{3.1}$$

while when $i = 3$ and 4, equations $\underset{\vee}{\Gamma^s_{is}} = 0$ give

$$\beta(\mu + \nu) + \{\rho(k_2 B + k_1) + \sigma(k_1 B + k_2)\}/(1 - B^2) = 0, \tag{3.2}$$

where

$$\beta = \frac{(\rho + B\sigma)\left(1 + \dfrac{2A\rho^2}{(1+A^2)}\right)}{\left(1 + \dfrac{2A\sigma^2}{(1+A^2)}\right)\left(1 + \dfrac{2A\rho^2}{(1+A^2)}\right) - \left(-B + \dfrac{2A\rho\sigma}{(1+A^2)}\right)^2},$$

and $\mu = -B_1/(1-B^2)$, $\nu = -B_2/(1-B^2)$.

Thus equations (3.1) and (3.2) are necessary conditions in order that Buchdahl's gauge-invariant field equation (1.4) be satisfied.

4. Tensors P_{ij} and G_{ij}. The gauge-invariant Einstein tensor G_{ij}, as given by Buchdahl [2], is

$$G_{ij} = P_{ij} - (k_{i,j} + k_{j,i} - 2k_1 L^1{}_{(ij)}) - 2\,\gamma^1{}_{ij}\,k_1 + \gamma^1{}_{ij,1} + \\ + L^1{}_{\underline{m1}}\,\gamma^m{}_{ij} - L^m{}_{i1}\,\gamma^1{}_{mj} - L^m{}_{1j}\,\gamma^1{}_{im} + \gamma^m{}_{i1}\,\gamma^1{}_{mj}\,. \tag{4.1}$$

Using the components of $L^k{}_{ij}$ from [7] into (1.6) we get the components of P_{ij} as follows:

$$\begin{aligned}
P_{11} &= -\mu_{,2} - B\,\mu\,\nu + 2A\mu\,(BM\mu + L\,\nu)/(1 + A^2) + \\
&\quad + 4A^2 M^2 \mu^2/(1 + A^2)^2 - (2A/(1 + A^2))\,(M\,\mu)_{,1}\,, \\
P_{22} &= -\nu_{,1} - B\,\mu\,\nu + 2A\,\nu\,(M\,\mu + BL\,\nu)/(1 + A^2) + \\
&\quad + 4A^2 L^2 \nu^2/(1 + A^2)^2 - (2A/(1 + A^2))\,(L\,\nu)_{,2}\,, \\
P_{33} &= -\varphi_{,3} - \beta^2(\mu^2 + \nu^2) - 2\alpha^2\,\mu\,\nu - A\,\varphi\,\psi - \\
&\quad - 4\,(M^2\mu^2 + 2BLM\,\mu\nu + L^2\nu^2)/m - \\
&\quad - 2B\,\{M\,\mu^2 + B\,\mu\,\nu\,(L + M) + L\,\nu^2\}/(1 - B^2) - \\
&\quad - 2\left\{\frac{(M\mu + BL\,\nu)}{(1 - B^2)}\right\}_{,1} - 2\left\{\frac{(BM\mu + L\nu)}{(1 - B^2)}\right\}_{,2}, \\
P_{44} &= \varphi_{,3} - \beta^2(\mu^2 + \nu^2) - 2\alpha^2\,\mu\,\nu + A\,\varphi\,\psi + \\
&\quad + 4(M^2\mu^2 + 2BLM\,\mu\nu + L^2\nu^2)/m - \\
&\quad - 2B\,\{M\,\mu^2 + B\,\mu\,\nu\,(L + M) + L\,\nu^2\}/(1 - B^2) - \\
&\quad - 2\left\{\frac{(M\mu + BL\,\nu)}{(1 - B^2)}\right\}_{,1} - 2\left\{\frac{(BM\mu + L\nu)}{(1 - B^2)}\right\}_{,2}, \\
P_{12} &= P_{21} = (B\mu)_{,2} + \mu\,\nu + 4A^2 LM\,\mu\,\nu/(1 + A^2)^2 - \\
&\quad - (A/(1 + A^2))\,\{(L\,\nu)_{,1} + (M\mu)_{,2}\}\,, \\
P_{34} &= P_{43} = (A\,\varphi)_{,3} + \beta^2(\mu^2 + \nu^2) + 2\alpha^2\,\mu\,\nu - \varphi\,\psi + \\
&\quad + 4A\,(M^2\mu^2 + 2BLM\,\mu\nu + L^2\nu^2)/m\,, \\
P_{13} &= -J_1 + J_3\,, \quad P_{31} = J_1 + J_3\,, \\
P_{14} &= J_1 + J_4\,, \quad P_{41} = -J_1 + J_4\,, \\
P_{23} &= -J_2 + J_5\,, \quad P_{32} = J_2 + J_5\,, \\
P_{24} &= J_2 + J_6\,, \quad P_{42} = -J_2 + J_6\,,
\end{aligned} \tag{4.2}$$

where

$$L = \rho\alpha + \sigma\beta\,, \quad M = \sigma\alpha + \rho\beta\,, \quad \varphi = A_4/(1+A^2)\,, \quad \psi = A_3/(1+A^2)\,,$$

$$m = (1+A^2)(1-B^2)\,,$$

$$J_1 = (\beta\mu)_{,1} + (\alpha\mu)_{,2} - (1-B)\,\alpha\,\mu\,\nu\,,$$

$$J_2 = (\alpha\nu)_{,1} + (\beta\nu)_{,2} - (1-B)\,\alpha\,\mu\,\nu\,,$$

$$J_3 = \mu\,[M\{(1-A)A\varphi + (1+A)\psi\}/(1+A^2) + + (M/(1+A^2))_{,3} + ((1-A)M/(1+A^2))_{,4}]\,,$$

$$J_4 = \mu\,[M\{(1-A)\varphi - (1+A)A\psi\}/(1+A^2) - - ((1+A)M/(1+A^2))_{,3} - (M/(1+A^2))_{,4}]\,,$$

$$J_5 = \nu\,[L\{(1-A)A\varphi + (1+A)\psi\}/(1+A^2) + + (L/(1+A^2))_{,3} + ((1-A)L/(1+A^2))_{,4}]\,,$$

$$J_6 = \nu\,[L\{(1-A)\varphi - (1+A)A\psi\}/(1+A^2) - - ((1+A)L/(1+A^2))_{,3} - (L/(1+A^2))_{,4}]\,,$$

$$\alpha = \frac{-(\rho + B\sigma)\left(-B + \dfrac{2A\rho\sigma}{(1+A^2)}\right)}{\left(1 + \dfrac{2A\sigma^2}{(1+A^2)}\right)\left(1 + \dfrac{2A\rho^2}{(1+A^2)}\right) - \left(-B + \dfrac{2A\rho\sigma}{(1+A^2)}\right)^2}\,,$$

and the Indices 1,2,3,4 after a letter denote partial differentiation with respect to x, y, z and t respectively.

The components of $L^s_{\underline{is}}$ are given by

$$\begin{aligned}
&L^1_{11} = B\mu\,, \quad L^2_{11} = \mu\,, \quad L^1_{22} = \nu\,, \quad L^2_{22} = B\nu\,,\\
&L^1_{33} = L^1_{44} = 2\,(M\mu + BL\nu)/(1-B^2)\,,\\
&L^2_{33} = L^2_{44} = 2\,(BM\mu + L\nu)/(1-B^2)\,,\\
&L^3_{33} = A\psi\,, \quad L^4_{33} = \psi\,,\\
&L^3_{44} = -\varphi\,, \quad L^4_{44} = A\varphi\,,\\
&L^3_{\underline{13}} = -L^3_{\underline{14}} = -(1+A)M\mu/(1+A^2)\,,\\
&L^4_{\underline{13}} = -L^4_{\underline{14}} = -(1-A)M\mu/(1+A^2)\,,\\
&L^3_{\underline{23}} = -L^3_{\underline{24}} = -(1+A)L\nu/(1+A^2)\,,\\
&L^4_{\underline{23}} = -L^4_{\underline{24}} = -(1-A)L\nu/(1+A^2)\,.
\end{aligned} \tag{4.3}$$

Substituting from equations (2.14), (3.1), (3.2), (4.3) and the components of $L^k{}_{ij}$ from [7] in equation (4.1), we find the components of G_{ij} as follows:

$$G_{11} = P_{11} + I_1 - \frac{3}{2} k_{1,1} + \frac{k_1^2}{2} - \{B(k_1 B + k_2) / 2(1 - B^2)\}_{,1} -$$

$$- \{(k_1 B + k_2)/2(1 - B^2)\}_{,2} - \{k_1 - B(2k_1 B + k_2)\} \cdot \frac{k_1 + AM\mu/(1+A^2) + B\mu/2}{1 - A^2} +$$

$$+ (k_1 B + k_2) \cdot \frac{k_2 - B\nu/2 + AL\nu/(1 + A^2) + \mu(1 - 2B^2)}{1 - B^2} +$$

$$+ (k_4 A - k_3) \cdot \frac{A(\varphi + \psi) - (k_3 + k_4)}{2(1 + A^2)} + \frac{k_1 - B(2k_1 B + k_2)^2 + (k_2 B + k_1)^2}{4(1 - B^2)^2} -$$

$$- \frac{(k_1 B + k_2)^2}{2(1 - B^2)^2} + \frac{\rho^2(k_4 A - k_3)^2}{4(1 + A^2)^2},$$

$$G_{22} = P_{22} + I_1 - \frac{3}{2} k_{2,2} + \frac{k_2^2}{2} + 2\mu(k_1 B + k_2) + \frac{2AL\nu k_2}{1 + A^2} + \tag{4.4}$$

$$+ \frac{(k_4 A - k_3)\{A(\varphi + \psi) - (k_3 + k_4)\}}{2(1 + A^2)} - \{(k_1 B + k_1) / 2(1 - B^2)\}_{,1} -$$

$$- \{B(k_2 B + k_1) / 2(1 - B^2)\}_{,2} + \frac{\{k_2 - B(2k_2 B + k_1)\} + (k_1 B + k_2)^2}{4(1 - B^2)^2} -$$

$$- \frac{\sigma^2(k_4 A - k_3)^2}{(1 + A^2)^2} - \frac{\{k_2 - B(2k_2 B + k_1)\}(B\nu + 2k_2)}{2(1 - B^2)} +$$

$$+ \frac{(k_2 B + k_1)(2k_1 - 2\nu - B\mu)}{2(1 - B^2)} - \frac{(k_2 B + k_1)^2}{2(1 - B^2)^2} +$$

$$+ A \frac{M\mu(k_2 B + k_1) - L\nu\{k_2 - B(2k_2 B + k_1)\}}{m},$$

$$G_{12} = G_{21} = P_{12} - BI_1 - (k_{1,2} + k_{2,1}) + \frac{k_1 k_2}{2} - \frac{3}{4} \cdot \frac{B(k_1^2 + k_2^2) + 2k_1 k_2}{1 - B^2} -$$

$$- \frac{B(k_4 A - k_3)\{A(\varphi + \psi) - (k_3 + k_4)\}}{2(1 + A^2)} + \{(k_1 B + k_2) / 2(1 - B^2)\}_{,1} +$$

$$+ \{(k_2 B + k_1) / 2(1 - B^2)\}_{,2} - \frac{AB\{k_1(M\mu + BL\nu) + k_2(L\nu + BM\mu)\}}{m} +$$

$$+ \frac{\mu(k_2 B + k_1) + \nu(k_1 B + k_2)}{2(1 - B^2)} + \frac{(k_1 B + k_2)(k_2 B + k_1)}{2(1 - B^2)^2} -$$

$$- \frac{(k_1^2 + k_2^2)(1 + B^2) + 4B k_1 k_2}{4(1 - B^2)^2},$$

similar equations for G_{33}, G_{44}, G_{13}, G_{31}, G_{14}, G_{41}, G_{23}, G_{32}, G_{24}, G_{42}, G_{34} and G_{43} are omitted for brevity's sake, where

$$I_1 = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left\{\frac{(k_4 A - k_3)}{(1 + A^2)}\right\},$$

$$I_2 = \frac{1}{2}\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial t}\right)\left\{\frac{(k_4 A - k_3)}{(1 + A^2)}\right\},$$

$$I_3 = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left\{\frac{\rho (k_4 A - k_3)}{(1 + A^2)}\right\},$$

$$I_4 = \frac{1}{2}\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left\{\frac{\sigma (k_4 A - k_3)}{(1 + A^2)}\right\},$$

$$I_5 = \frac{3}{4}(k_4 A - k_3)(k_3 + k_4)/(1 + A^2).$$

5. Solution of Equation (1.5). Using (4.4) in (1.5), we get

$$P_{11} + I_1 - \frac{3}{2}k_{1,1} - \{B(k_1 B + k_2)/2(1 - B^2)\}_{,1} + \frac{k_1^2}{2} -$$

$$- \{(k_1 B + k_2)/2(1 - B^2)\}_{,2} - \frac{\{k_1 - B(2k_1 B + k_2)\}\{k_1 + AM\mu/(1 - A^2) + B\mu/2\}}{(1 + A^2)} +$$

$$+ \frac{(k_1 B + k_2)\{k_2 - B\nu/2 + AL\nu/(1 + A^2) + \mu(1 - 2B^2)\}}{1 - B^2} +$$

$$+ \frac{(k_4 A - k_3)\{A(\varphi + \psi) - (k_3 + k_4)\}}{2(1 + A^2)} + \frac{\{k_1 - B(2k_1 B + k_2)^2\} + (k_2 B + k_1)^2}{4(1 - B^2)^2} -$$

$$- \frac{(k_1 B + k_2)^2}{2(1 - B^2)^2} + \frac{\rho^2 (k_4 A - k_3)^2}{4(1 + A^2)^2} = 0,$$

$$P_{22} + I_1 - \frac{3}{2}k_{2,2} + \frac{k_2^2}{2} + 2\mu(k_1 B + k_2) + \frac{2AL\nu k_2}{1 + A^2} + \qquad (5.1)$$

$$+ \frac{(k_4 A - k_3)\{A(\varphi + \psi) - (k_3 + k_4)\}}{2(1 + A^2)} - \{(k_2 B + k_1)/2(1 - B^2)\}_{,1} -$$

$$- \{B(k_2 B + k_1)/2(1 - B^2)\}_{,2} + \frac{\{k_2 - B(2k_2 B + k_1)\} + (k_1 B + k_2)^2}{4(1 - B^2)^2} -$$

$$- \frac{\sigma^2 (k_4 A - k_3)^2}{(1 + A^2)^2} - \frac{\{k_2 - B(2k_2 B + k_1)\}(B\nu + 2k_2)}{2(1 - B^2)} +$$

$$+ \frac{(k_2B + k_1)(2k_1 - 2\nu - B\mu)}{2(1 - B^2)} - \frac{(k_2B + k_1)^2}{2(1 - B^2)^2} +$$

$$+ \frac{A[M\mu(k_2B + k_1) - L\nu\{k_2 - B(2k_2B + k_1)\}]}{m} = 0,$$

$$P_{12} - BI_1 - (k_{1,2} + k_{2,1}) + \frac{k_1 k_2}{2} - \frac{3}{4} \cdot \frac{B(k_1^2 + k_2^2) + 2k_1 k_2}{1 - B^2} -$$

$$- \frac{B(k_4A - k_3)\{A(\varphi + \psi) - (k_3 + k_4)\}}{2(1 + A^2)} + \{(k_1B + k_2)/2(1 - B^2)\}_{,1} +$$

$$+ \{(k_2B + k_1)/2(1 - B^2)\}_{,2} - \frac{AB\{k_1(M\mu + BL\nu) + k_2(L\nu + BM\mu)\}}{m} +$$

$$+ \frac{\mu(k_2B + k_1) + \nu(k_1B + k_2)}{2(1 - B^2)} + \frac{(k_1B + k_2)(k_2B + k_1)}{2(1 - B^2)^2} -$$

$$- \frac{(k_1^2 + k_2^2)(1 + B^2) + 4Bk_1 k_2}{4(1 - B^2)^2} = 0, \text{ etc.}$$

There are various possibilities under which the solution of field equation (1.5) may be considered. However we shall consider the solution in case when

$$k_1 = k_2 = 0, \; k_3 \text{ and } k_4 \text{ are functions of } (z - t), \tag{5.2}$$

$$A = A(z - t), \; B = B(x - y). \tag{5.3}$$

With this choice of k_i, A and B, equation (3.2) is identically satisfied. Using (5.2) and (5.3), equations (5.1) reduce to

$$P_{11} = -\frac{P_{12}}{B} - \frac{\rho^2(k_4A - k_3)^2}{4(1 + A^2)^2},$$

$$P_{22} = -\frac{P_{12}}{B} + \frac{\sigma^2(k_4A - k_3)^2}{(1 + A^2)^2},$$

$$P_{33} = \frac{3}{2}k_{3,3} + \frac{k_3^2}{4} - \{A(k_4A - k_3)/2(1 + A^2)\}_{,3} -$$

$$- \{(k_4A - k_3)/2(1 + A^2)\}_{,4} + \frac{(k_4A - k_3)(k_3A + 3k_4 + 4A_{,4})}{4(1 + A^2)},$$

$$P_{44} = \frac{3}{2}k_{4,4} + \frac{k_4^2}{4} + \{(k_4A - k_3)/2(1 + A^2)\}_{,3} -$$

$$- \{A(k_4A - k_3)/2(1 + A^2)\}_{,4} - \frac{(k_4A - k_3)\{(1 - A)k_3 + 4k_4\}}{4(1 + A^2)},$$

$$P_{13} = -\rho I_5 + \frac{(k_4 A - k_3)\{\rho A(k_4 A - k_3 + A_4) - (3 + A^2) M\nu\}}{2(1 + A^2)^2}, \tag{5.4}$$

$$P_{31} = \rho I_5 + \frac{\rho(k_4 A - k_3)\{(1 - A) A_4 - A(k_4 A - k_3)\}}{2(1 + A^2)^2} - \frac{M\nu(k_4 A - k_3)}{1 + A^2},$$

$$P_{14} = -\frac{\rho P_{12}}{B} + \frac{\rho(1 + A) A_4 (k_4 A - k_3)}{2(1 + A^2)^2} - \frac{M\nu(k_3 A + 2k_4)}{1 + A^2};$$

$$P_{41} = \frac{\rho P_{12}}{B} - \frac{\rho(1 - A) A_4 (k_4 A - k_3)}{2(1 + A^2)^2} - \frac{M\nu(k_4 A - k_3)}{1 + A^2} + \frac{B\mu k_4}{2},$$

$$P_{23} + P_{32} = \frac{2L\nu(k_4 A - k_3)}{1 + A^2},$$

$$P_{23} - P_{32} = -2\sigma I_5 + \frac{\sigma(k_4 A - k_3)\{A(k_4 A - k_3) - 2(1 - A) A_4\}}{2(1 + A^2)^2},$$

$$P_{24} = -\frac{\sigma P_{12}}{B} + \frac{(k_4 A - k_3)\{\sigma(1 + A) A_4 / (1 + A^2) + \alpha\nu\rho\}}{2(1 + A^2)} +$$

$$+ \frac{L\nu\{(k_4 A - k_3)(1 + 5A^2) + k_4 A(1 + A^2)\}}{2(1 + A^2)^2},$$

$$P_{42} = \frac{\sigma P_{12}}{B} - \frac{\sigma(1 + A) A_4 (k_4 A - k_3)}{2(1 + A^2)^2} + \frac{(1 + A) L\nu k_4}{1 + A^2},$$

$$P_{34} = (k_{3,4} + k_{4,3}) + I_5 - I_2 - \frac{k_3 k_4}{2} + \frac{(k_4 A - k_3)^2}{2(1 + A^2)^2}$$

and

$$(B\rho + \sigma) L = (B\sigma + \rho) M,$$

where

$$I_2 = \frac{\partial}{\partial z}\left\{\frac{(k_4 A - k_3)}{(1 + A^2)}\right\} \quad \text{and} \quad I_5 = \frac{3}{4} \cdot \frac{(k_4 A - k_3)(k_3 + k_4)}{(1 + A^2)}.$$

Thus the solutions of Buchdahl's field equations are given by equations (1.8), (3.1), (5.2), (5.3) and (5.4).

When $k_i = 0$, we have the solutions of Einstein's strong field equations discussed in [7].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GORAKHPUR
U.P., INDIA

ÖZET

Bu çalışmada, temel tensörü asimetrik olan alan kuramlarının invaryant ölçümlü genelleştirmelerinin dalga çözümleri elde edilmektedir.