ON SOME (f, g) - LINEAR CONNECTIONS

N. SOARE

Universitatea Bucureşti, Facultatea de Matematica, Str. Academiei 14,
70109 Bucureşti-ROMANIA

Summary : In this note we study the (f, g) - structures determined by a tensor field of type (1,1) so that $f^{2\nu+3} + f = 0$, and g is a Riemannian structure satisfying a supplementary condition.

BAZI (f, g) - LİNEER BAĞLANTILAR HAKKINDA

Özet : Bu çalışmada, (1,1) tipinde bir tensör alanı tarafından belirlenen (f, g) - yapıları incelenmektedir. Burada $f^{2\nu+3} + f = 0$ olup, g ek bir koşul gerçekleyen bir Riemann yapısıdır.

INTRODUCTION

Let M be a Riemannian manifold with Riemannian metric g, $\mathscr{C}(M)$ the affin modul of the linear connections on M, $\mathscr{T}^r_s(M)$ - the modul of the tensors of type (r, s) : for $\mathscr{T}^1_0(M)$ and $\mathscr{T}^0_1(M)$ are used the notations $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$ respectively. All the objects are of class C^∞.

Definition 1.1. We call $f(2\nu+3,1)$ - structure on M, a non-null field of tensors $f \in \mathscr{T}^1_1(M)$, of rank r, where r is constant everywhere, so that

$$f^{2\nu+3} + f = 0.$$

If M is a $f(2\nu+3,1)$ - manifold, that is, if M is an n-dimensional Riemannian manifold, equiped with a $f(2\nu + 3,1)$ - structure, then for

$$l = -f^{2\nu+2}, \quad m = f^{2\nu+2} + I \tag{1.1}$$

(I denoting the identity operator) we have

$$f\,l = l\,f = l, \quad f\,m = m\,f = 0, \quad f^{2\nu+2}\,l = -l, \quad f^{2\nu+2}\,m = 0 \tag{1.2}$$

and

$$l + m = I, \quad l\,m = m\,l = 0, \quad l^2 = l, \quad m^2 = m. \tag{1.3}$$

Thus the operators l and m are complementary projection operators on M.

The Riemannian structure g on M can be considered a $\mathfrak{X}^*(M)$-valued differential 1-form and we'll have $g: \mathfrak{X}(M) \to \mathfrak{X}^*(M)$, $g(X) = g_X$, where $g_X(Y) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$. If $f \in \mathcal{T}_1^1(M)$, then ${}^t f$ is the transpose of f, ${}^t f: \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$, ${}^t f(\theta) = \theta \circ f$, $\forall \theta \in \mathfrak{X}^*(M)$.

Definition 1.2. We call (f, g)-structure on M, a couple made up a $f(2\nu+3,1)$-structure and a Riemannian structure g so that

$$ {}^t f^{\nu+1} \circ g \circ f^{\nu+1} = g \circ l. \tag{1.4} $$

Theorem 1.1. Let M be a paracompact differential manifold with a $f(2\nu+3,1)$-structure. Then, there is a (f, g)-structure.

Proof. In truth, if γ is a Riemannian metric, fixed on M, then

$$ g = \frac{1}{2}(\gamma + {}^t f^{\nu+1} \circ \gamma \circ f^{\nu+1} - \gamma \circ m - {}^t m \circ \gamma + 3\, {}^t m \circ \gamma \circ m) \tag{1.5} $$

verifies the condition (1.4).

Proposition 1.1. For a $(f\ g)$-structure on M and l, m defined by the equations (1.1) we have

$$ \begin{aligned} g \circ f^{\nu+1} &= -{}^t f^{\nu+1} \circ g\ , \quad g^{-1} \circ {}^t f^{\nu+1} = -f^{\nu+1} \circ g^{-1} \\ g \circ m &= {}^t m \circ g \quad\quad\ \ , \quad g^{-1} \circ {}^t m = m \circ g^{-1}. \end{aligned} \tag{1.6} $$

Proposition 1.2. $\omega = g \circ f^{\nu+1}$ is a differential 2-form on M.

Definition 1.3. We call Obata operators associated to $f(2\nu+3,1)$-structure, the applications $A^{(2\nu+3,1)}, A^{(2\nu+3,1)^*}: \mathcal{T}_1^1(M) \to \mathcal{T}_1^1(M)$ defined by

$$ \begin{aligned} A^{(2\nu+3,1)}(w) &= \frac{1}{2}(w - mw - wm + 3mwm - f^{\nu+1} w f^{\nu+1}) \\ A^{(2\nu+3,1)^*}(w) &= w - A^{(2\nu+3,1)}(w). \end{aligned} \tag{1.7} $$

We also consider the Obata operators [6] associated to g:

$$ \begin{aligned} B(u) &= \frac{1}{2}(u - g^{-1} \circ {}^t u \circ g) \\ B^*(u) &= \frac{1}{2}(u + g^{-1} \circ {}^t u \circ g). \end{aligned} \tag{1.8} $$

We can demonstrate

Proposition 1.3. For a (f, g)-structure on M and for $A^{(2\nu+3,1)^*}$, $A^{(2\nu+3,1)}$ and B, B^* defined by (1.7) and (1.8) we have:

1) $A^{(2\nu+3,1)}$ and $A^{(2\nu+3,1)^*}$ are complementary projections on $\mathcal{T}_1^1(M)$.

2) B and B^* commute pairwise with $A^{(2\nu+3,1)}$ and $A^{(2\nu+3,1)^*}$.

3) $A^{(2\nu+3,1)} \circ B$ and $A^{(2\nu+3,1)^*} \circ B^*$ are projections on $\mathcal{T}_1^1(M)$.

4) $\operatorname{Ker} A^{(2\nu+3,1)^*} \cap \operatorname{Ker} B^* = Im(A^{(2\nu+3,1)} \circ B)$.

In truth, by simple calculation, we obtain the result 1).

The affirmation 2) is true, because, taking into account the relations (1.6), we have:

$$(A^{(2\nu+3,1)} \circ B - B \circ A^{(2\nu+3,1)})\ (u) =$$

$$= \frac{1}{4}(m \circ g^{-1} \circ {}^t u \circ g - g^{-1} \circ {}^t m \circ {}^t u \circ g) + (g^{-1} \circ {}^t u \circ g \circ m - g^{-1} \circ {}^t u \circ {}^t m \circ g) -$$

$$- 3\,(m \circ g^{-1} \circ {}^t u \circ g \circ m - g^{-1} \circ {}^t m \circ {}^t u \circ {}^t m \circ g) +$$

$$+ (f^{\nu+1} \circ g^{-1} \circ {}^t u \circ g \circ f^{\nu+1} - {}^t f^{\nu+1} \circ {}^t u \circ {}^t f^{r+1} \circ g) = 0,$$

for every $u \in \mathcal{T}_1^1(M)$, or

$$A^{(2\nu+3,1)} \circ B = B \circ A^{(2\nu+3,1)}.$$

Thus we have the relations

$$A^{(2\nu+3,1)} \circ B^* = B^* \circ A^{(2\nu+3,1)},$$

$$A^{(2\nu+3,1)^*} \circ B = B \circ A^{(2\nu+3,1)^*}.$$

The above mentioned relations give us the possibility to formulate [10]:

Proposition 1.4. The system of tensorial equations

$$A^{(2\nu+3,1)^*}(u) = a, \quad B^*(u) = b \tag{1.9}$$

has a solution $u \in \mathcal{T}_1^1(M)$, if and only if

$$\begin{aligned} &A^{(2\nu+3,1)}(a) = 0, \quad B(b) = 0, \\ &A^{(2\nu+3,1)^*}(b) = B^*(a). \end{aligned} \tag{1.10}$$

If the conditions (1.10) are fulfilled, then the general solution of the system (1.9) is

$$u = a + A^{(2\nu+3,1)}(b) + (A^{(2\nu+3,1)} \circ B)\ (w)$$

for every $w \in \mathcal{T}_1^1(M)$.

2. (f, g) - LINEAR CONNECTIONS

In the following paragraphs, $\overset{\circ}{\nabla} \in \mathscr{C}(M)$ will be a linear connection fixed on M. Every tensor field $u \in \mathscr{T}_1^1(M)$ may be considered as a field of $\mathfrak{X}(M)$ - valued differential l-forms. So, if ∇ is a linear connection on M, then we'll note with D and $\tilde{D}$ the associated connections, acting on the $\mathfrak{X}(M)$ - valued differential l-forms and respectively on the differential l-forms $g : \mathfrak{X}(M) \rightarrow \mathfrak{X}^*(M)$:

$$D_X u = \nabla_X u - u \nabla_X, \quad \forall X \in \mathfrak{X}(M) \tag{2.1}$$

$$D_X g = {}^t\nabla_X \circ g - g \circ \nabla_X, \quad \forall X \in \mathfrak{X}(M) \tag{2.2}$$

where

$$({}^t\nabla_X g)\ (Y, Z) = X g(Y, Z) - g(\nabla_X Y, Z), \ \forall X, Y, Z \in \mathfrak{X}(M).$$

Definition 2.1. A linear connection ∇ on M is called (f, g) - linear connection if

$$D_X f = 0, \quad \tilde{D}_X g = 0, \quad \forall X \in \mathfrak{X}(M). \tag{2.3}$$

Of course, for every $(f\ g)$ - linear connection, we have

$$\begin{aligned} &D_X l = \nabla_X l - l \nabla_X = 0, \quad D_X m = \nabla_X m - m \nabla_X = 0 \\ &D_X f^k = \nabla_X f^k - f^k \nabla_X = 0, \quad k \text{ natural number,} \end{aligned} \tag{2.4}$$

for every $X \in \mathfrak{X}(M)$. We see that D and $\tilde{D}$ commute with the operators $A^{(2\nu+3,1)}$, $A^{(2\nu+3,1)*}$, B and B^*.

We take

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X,$$

$X \in \mathfrak{X}(M)$, $V \in \mathscr{T}_2^1(M)$, $V_X Y = V(X, Y)$ and find the tensor field V so that ∇ satisfies the conditions (2.3).

∇ will be a (f, g) - linear connection if and only if the field V verifies the system :

$$V_X \circ f - f \circ V_X = - \overset{\circ}{D}_X f \quad {}^tV_X \circ g + g \circ V_X = \overset{\circ}{\tilde{D}}_X g.$$

This system is equivalent with the system

$$\begin{aligned} &A^{(2\nu+3,1)*}(V_X) = -\frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X f - 3\, m \circ \overset{\circ}{D}_X m) \\ &B^*(V_X) = \frac{1}{2} g^{-1} \circ \overset{\circ}{\tilde{D}}_X g\,. \end{aligned} \tag{2.5}$$

Applying the proposition 1.4, it becomes evident that the system (2.5) has solutions and the general solution is

$$V_X = -\frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X m - 3\, m \circ \overset{\circ}{D}_X m) +$$

$$+ \frac{1}{4}(\tilde{\overset{\circ}{D}}_X g - {}^t f^{\nu+1} \circ \tilde{\overset{\circ}{D}}_X g \circ f^{\nu+1} - \tilde{\overset{\circ}{D}}_X g \circ m - {}^t m \circ \tilde{\overset{\circ}{D}}_X g + 3\, {}^t m \circ \tilde{\overset{\circ}{D}}_X g \circ m) +$$

$$+ (A^{(2\nu+3,1)} \circ B)\,(W_X), \quad W \in \mathcal{T}_2^1(M).$$

Thus we have

Theorem 2.1. There are (f, g) - linear connections; one of them is

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X,$$

where $\overset{\circ}{\nabla}$ is an arbitrary linear connection, fixed on M, and V_X is given by (2.6), W being an arbitrary tensor field.

If $\overset{\circ}{\nabla}$ is the Levi - Civita connection of g, then we have $\tilde{\overset{\circ}{D}}_X\, g = 0$ and the theorem 2.1 becomes

Theorem 2.2. For every (f, g) - structure, the following linear connection

$$\overset{c}{\nabla}_X = \overset{\circ}{\nabla}_X - \frac{1}{2}(f \circ \overset{\circ}{D}_X f + \overset{\circ}{D}_X m - 3\, m \circ \overset{\circ}{D}_X m), \ \forall X \qquad (2.8)$$

where $\overset{\circ}{\nabla}$ is the Levi - Civita connection of g, has the following characteristics:

1) $\overset{c}{\nabla}$ is dependent uniquely on f, and g;

2) $\overset{c}{\nabla}$ is a (f, g) - linear connection.

The linear connection $\overset{c}{\nabla}$ will be called the (f, g) - canonic connection.

Theorem 2.3. The set of all the (f, g) - linear connections is given by

$$\overline{\nabla}_X = \nabla_X + (A^{(2\nu+3,1)} \circ B)\,(W_X), \quad W \in \mathcal{T}_2^1(M), \qquad (2.9)$$

where ∇ is a (f, g) - linear connection, in particular $\nabla = \overset{c}{\nabla}$.

Observing that (2.9) can be considered as a transformation of (f, g) - linear connections, we have:

Theorem 2.4. The set of the transformations of (f, g) - linear connections and the multiplication of the applications is an abelian group, noted with $G(f, g)$, isomorph with the additive group of the tensors $W \in \mathcal{T}_2^1(M)$, which have the characteristic

$$W_X \in Im\,(A^{(2\nu+3,1)} \circ B) = \text{Ker}\, A^{(2\nu+3,1)^*} \cap B^*,$$

for every $X \in \mathfrak{X}(M)$.

3. THE INTEGRABILITY OF THE (f, g) - STRUCTURES

The f- structure is called integrable if in every point of M there is an admissible map where f has constant coefficients in natural frames.

It is known [1], that a $f(2\nu+3,1)$ - structure is integrable if and only if the tensor $N \in \mathcal{T}_2^1(M)$ given by

$$N(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2 [X, Y] \tag{3.1}$$

is zero.

We find out with no difficulty that we have :

Proposition 3.1. The tensor of integrability of the $f(2\nu+3,1)$ - structure can be expressed thus

$$N(X, Y) = -f^2\, T(X, Y) - T(fX, fY) + fT(fX, Y) + fT(X, fY). \tag{3.2}$$

To a (f, g) - structure, besides the tensor N given by (3.2) we associate a second tensor of integrability K, given by

$$\begin{aligned} K(X, Y, Z) &= d\,\omega(X, Y, Z) = \\ &= \omega(T(X, Y), Z) + \omega(T(Y, Z), X) + \omega(T(Z, X), Y) \end{aligned} \tag{3.3}$$

where ω is the 2-form from the proposition 1.2:

$$(X, Y) = g(f^{\nu+1} X, Y) = - g(X, f^{\nu+1} Y). \tag{3.4}$$

From (3.1) and (3.3) we have

Theorem 3.1. The tensors of integrability N and K of a (f, g) - structure are invariant in comparison with the transformations of the group $G(f, g)$.

It takes place the following theorem:

Theorem 3.2. If there is a (f, g) - semi-symmetric connection (in particular (f, g) - symmetric connection), then $N = 0$ and $K = 0$.

Proof. In truth, $T(X, Y) = \sigma(X)\, Y - \sigma(Y)\, X$, $\sigma \in \mathfrak{X}^*(M)$ imply

$$\begin{aligned} -f^2 T(X, Y) &= - \sigma(X) f^2(Y) + \sigma(X) f^2(X), \\ - T(fX, fY) &= - \sigma(fX) f(Y) + \sigma(fY) f(X) \\ fT(fX, Y) &= \sigma(fX) f(Y) - \sigma(Y) f^2(Y) \\ fT(X, fY) &= \sigma(X) f^2(Y) - \sigma(fY) f(X). \end{aligned}$$

Substituting these relations in (3.2) and (3.3) we have respectively $N(X, Y)=0$ and $K(X, Y, Z) = 0$, for every $X, Y, Z \in \mathfrak{X}(M)$.

REFERENCES

[1] DEMETROPOULOU-PSOMOPOULOU, D. : *On integrability conditions of a structure satisfying* $F^{2\nu+3} + F = 0$, Tensor, N.S., 42 (1985), 252-257.

[2] DEMETROPOULOU-PSOMOPOULOU, D. : *Linear connections on manifolds admitting an* $f(2\nu + 3,1)$*-structure or an* $f(2\nu + 3, -1)$*-structure*, Tensor, N.S., 47 (1988), 235-238.

[3] GOLDBERG, S. and YANO, K. : *Globally framed f-manifolds*, *Illinois Journal of Math.*, 15 (1971), No. 3, 456-474.

[4] ISHIHARA, S. and YANO, K. : *On integrability conditions of a structure satisfying* $f^3 + f = 0$, Quart. J. Math., 15 (1964), 217-222.

[5] MIRON, R. et ATANASIU, GH. : *Sur les* (f, g) *- connexions lineaires et l'integrabilité des* (f, g) *- structures*, Bull. Math. Soc. Sci. Math. R.S. Roumanie, (78) 30 (1986), 245-256.

[6] OBATA, M. : *Affine connections on manifolds with almost complex, quaternionic or Hermitian structure*, Jap. Journ. Math., 26 (1956), 43-77.

[7] PETRAKIS, A. : *Structure f qui satisfait la relation* $f^{2\nu+3} - f = 0$, Tensor, N.S., 41 (1984), 103-107.

[8] SOARE, N. : *Sur quelgues f-connexions linéaire*, Riv. Mat. Univ. Parma, (4) 14 (1988), 143-150.

[9] TAMIA-DIMOPOULOU, P. : *On direction dependent f-structure satisfying* $f^s + f^t = 0$, Riv. Mat. Univ. Parma, (4) 17 (1991), 279-293.

[10] WILDE, A.C. : *Commutative Projection Operators*, Atti Sem. Mat. Fis. Univ. Modena, **XXXV** (1987), 167-172.