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ON A SPECIAL BPR Fn-SPACE

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In this paper it has been investigated the basic properties of the space under the conditions

$$H^i{}_{hjk\,(s)\,(m)} = b_{sm}\, H^i{}_{hjk}\,,$$
$$v^i{}_{(j)} = \psi_j\, v^i\,,$$
$$H_{hj} = \psi_h\, \varepsilon_j\,.$$

1. Introduction. Let us consider an n-dimensional affinelly connected Finsler space Fn [4] [1)] equipped with a linear symmetric Berwald's connection coefficient $G^i_{hk}(x, \dot{x})$. The covariant derivative of any tensor field $T^i{}_j(x, \dot{x})$ with respect to $G^i{}_{hk}(x, \dot{x})$ is given by

$$T^i{}_{j(k)} = \partial_k T^i_j - \dot{\partial}_h T^i_j\, G^h_k - T^i_h\, G^h{}_{jk} + T^h_j\, G^i_{hk}\,. \tag{1.1}$$

The well known commutation formula involving the above covariant derivative is characterized by

$$2\ T^i{}_{j[(h)\,(k)]} = -\,\dot{\partial}_s T^i_j\, H^s{}_{\gamma hk}\, \dot{x}^\gamma - T^i_s\, H^s{}_{hjk} + T^s_j\, H^i{}_{shk}{}^{2)}\,, \tag{1.2}$$

where

$$H^i{}_{hjk}(x, \dot{x}) \overset{\text{def.}}{=} 2\,\{\partial_{[k}\, G^i{}_{j]h} - G^i{}_{\gamma h[j}\, G^\gamma{}_{k]} + G^\gamma{}_{h[j}\, G^i{}_{k]\gamma}\} \tag{1.3}$$

is called Berwald's curvature tensor and satisfies the following identities [4] :

$$H_{hj} = H^i{}_{hji} \tag{1.4}$$

and

$$H^i{}_{hjk} = -\,H^i{}_{hkj}\,. \tag{1.5}$$

In an Fn, if the Berwald's curvature tensor satisfies the following relation [1] :

$$H^i{}_{hjk\,(s)\,(m)} = b_{sm}\, H^i{}_{hjk}\,, \tag{1.6}$$

1) Numbers in square brackets refer to the references given at the end of the paper.

2) $2\,A_{[hk]} = A_{hk} - A_{kh}$.

where b_{sm} means in general a non-symmetric and non-vanishing covariant tensor, then the space is called bi-projective recurrent Finsler space or BPR Fn-space.

In what follows we shall assume to put the following two conditions in our space [2] :

$$v^i_{(j)} = \psi_j v^i \tag{1.7}$$

and

$$H_{hj} = \psi_h \varepsilon_j , \tag{1.8}$$

where ε_j means a suitable covariant vector.

In fact when the space under consideration admits a projective affine motion $\overline{x}^i = x^i + v^i(x)\,dt$, characterized by (1.7) we have a resolved form of projective Ricci tensor $H_{hj}(x, \dot{x})$ of the form (1.8) [2]. In this paper leaving the existence of projective affine motion of recurrent type out of consideration we dare to assume the existence of recurrent contravariant vector $v^i(x)$ given by (1.7) and in addition the resolvability of $H_{hj}(x, \dot{x})$.

In the following we shall study on the basic properties of the space under the conditions (1.6), (1.7) and (1.8).

Differentiating (1.7) covariantly with respect to x^k and remembering the formula (1.7) itself, we get

$$v^i_{(j)(k)} = (\psi_{j(k)} + \psi_j \psi_k) v^i . \tag{1.9}$$

Commutating the above formula with respect to the indices j and k and using the commutation formula (1.2), we have

$$H^i_{sjk} v^s = E_{jk} v^i , \tag{1.10}$$

where

$$E_{jk} \overset{\text{def.}}{=} (\psi_{j(k)} - \psi_{k(j)}) . \tag{1.11}$$

Applying the fundamental definition (1.6) to the so-called projective Ricci tensor $H_{hj}(x, \dot{x})$, we find

$$H_{hj(s)(m)} = b_{sm} H_{hj} . \tag{1.12}$$

In view of the condition (1.8) the last formula reduces to

$$\psi_{h(s)(m)} \varepsilon_j + \psi_{h(s)} \varepsilon_{j(m)} + \psi_{h(m)} \varepsilon_{j(s)} + \psi_h \varepsilon_{j(s)(m)} = b_{sm} \psi_h \varepsilon_j . \tag{1.13}$$

Commutating the indices s and m in the above result and using the commutation formula (1.2), we obtain

$$- \varepsilon_j \psi_\gamma H^\gamma_{hsm} - \psi_h \varepsilon_\gamma H^\gamma_{jsm} = \Omega_{sm} \psi_h \varepsilon_j , \tag{1.14}$$

where

$$\Omega_{sm} \stackrel{\text{def.}}{=\!=} (b_{sm} - b_{ms}) . \tag{1.15}$$

Transvecting the formula (1.14) by v^h and summing over the index h and noting the equation (1.10), we get

$$\psi (\Omega_{sm} \varepsilon_j + \varepsilon_\gamma H^\gamma{}_{jsm} + \varepsilon_j E_{sm}) = 0 , \tag{1.16}$$

where

$$\psi (x) \stackrel{\text{def.}}{=\!=} \psi_h v^h . \tag{1.17}$$

Thus, we have to discuss here the next two cases :

$$\text{a)} \quad \Omega_{sm} \varepsilon_j + \varepsilon_\gamma H^\gamma{}_{jsm} + \varepsilon_j E_{sm} = 0 \quad \text{and} \quad \text{b)} \quad \psi = 0 . \tag{1.18}$$

2. **The case of $\varepsilon = 0$.** The above case can be obtained from (1.18a) by putting

$$\varepsilon = \varepsilon_s v^s . \tag{2.1}$$

We shall show the fact $\varepsilon = 0$. Transvecting the former case (1.18a) by v^j and using the equations (1.10) and (2.1), we can get

$$\varepsilon (\Omega_{sm} + 2 E_{sm}) = 0 . \tag{2.2}$$

Consequently, in the present case (1.18a), we have to consider two cases :

$$\text{a)} \quad \varepsilon = 0 \quad \text{and} \quad \text{b)} \quad \Omega_{sm} + 2 E_{sm} = 0 . \tag{2.3}$$

Multiplying the latter case (2.3b) by v^i and remembering the formula (1.10), we obtain

$$\Omega_{sm} v^i + 2 H^i{}_{\gamma sm} v^\gamma = 0 . \tag{2.4}$$

Contracting the above equality with respect to the indices i and m and noting (1.4), we have

$$\Omega_{sm} v^m + 2 H_{\gamma s} v^\gamma = 0 . \tag{2.5}$$

On account of the equations (1.8) and (1.17), the above result can be re-written as

$$\Omega_{sm} v^m + 2 \psi \varepsilon_s = 0 . \tag{2.6}$$

Again transvecting the last formula by v^m and using (2.1), we find

$$\Omega_{sm} v^m v^s + 2 \varepsilon \psi = 0 \tag{2.7}$$

or

$$\varepsilon \psi = 0 . \tag{2.8}$$

In this way $\varepsilon = 0$, i.e. the second case means the first case and (1.18a) may be replaced by (1.18b). By this reason there exists only one case of $\varepsilon = 0$.

In view of the basic condition (1.6) and the commutation formula (1.2), we can get

$$\Omega_{sm} H^i{}_{hjk} = - \dot{\partial}_\gamma H^i{}_{hjk} H^\gamma{}_{sm} + H^\gamma{}_{hjk} H^i{}_{\gamma sm} - H^i{}_{\gamma jk} H^\gamma{}_{hsm} - H^i{}_{h\gamma k} H^\gamma{}_{jsm} -$$
$$- H^i{}_{hj\gamma} H^\gamma{}_{ksm}, \tag{2.9}$$

where we have used (1.15).

Contracting the last formula with respect to the indices i and k, and remembering the equation (1.4), we find

$$\Omega_{sm} H_{hj} = - H_{h\gamma} H^\gamma{}_{jsm} - H_{\gamma j} H^\gamma{}_{hsm} - \dot{\partial}_\gamma H^i{}_{hji} H^\gamma{}_{sm}. \tag{2.10}$$

Transvecting the last formula by v^h and noting the equations (1.8), (1.10) and (1.17), we obtain

$$\varepsilon_\gamma H^\gamma{}_{jsm} = - (\Omega_{sm} + E_{sm}) \varepsilon_j, \tag{2.11}$$

where we have neglected the non-zero scalar $\psi(x)$.

Introducing the last formula into the left-hand side of the equality (1.14), we get

$$\varepsilon_j (\psi_\gamma H^\gamma{}_{hsm} - \psi_h E_{sm}) = 0. \tag{2.12}$$

Thus there occur the following two cases to be discussed:

$$\text{a)} \quad \varepsilon_j = 0 \quad \text{and} \quad \text{b)} \quad \psi_\gamma H^\gamma{}_{hsm} = \psi_h E_{sm}. \tag{2.13}$$

The Bianchi's identity for the Berwald's curvature tensor $H^i{}_{hjk}(x, \dot{x})$ is given by

$$H^i{}_{hjk} + H^i{}_{jkh} + H^i{}_{khj} = 0. \tag{2.14}$$

Transvecting the above identity by ψ_i and using the latter case (2.13b), we get

$$\psi_h E_{jk} + \psi_j E_{kh} + \psi_k E_{hj} = 0. \tag{2.15}$$

On account of the equations (2.3b), (2.5) and the fact that $E_{sm} = - E_{ms}$, we can deduce

$$E_{ms} v^m = - H_{\gamma s} v^\gamma. \tag{2.16}$$

By virtue of the relations (1.8) and (1.17), the last formula yields

$$E_{ms} v^m = - \psi \varepsilon_s. \tag{2.17}$$

Transvecting the equality (2.15) by v^h and noting the formula (2.16) and the fact that $E_{hj} = - E_{jh}$, we have

$$\psi_h v^h E_{jk} = - \psi_j H_{hk} v^h - \psi_k H_{hj} v^h. \tag{2.18}$$

In view of the equations (1.8), (1.17) and (2.16), the last formula reduces to

$$E_{jk} = \psi_k \varepsilon_j - \psi_j \varepsilon_k , \tag{2.19}$$

where we have neglected $\psi(x)$.

By virtue of the formula (1.8), we can deduce

$$H_{hj} - H_{jh} = \psi_h \varepsilon_j - \psi_j \varepsilon_h . \tag{2.20}$$

With the help of the equations (1.11) and (2.19), the above result can be re-written as

$$H_{hj} - H_{jh} = \psi_{j\,(h)} - \psi_{h\,(j)} . \tag{2.21}$$

In view of the equations (1.10) and (2.13b), we can conclude

$$\psi_\gamma H^\gamma{}_{hsm} v^i = \psi_h H^i{}_{\gamma sm} v^\gamma . \tag{2.22}$$

On account of the basic formula (1.7), the above equality takes the form :

$$H^\gamma{}_{hsm} v^i{}_{(\gamma)} - H^i{}_{\gamma sm} v^\gamma{}_{(h)} = 0 . \tag{2.23}$$

By virtue of the commutation formula (1.2), the last formula reduces to

$$(v^i{}_{(h)})_{(s)\,(m)} - (v^i{}_{(h)})_{(m)\,(s)} = 0 . \tag{2.24}$$

Consequently we can imagine the existence of a gradient vector λ_s and we are able to put

$$v^i{}_{(h)\,(s)} = \lambda_s v^i{}_{(h)} . \tag{2.25}$$

With the help of the last definition the formula (1.9) can be re-written as

$$\lambda_k \psi_j = \psi_{j\,(k)} + \psi_j \psi_k , \tag{2.26}$$

where we have used (1.7) and neglected the non-zero $v^i(x)$.

Transvecting the last equality by v^j and remembering the equation (1.17), we get

$$\begin{aligned} \psi \lambda_k &= \psi_{j(k)} v^j + \psi_k \psi \\ &= (\psi_j v^j)_{(k)} + \psi_k \psi - \psi_j v^j{}_{(k)} \\ &= \psi_{(k)} + \psi \psi_k - \psi_j v^j \psi_k \\ &= \psi_{(k)} + \psi \psi_k - \psi \psi_k \\ &= \psi_{(k)} , \end{aligned} \tag{2.27}$$

where we have used (1.7) also.

In this way, the existence of λ_j is examined and we have here a characteristic condition on $v^i{}_{(h)}$:

$$v^i{}_{(h)\,(j)} = \lambda_j v^i{}_{(h)} \quad , \quad \lambda_j = \psi_{(j)} / \psi . \tag{2.28}$$

On the other hand, in view of the condition (2.13a), the supposition (1.8) takes the form :

$$H_{hj} = 0 \ . \tag{2.29}$$

Summarizing the above all consideration, we can have :

Theorem 2.1. In a BPR Fn-space admitting a contravariant vector $v^i(x)$ characterized by (1.7) and having a disjoint projective Ricci tensor $H_{hj}(x, \dot{x})$ of the form (1.8), there exists a case of $\varepsilon_s v^s = 0$. In this case if $\varepsilon_s = 0$, then we have the vanishing of projective Ricci tensor H_{hj} and if $\varepsilon_s \neq 0$, we have (2.20). The mixed tensor $v^i_{(h)}$ itself is a recurrent one characterized by (2.28).

3. The case of $\psi = 0$. Let us consider the case (1.18b), then using the analogous methods used in § 2 with the help of the formula (2.10), we can easily conclude

$$\psi_\gamma H^\gamma{}_{hsm} = -(\Omega_{sm} + E_{sm})\,\psi_h \ . \tag{3.1}$$

Introducing the last formula into the left hand side of (1.14), we obtain

$$\psi_h(\varepsilon_\gamma H^\gamma{}_{jsm} - \varepsilon_j E_{sm}) = 0 \ . \tag{3.2}$$

In this way we have here two cases to discuss :

$$\text{a)}\quad \psi_h = 0 \quad \text{and} \quad \text{b)}\quad \varepsilon_\gamma H^\gamma{}_{jsm} = \varepsilon_j E_{sm} \ . \tag{3.3}$$

In view of the former case (3.3a) the suppositions (1.7) and (1.8) reduce to

$$v^i{}_{(j)} = 0 \quad \text{and} \quad H_{hj} = 0 \ . \tag{3.4}$$

By virtue of the latter case (3.3b) and the identity (2.14), we can deduce

$$\varepsilon_h E_{jk} + \varepsilon_j E_{kh} + \varepsilon_k E_{hj} = 0 \ . \tag{3.5}$$

With the help of the equations (1.8), (1.17), (2.16) and (3.3a), we can get

$$E_{hj} v^h = -H_{\gamma j} v^\gamma = -\psi_\gamma \varepsilon_j v^\gamma = -\psi \varepsilon_j = 0 \ . \tag{3.6}$$

Thus, transvecting the equality (3.5) by v^h and using the equations (2.1) and (3.6), we obtain

$$\varepsilon E_{jk} = 0 \ , \tag{3.7}$$

where we have also used the fact that $E_{hj} = -E_{jh}$.

In this way, we have

$$E_{jk} = 0 \ . \tag{3.8}$$

In view of the last formula, the basic definition (1.11) takes the form :

$$\psi_{j(k)} = \psi_{k(j)} \, . \tag{3.9}$$

Thus, we can have :

Theorem 3.1. When $\psi = 0$ in our space, there exist two cases (3.3). The former case (3.3a) satisfies (3.4) and the latter one satisfies (3.9).

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ÖZET

Bu çalışmada uzayın

$$H^i{}_{hjk(s)(m)} = b_{sm} H^i{}_{hjk} \, ,$$

$$v^i{}_{(j)} = \psi_j v^i \, ,$$

$$H_{hj} = \psi_h \varepsilon_j$$

koşulları altında bazı temel özellikleri incelenmektedir.