İstanbul Üniv. Fen Fak. Mec. Seri A, 43 (1978), 181-188

SOME THEOREMS ON AFFINE MOTION IN A RECURRENT FINSLER SPACE

A. KUMAR

In this paper it has been investigated the case $\gamma_s v^s = 0$ occured when $\phi_h \neq$ gradient vector and hence completed authors earlier results on affine motions in recurrent Finsler space.

1. Introduction. Let us consider an n-dlmensional affinelly connected and non-flat Finsler space Fn [6] [1] having symmetric Cartan's connection coefficient $\Gamma_{hk}^{*i}(x, \dot{x})$. The covariant derivative of any tensor field $T_j^i(x, \dot{x})$ with respect to $\Gamma_{hk}^{*i}(x, \dot{x})$ is given by

$$T^i_{j|k} = \partial_k T_j^i - \dot{\partial}_s T_j^i G_k^s + T_j^s \Gamma_{sk}^{*i} - T_s^i \Gamma_{jk}^{*s} \,{}^{2)} . \tag{1.1}$$

The well known commutation formula involving the Cartan's covariant derivative is given by

$$2\, T^i_{j\,|\,|hk]} = -\, \dot{\partial}_\gamma K^\gamma_{shk} \dot{x}^s + T_j^s K^i_{shk} - T_s^i K^s_{jhk} \,{}^{3)} , \tag{1.2}$$

where

$$K^i_{hjk}(x, \dot{x}) \overset{\text{def.}}{=\!=} 2\, \{\partial_{[k} \Gamma_{j]h}^{*i} - \dot{\partial}_s \Gamma_{h[j}^{*i} G^s_{k]} + \Gamma_{h[j}^{*s} \Gamma_{k]s}^{*i}\} \tag{1.3}$$

is called Cartan's curvature tensor field and satisfies the following identities [6]:

$$K^i_{hjk} + K^i_{jkh} + K^i_{khj} = 0, \tag{1.4}$$

$$K_{hj} = K^i_{hji} \tag{1.5}$$

and

$$K^i_{hjk} = -\, K^i_{hkj} . \tag{1.6}$$

Let us consider an infinitisimal point transformation

$$\bar{x}^i = x^i + v^i(x)\, dt, \tag{1.7}$$

---

[1] Numbers in square brackets refer to the references given at the end of the paper.

[2] $\partial_k \equiv \partial/\partial x^k$ and $\dot{\partial}_h \equiv \partial/\partial \dot{x}^h$.

[3] $2A_{[hk]} = A_{hk} - A_{kh}$.

where $v^i(x)$ is any vector field and dt is an infinitesimal point constant. In view of the above point transformation and Cartan's covariant derivative the Lie-derivatives of T_j^i and Γ_{hk}^{*i} are given by

$$\pounds v\, T_j^i = T^i_{j\,|\,h} v^h - T_j^h v^i_{\,|\,h} + T_h^i v^h_{\,|\,j} + \dot\partial_h T_j^i v^h_{\,|\,\gamma}\, \dot x^\gamma \tag{1.8}$$

and

$$\pounds v\, \Gamma_{hk}^{*i} = v^i_{\,|\,jk} - K^i_{jkh} v^h + \dot\partial_s \Gamma_{jk}^{*i} v^s_{\,|\,\gamma}\, \dot x^\gamma \tag{1.9}$$

respectively.

In an Fn, if the Cartan's curvature tensor field K^i_{hjk} satisfies the relation

$$K^i_{hjk\,|\,s} = \gamma_s K^i_{hjk}, \tag{1.10}$$

where $\gamma_s(x)$ is any vector field, then the space is called recurrent Finsler space of first order and ∂_s is called recurrence vector field.

In a recurrent Finsler space, when we consider an infinitesimal affine motion of the general form [4]:

$$\bar x^i = x^i + v^i(x)\,dt,\ v^i_{\,|\,h} = \phi_h(x)\, v^i, \tag{1.11}$$

there exists a case characterized by

$$\phi_s v^s = \text{const.} \tag{1.12}$$

under the assumptions

$$\text{a) } K_{hs} v^s = 0 \quad \text{and} \quad \text{b) } (\gamma_s + \phi_s) v^s \neq 0. \tag{1.13}$$

In view of the equation (1.12), we can find

$$(\phi_{h\,|\,k} - \phi_{k\,|\,h}) \gamma_s v^s = 0\,. \tag{1.14}$$

From this relation we can have here two cases :

$$\text{a) } \gamma_s v^s = 0 \quad \text{and} \quad \text{b) } \phi_h = \text{gradient vector.} \tag{1.15}$$

The present author has already shown these facts in his paper [4] and, under $\gamma_s v^s \neq 0$ discussed about (1.15b) deeply, but he has kept mum about the former case (1.15a) occured when $\phi_h \neq$ gradient vector. So in this short paper we shall try to investigate this case and bring our long study on affine motions in recurrent Finsler space to a close.

2. Characteristic condition. If a recurrent Finsler space admits an infinitesimal affine motion (1.7), with respect to the vector field $v^i(x)$, we have

$$\text{a)}\quad \Gamma_{jk}{}^{*i} = 0 \quad \text{i.e.} \quad \text{b)}\quad v^i{}_{|jk} = K^i{}_{jkh} v^h + \dot{\partial}_s \Gamma_{jk}{}^{*i} v^s{}_{|\gamma} \dot{x}^\gamma \tag{2.1}$$

and its integrability condition :

$$\pounds v\, K^i{}_{jkh} = 0 \,. \tag{2.2}$$

In the present case, we assume (2.1) and (2.2) under which we devise to find the characteristic condition for the affine motion (1.11) with (1.15a).

By virtue of the definition (1.8), the Lie-derivative of the Cartan's curvature tensor field $K^i{}_{hjk}(x, \dot{x})$ can be written as

$$\begin{aligned}\pounds v\, K^i{}_{hjk} = K^i{}_{hjk\,|\,s} v^s - K^s{}_{hjk} v^i{}_{|h} + K^i{}_{sjk} v^s{}_{|h} + K^i{}_{hsk} v^s{}_{|j} + K^i{}_{hjs} v^s{}_{|k} + \\ + \dot{\partial}_s K^i{}_{hjk} v^s{}_{|\gamma} \dot{x}^\gamma .\end{aligned} \tag{2.3}$$

In view of the commutation formula (1.2) and the equations (1.10), (1.11) and (2.1b) we find

$$\begin{aligned}\pounds v\, K^i{}_{hjk} &= \gamma_s v^s K^i{}_{hjk} - K^s{}_{hjk} v^i \phi_s + K^i{}_{sjk} v^s \phi_h + K^i{}_{hsk} v^s \phi_j + K^i{}_{hjs} v^s \phi_h = 0 \quad (2.4)\\ &= \gamma_s v^s K^i{}_{hjk} - v^i (\phi_{h|jk} - \phi_{h|kj}) + \phi_h (v^i{}_{|jk} - v^i{}_{|kj}) - \phi_j v^i{}_{|hk} + \phi_k v^i{}_{|hj} = 0\\ &= \gamma_s v^s K^i{}_{hjk} - v^i (\phi_{h|jk} - \phi_{h|kj}) + \phi_h (v^i{}_{|jk} - v^i{}_{|kj}) -\\ &\quad - \phi_j (\phi_{h|k} + \phi_h \phi_k) v^i + \phi_k (\phi_{h|j} + \phi_h \phi_j) v^i = 0\\ &= \gamma_s v^s K^i{}_{hjk} - (\phi_{h|j} + \phi_h \phi_j)_{|k} v^i + (\phi_{h|k} + \phi_h \phi_k) v^i = 0\\ &= \gamma_s v^s K^i{}_{hjk} - v^i \{ (\phi_{h|j} + \phi_h \phi_j)_{|k} - (\phi_{h|k} + \phi_h \phi_h)_{|j} \} = 0.\end{aligned}$$

Consequently, if (1.15a) will be the case the last formula yields

$$(\phi_{h|j} + \phi_h \phi_j)_{|k} = (\phi_{h|k} + \phi_h \phi_k)_{|j} \,. \tag{2.5}$$

And, if we have (2.5), we can have

$$\gamma_s v^s K^i{}_{hjk} = 0 \,. \tag{2.6}$$

Since the space under consideration is a non-flat one, therefore, the above equation reduces to (1.15a). In this way, we have :

Conclusion 2.1a. In a general recurrent Finsler space, when the equation of affine motion (2.1a) is integrable, in order that the motion (1.11) has a form satisfying (1.15a), it is necessary and sufficient that we have the commutative condition (2.5).

Conclusion 2.1b. In a general recurrent Finsler space, in order that the equation (2.1a) of affine motion (1.11) with (1.15a) is integrable, it is necessary and sufficient that we have the condition (2.5).

3. Continued discussion. In an affinelly connected Finsler space, the Bianchi's second identity for Cartan's curvature tensor $K^i{}_{hjk}$ takes the form :

$$K^i{}_{hjk|s} + K^i{}_{hks|j} + K^i{}_{hsj|k} = 0. \tag{3.1}$$

In view of basic definition (1.10), the above equality reduces to

$$\gamma_s K^i{}_{hjk} + \gamma_j K^i{}_{hks} + \gamma_k K^i{}_{hsj} = 0. \tag{3.2}$$

Transvecting the above identity by v^s and taking care of the hypothesis (1.15a), we find

$$\gamma_j K^i{}_{hks} v^s = \gamma_k K^i{}_{hjs} v^s, \tag{3.3}$$

where we have used (1.6).

In view of the basic assumption (2.ab), the last formula can be re-written as

$$\gamma_j (v^i{}_{|hk} - \dot{\partial}_s \Gamma_{hk}^{*i} v^s{}_{|\gamma} \dot{x}^\gamma) = \gamma_k (v^i{}_{|hj} - \dot{\partial}_s \Gamma_{hj}^{*i} v^s{}_{|\gamma} \dot{x}^\gamma). \tag{3.4}$$

Remembering the latter part of (1.11), the above equality yields

$$\gamma_j (\phi_{h|k} + \phi_h \phi_k) v^i = \gamma_k (\phi_{h|j} + \phi_h \phi_j). \tag{3.5}$$

Neglecting the non-zero $v^i(x)$ from the above formula, we obtain

$$\gamma_j (\phi_{h|k} + \phi_h \phi_k) = \gamma_k (\phi_{h|j} + \phi_h \phi_j). \tag{3.6}$$

Conversely, if we have (3.6), multiply it by v^i and noting (1.11), we have (3.4) i.e. (3.3). In view of (3.2) and (3.3), we get (2.6) from which we have (1.15a).

Thus, we have :

Conclusion 3.1. When a recurrent Finsler space admits an affine motion of the general recurrent form (1.11), in order to have (1.15a) always, it is necessary and sufficient that we have (3.6). Thus upon the same fact, we have obtained two conclusions (2.1) and (3.1). In what follows we shall discuss about the mutual relation existing between these conclusions. For this purpose we shall be given our study from the second conclusion, i.e. (3.6).

The recurrent Finsler space under consideration is not a symmetric one, say $\gamma_s \neq 0$, so according to the assumption (3.6), we can suppose a suitable vector B_h such that

$$\phi_{h|k} + \phi_h \phi_k = B_h \gamma_k. \tag{3.7}$$

Introducing the latter part of the basic condition (1.11) into the above equation, we find

$$v^i{}_{|hk} = B_h \Upsilon_k v^i. \tag{3.8}$$

Differentiating (3.2) covariantly with respect to x^m in the sense of Cartan and using the fundamental assumption (1.10) and the equality (3.2) itself, we get

$$\Upsilon_{s|m} K^i{}_{hjk} + \Upsilon_{j|m} K^i{}_{hks} + \Upsilon_{k|m} K^i{}_{hsj} = 0. \tag{3.9}$$

On the other hand, taking the covariant derivative of the basic supposition (1.15a) with respect to x^m and taking notice of the latter part of (1.11) and (1.15a) itself, we obtain

$$\Upsilon_{s|m} v^s + \Upsilon_s v^s \Upsilon_m = 0 \tag{3.10}$$

or

$$\Upsilon_{s|m} v^s = 0. \tag{3.11}$$

Now, transvecting the equality (3.1) by and noting the relations (2.1b) and (3.11), we have

$$\Upsilon_{s|m} (v^i{}_{|hj} - \dot{\partial}_n \Gamma_{hj}^{*i} v^n{}_{|\Upsilon} \dot{x}^\Upsilon) = \Upsilon_{j|m} (v^i{}_{|hs} - \dot{\partial}_n \Gamma_{hs}^{*i} v^n{}_{|\Upsilon} \dot{x}^\Upsilon). \tag{3.12}$$

By virtue of the equations (1.11) and (3.8), the last formula takes the form :

$$(\Upsilon_{s|m} \Upsilon_j - \Upsilon_{j|m} \Upsilon_s) B_h = 0, \tag{3.13}$$

where we have neglected the non-zero $v^i(x)$.

In view of the above equality, we can find here two cases :

$$\text{a)} \quad B_h = 0 \quad \text{and} \quad \text{b)} \quad \Upsilon_{s|m} \Upsilon_j = \Upsilon_{j|m} \Upsilon_s. \tag{3.14}$$

4. Essential discussion. The above all discussions stand only on the assumption (1.15a) derived from $\phi_h v^h =$ const. under $(\phi_{h|k} \neq \phi_{k|h})$. In the present case of (1.15a), we shall conversely show that $\phi_h v^h =$ const. holds good.

Now, if (3.14a) will be the case, from (3.7), we can have

$$(\phi_{h|k} + \phi_h \phi_k) v^h = (B_h v^h) \Upsilon_k = 0, \quad \text{i.e.} \quad \phi_h v^h = \text{const.} \tag{4.1}$$

For the non vanishing property of v^h the above formula reduces to

$$\phi_{h|k} + \phi_h \phi_k = 0. \tag{4.2}$$

In view of the last formula we can get (1.15b). However, under $\phi_h v^h$=const, we can consider only a case in which we have $\phi_{h|k} \neq \phi_{k|h}$ and (1.15a), so (3.14a) must be an exceptional condition.

Then let us consider now (3.14b). In this case, at first, we have to test also the fact : $\phi_h v^h =$ const. Before the discussion of this fact, we shall find a few important conditions. In such a case, we can suppose the existence of a covariant vector E_k such as

$$\gamma_{h|k} = \gamma_h E_k . \tag{4.3}$$

Now, in view of the latter part of (1.11), comparing the equations (2.1b) and (3.8), we can deduce

$$K^i_{jkh} v^h = B_j \gamma_k v^i . \tag{4.4}$$

Differentiating the above formula covariantly with respect to x^m and using the fundamental definition (1.10) and the latter part of (1.11), we get

$$(\gamma_m + \phi_m) K^i_{jkh} v^h = B_{j|m} \gamma_k v^i + B_j \gamma_{k|m} v^i + B_j \gamma_k \phi_m v^i . \tag{4.5}$$

With the help of the equations (4.3) and (4.4), the last formula can be re-written as

$$(\gamma_m + \phi_m) B_j \gamma_k v^i + B_{j|m} \gamma_k v^i + B_j \gamma_k E_m v^i + B_j \gamma_k \phi_m v^i . \tag{4.6}$$

Now, neglecting the non-zero terms γ_k and v^i from the last formula, we find

$$B_{j|m} = (\gamma_m - E_m) B_j . \tag{4.7}$$

By virtue of the equations (3.7), (4.3) and (4.7), we can deduce the following interesting relation :

$$(\phi_{h|k} + \phi_h \phi_k)_{|m} = (B_h \gamma_k)_{|m} = (\gamma_m - E_m) B_h \gamma_k + B_h \gamma_k E_m \tag{4.8}$$

or

$$(\phi_{h|k} + \phi_h \phi_k)_{|m} = B_h \gamma_k \gamma_m . \tag{4.9}$$

This yields the condition (2.5). In view of (3.7), the above equality can be re-written as

$$(\phi_h + \phi_h \phi_k)_{|m} = \gamma_m (\phi_{h|k} + \phi_h \phi_k) . \tag{4.10}$$

Now, we shall discuss about the condition $\phi_h v^h = \text{const.}$. As the author has shown in [4], when we consider an affine motion of recurrent form (1.11), we have always

$$\text{a)} \quad \pounds v \, \phi_k = 0, \quad \text{b)} \quad \pounds v \, \gamma_h = 0 \quad \text{and} \quad \text{c)} \quad \pounds v \, \phi_{h|k} = (\pounds v \, \phi_k)_{|k} = 0 . \tag{4.11}$$

Next, operating the both sides of (3.7) by $\pounds v$ and using the last identities, we get

$$\gamma_k \pounds v \, B_h = 0 \quad \text{or} \quad \pounds v \, B_h = 0 . \tag{4.12}$$

In view of the fundamental formula (1.8) and the latter part of (1.11), the last relation can be written as

$$\pounds v \, B_h = B_{h|k} v^k + B_s v^s{}_{|h} = B_{h|k} v^k + B_s v^s \phi_h = 0 . \tag{4.13}$$

With the help of (1.8) and (4.11b), we can have

$$\gamma_{h|k}\, v^k + \gamma_s\, v^s\, \phi_h = 0\,. \tag{4.14}$$

By virtue of basic condition (1.15a), the above result takes the form :

$$\gamma_{h|k}\, v^k = 0\,. \tag{4.15}$$

Transvecting the formula (4.3) by v^k and taking notice of the above result, we get

$$\gamma_h\, E_k\, v^k = 0\,, \quad \text{i.e.} \quad E_k\, v^k = 0\,. \tag{4.16}$$

Next, let us multiply the equation (4.7) by v^m and using the assumption (1.15a) and (4.16), we obtain

$$B_{j)m}\, v^m = 0 \cdot \tag{4.17}$$

Hence, when we have (1.15a), we can get an important property :

$$\phi_j\, B_h\, v^h = 0 \quad \text{or} \quad B_h\, v^h = 0\,. \tag{4.18}$$

Thus, transvecting the formula (3.7) by v^h and noting (1.11) and the above relation, we find

$$\phi_{h|k}\, v^h + \phi_h\, \phi_k\, v^h = 0\,, \tag{4.19}$$

i.e.

$$(\phi_h\, v^h)_{|k} = 0 \quad \text{or} \quad \phi_k\, v^h = \text{const.}. \tag{4.20}$$

This completes the proof of $\phi_h\, v^h = \text{const.}$.

Now under (3.14b), i.e. (4.3) and (3.7) followed from (3.6) respectively we have found the condition (4.10). And with the help of the equation we can deduce

$$(\phi_{h|k} - \phi_{k|h})_{|m} = \gamma_m\, (\phi_{h|k} - \phi_{k|l})\,. \tag{4.21}$$

Namely in the present case (4.20) and (1.15a) we are able to say that the antisymmetric tensor $(\phi_{h|k} - \phi_{k|h})$ is not a zero tensor but a recurrent one with respect to the given vector γ_m in the space.

Consequently, in order to avoid getting $(\phi_{h|k} = \phi_{k|l})$ appearing in (1.15b) that is, to get (1.15a) purely, it is sufficient to assume the non-parallel property of $(\phi_{h|k} - \phi_{k|l})$ instead of $K^i{}_{hjk}\, v^h \neq 0$ giving

$$(\phi_{h|k} - \phi_{k|h})\, v^i \neq 0\,, \quad \text{i.e.} \quad \phi_{h|k} \neq \phi_{k|h}\,. \tag{4.22}$$

Next, we have found in the above discussions that (2.5) and (4.10) followed from (3.6) and (3.7) may be proved conversely by use of a set of (2.5) and (4.10). Hence (3.6) and the set of (2.6) and (4.10) are equivalent recurrent affine motions.

In this way we can state :

Conclusion 4.1. In a non-symmetric and non-flat recurrent in admitting affine motion, in order that we have a recurrent affine motion (1.11) having (1.12) and satisfying (1.15a), it is necessary and sufficient that we put, in the space, the conditions (2.5) and (4.10), where $(\phi_{h|k} - \phi_{k|h})$ should be assumed to be a non-parallel tensor in the space.

Namely, comparing these conclusions for (1.15a), we can see that the condition (3.6) is more fruitful than the condition (2.5), so our pursued condition is (3.6), with an additional condition being $(\phi_{h|k} - \phi_{k|h})$ a non-parallel tensor.

Remark. The author expresses his sincere thanks to Professor K.Takano for the valuable help taken from his papers in writing the series of papers on affine motion.

REFERENCES

[1] KUMAR, A. : *Some theorems on affine motion in a recurrent Finsicr space* (Comm.).

[2] KUMAR, A. : *Some theorems on affine motion in a recurrent Finsler space*, II (Comm.).

[3] KUMAR, A. : *Some theorems on affine motion in a recurrent Finsler space*, III (Comm.).

[4] KUMAR, A. : *Some theorems on affine motion in a recurrent Finsler space*, IV (Comm.).

[5] KUMAR, A. : *Some theorems on affine motion in a recurrent Finsler space*, V (Comm.).

[6] RUND, H. : The differential geometry of Finsler space, Springer-Verlag (1959).

[7] TAKANO, K. : *On a remained affine motion in Y.C. Wong's space*, Tensor, N.S., 13 (1963), 155-162.

[8] YANO, K. : The theory of Lie derivatives and its application, Amsterdam (1957).

DEPARTMENT OF APPLIED SCIENCES
MADAN MOHAN MALAVIYA ENGINEERING COLLEGE
GORAKHPUR (273010) U.P. INDIA

ÖZET

Bu çalışmada ϕ_h nın gradiyent vektöründen farklı olduğu $\gamma_s v^s = 0$ hali incelenmekte ve böylece yazarın tekrarlı Finsler uzayındaki afin hareketlere dair, evvelce elde etmiş olduğu sonuçlar tamamlanmaktadır.