CONE MAXIMAL POINTS IN COMPACT SUBSETS OF TOPOLOGICAL VECTOR SPACES

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Summary : In this note we have investigated cone maximal points in compact subsets of topological vector spaces.

TOPOLOJİK VEKTÖR UZAYLARININ KOMPAKT ALT CÜMLELERİNDEKİ KONİK MAKSİMAL NOKTALAR

Özet : Bu çalışmada topolojik vektör uzaylarının kompakt alt cümlelerindeki konik maksimal noktalar araştırılmaktadır.

§ 1. PRELIMINARIES

Let E be a (real) vector space. By a *cone* in E we shall mean any part C of E with

(D_1^1) $C+C\subseteq C,\ \lambda C\subseteq C,\ \lambda\geqslant 0.$

Given such an object, denote

(D_2^1) $\mathrm{lin}\,(C)=C\cap(-C).$

Of course, this is the largest linear space included in C; when $\mathrm{lin}\,(C)=\{0\}$, the cone C will be called *pointed.*

Now, for any (nonempty) part Y of E, denote by $\max\,(Y, C)$ the subset of all $z\in Y$ with the maximal (modulo C) property

(MP) $w\in Y,\ z\leqslant w\ (\mathrm{mod}\ C)\Longrightarrow z\leqslant w\ (\mathrm{mod\ lin}\ (C)).$

Here, by $\leqslant(\mathrm{mod}\ C)$ we understand the *quasi-ordering* over E induced by C, in the usual way

(D_3^1) $x\leqslant y\ (\mathrm{mod}\ C)$ if and only if $y-x\in C.$

Let in the following $\mathcal{T}$ be a *linear topology* over E. We are interested to determine structural conditions upon C and $\mathcal{T}$ so that the following property -referred to as C is a *comp-max cone*- be valid:

(CM) $\max\,(H, C)$ is nonempty, for each (nonempty) compact part H of E.

Here, the term "compact" is taken as in Kelley [6, Ch. 5, § 1] ; *i.e.*,

(D_4^1) each net in H admits an accumulation point in H.

To give a practical motivation of this, we note (cf. Penot [8]) that any maximal (mod C) element of a generic subset in E may be deemed as a Pareto efficient point of the associated multicriterion optimization problem. But, our interest has also a theoretical motivation; because the problem we already formulated is ultimately a (linear) topological version of the Zorn maximality principle. In this perspective, the following 1954 result in Ward [14] must be considered as a first (and basic) answer to it:

Theorem 1. Suppose that

(K_1) C is a closed cone.

Then, C has the comp-max property, in the sense

$$\begin{aligned} &\max (H, C) \text{ is nonempty and cofinal (mod } C) \text{ in } H, \\ &\text{for each (nonempty) compact part } H \text{ of } E. \end{aligned} \tag{1.1}$$

Now, for the above precised reasons, it is natural to ask of whether or not is Theorem 1 extendable beyond the closedness context. Note that any such extension is purely technical; because, as results from the paper by Borwein [1], Theorem 1 is actually equivalent to the Axiom of Choice. The answer is positive. To state it, call the cone C in E, *admissible*, when

(D_5^1) $\left\{ \begin{array}{l} L = \text{closed subspace of lin cl}(C) \text{ and cl}(L \cap C) = \text{linear} \\ \text{subspace imply } L \cap C = \text{linear subspace} \end{array} \right.$

(Of course, this implication must be checked only if cl$(L \cap C)$ is not reduced to the null subspace; since, otherwise, it becomes trivial). The following 1986 statement by Sterna-Karwat [11] is basic to considerations below.

Theorem 2. Suppose that

(K_2) C is an admissible cone.

Then, necessarily, C has the comp-max property.

Concerning the relationships between these results, it is now clear -by the definitions involved- that (K_1) is a particular case of (K_2); hence, the statament above comprises the existence part of Theorem 1. On the other hand, call the underlying cone C in E, *non-flat*, when

(D_6^1) $\left\{ \begin{array}{l} \text{for each closed subspace } L \text{ of lin cl}(C) \text{ with cl}(L \cap C) = \\ = \text{nondegenerate linear subspace, } L \cap C \text{ has a nonempty} \\ \text{interior in cl}(L \cap C) \text{ (endowed with the relative (linear) topology).} \end{array} \right.$

Now, it turns out that

(K_3) C is a non-flat cone

is also a particular case of (K_2). In fact, let the closed subspace L of lin cl (C) be taken as in (D_5^1). Then, by the classical Eidelheit's separation theorem (see, e.g., Cristescu [4, Ch. 1, § 2]) it is not hard to see that

$$L\cap C=\mathrm{cl}\,(L\cap C)=\text{linear subspace}, \tag{1.2}$$

and the claim follows. Now, (K_3) is fulfilled when

(K_4) $C'=\{x\in C\,;\ x\neq 0\}$ is open (in E).

To verify this, let the closed subspace L of lin cl(C) be such that cl($L\cap C$) is a nondegenerate linear subspace. We thus have

$$L\cap C\neq\{0\}\ \text{(or, equivalently, } L\cap C'\neq\phi). \tag{1.3}$$

By the double inclusion

$$L\cap C'=(L\cap C)\cap C'\subseteq \mathrm{cl}\,(L\cap C)\cap C'\subseteq L\cap C' \tag{1.4}$$

we deduce $L\cap C'$ is open in cl($L\cap C$) (endowed with the relative (linear) topology); hence, the interior of $L\cap C$ in cl($L\cap C$) is nonempty, as claimed. The similar statament (Remark 2.2, (iii)) in the quoted paper by Sterna-Karwat corresponds to C' being -in addition- convex in (K_4). This is redundant, by the argument above; and moreover, it makes (K_3) be vacuously satisfied. Indeed, C must be pointed under such an assumption about C'. And so, combining with (1.2),

$$L\cap C=\mathrm{lin}\ (L\cap C)=L\cap \mathrm{lin}\ (C)=\{0\},$$

in contradiction with the admitted (in (D_6^1)) premise.

The above statements are non-trivial only if the ambient linear topology $\mathscr{T}$ over E is, roughly speaking, "not very strong". Precisely, denote by $\{E_i\,;i\in I\}$ the class of all finite dimensional subspaces of E. For each $i\in I$, we let $\mathscr{T}_i$ stand for the (unique) Hausdorff separated linear topology over E_i, and put

(D_7^1) $\mathscr{T}_f=$ the inductive limit of $\{\mathscr{T}_i\,;\ i\in I\}$.

This object -called the *convex core topology-* is actually the strongest locally convex topology over E; see Schaefer [9, Ch. 2, § 6] for details. Now, as a counterpart of Theorem 2, we have (cf. Sterna-Karwat [12]):

Theorem 3. Let the ambient linear topology $\mathscr{T}$ over E be stronger than the convex core topology $\mathscr{T}_f$. Then, any cone C in E has the comp-max property.

In particular, this necessarily happens under

(FD) E is finite dimensional,

whenever the ambient linear topology $\mathscr{T}$ is Hausdorf separated (cf. the remark above). But, if this topology is no more endowed with such a property, the conclusion in the statament above cannot be retained, in general, even if (FD) were admitted. This is shown by the following example: Let $E = R^2$ and $\mathscr{T}$, the locally convex topology over E introduced (in the standard way) by the seminorm

$$p(x, y) = |y|, \ (x, y) \in E.$$

Let also C be the cone in E defined as

$$C = \{(x, y) \in E \, ; \, x \geqslant 0, \, y \geqslant 0\}.$$

It is now clear that, for the compact subset (of E) $H = R \times [0, 1]$, one has $\max(H, C) = \phi$; hence the assertion. For other aspects we refer to the paper by Corley [3].

Now, with these preliminaries, it is our main aim in this exposition to show that, further enlargements of the statement above are still available; these will be discussed in Section 5. The main tools of such extensions are the relative type comp-max statements given in Section 4, and some technical facts involving admissible cones, presented in Section 3. These, in turn, are being founded on the Bourbaki fixed point principle (developed in Section 2). As a matter of fact, the obtained conclusions may be also put in a purely topological framework; we shall treat these questions elsewhere.

§ 2. THE BOURBAKI FIXED POINT PRINCIPLE

Let A be a nonempty set and $\leqslant$, an *ordering* over A. Let $f : A \to A$ be a *progressive* mapping; that is,

(D_1^2) $x \leqslant f(x)$, for all x in A.

Concerning the question of what can be said about the set Fix (f) (of all fixed points for this mapping), the following facts will be in effect for us. Call the ambient set A, *semi-complete*, when

(D_2^2) sup (X) exists, for each part X of A.

It is now clear that, with such a hypothesis about A, the set Fix(f) is not empty; in fact, sup(A) is an element of it. But, for the developments below, this will not suffice. Our objective is to determine, for any point a in A, the "shortest" iterative process starting from a, having as endpoint an element of Fix(f). In this direction, the following result due to Bourbaki [2] must be noted:

Proposition 1. Let $(A, \leqslant)$ be semi-complete and f be a progressive self-mapping of A. Then, for each a in A there may be determined a well ordered part $B = B(a)$ of A, with the properties

$$a \in B, \ f(B) \subseteq B \text{ and } \sup(X) \in B \text{ whenever } X \subseteq B \tag{2.1}$$

$$u, v \in B \Longrightarrow u \geqslant v \text{ or } f(u) \leqslant v \text{ (i.e., } x \to f(x) \text{ is the immediate success or mapping of } B) \tag{2.2}$$

$$\begin{aligned}&\sup(B) \text{ is the only fixed point of } f \text{ in } B\\&(\text{that is, } x\in B,\ x\neq\sup(B)\Longrightarrow x<f(x)).\end{aligned} \tag{2.3}$$

Actually, B may be defined as the intersection of all nonempty parts Y of Y fulfiling (2.1) (with Y in place of B); see the quoted paper for details. Now, in view of (2.2), the iterative process we are looking for is that defined by the well ordered set B. To explain this, we need some preliminary facts. Let W stand for the class of all ordinals; it has a contradictory character, by the well known Burali-Forti paradoxe (see, e.g., Sierpinski [10, Ch. 14, § 2]). However, when one restricts the considerations to a Grothendieck universe $\mathscr{G}$ (introduced as in Hasse and Michler [5, Ch. 1, § 2]) this contradictory character is removed for the class $W(\mathscr{G})$ of all *admissible* (modulo $\mathscr{G}$) ordinals; that is, ordinals generated by the well ordered (non-contradictory) sets in $\mathscr{G}$. In the following, we drop the subscript $(\mathscr{G})$ for simplicity. So, by an ordinal (in W) we shall actually mean a $\mathscr{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathscr{G}$. This will be referred to as an admissible ordinal (to indicate the fact that a generic universe $\mathscr{G}$ is considered in its construction). Clearly,

$$\xi = \text{admissible ordinal and } \eta\leqslant\xi \text{ imply } \eta = \text{admissible ordinal}. \tag{2.4}$$

Hence, in the formula

$$W(\lambda) = \{\xi\in W;\ \xi<\lambda\},\ \lambda\in W,$$

the set W in the brackets may be taken as the "absolute" set of all ordinals.

Let in the following the generic Grothendieck universe $\mathscr{G}$ be so large that A is a member/part of it. For each $a\in A$, let the *transfinite iterates* of f at this point be introduced as

$$(\mathrm{D}_3^2)\quad \begin{aligned}f^0(a) &= a\\ f^\lambda(a) &= f(f^{\lambda-1}(a)), \text{ if } \lambda \text{ is a first kind ordinal}\\ &= \sup\{f^\xi(a);\ \xi<\lambda\}, \text{ otherwise}\end{aligned}$$

(Here, by "ordinal" we actually mean "admissible ordinal". But, in the following, we shall not make any distinction between these; because, in view of the accepted hypothesis, only admissible ordinals are accepted). The definition above is meaningful, if we take into account the accepted conditions about A and f. Moreover,

$$f^\xi(a)\leqslant f^\eta(a), \text{ whenever } \xi\leqslant\eta\ ; \tag{2.5}$$

that is, the transfinite sequence $(f^\xi(a))$ increases. Now, in principle, it would be possible that such a sequence be non-stationary; that is,

$$f^\xi(a)<f^\eta(a), \text{ provided } \xi<\eta\,. \tag{2.5'}$$

Therefore, what the above result says, is that the transfinite sequence $(f^{\xi}(a))$ becomes *stationary* beyond a certain ordinal number $\beta = \beta(a)$ (which also depends on f and A), in the sense

$$f^{\beta}(a) = f^{\xi}(a), \text{ for all } \xi \geqslant \beta. \tag{2.6}$$

Precisely, let γ denote the order type of $(B, \leqslant)$. Hence, B is order isomorphic with $W(\gamma)$. And this, in conjunction with (2.3), shows γ is necessarily a first kind ordinal (that is, $\beta = \gamma - 1$ exists) and proves the assertion above, in view of (2.2).

Remark 1. The ordinal in question is necessarily admissible with respect to the ambient Grothendieck universe $\mathscr{G}$ which includes/contains A. In fact,

$$\mathscr{P}(A) = \text{the family of all subsets in } A$$

is a member/part of the same universe. Hence, any sub-family of $\mathscr{P}(A)$ -in particular, the one appearing in the definition of B- is again endowed with such a property. But, in this case, B is a member/part of the same universe; and then, the assertion follows.

Remark 2. The process of determining this ordinal is not depending on the Axiom of Choice. Nevertheless, it is true that, with the aid of this axiom, a more direct proof of the statement above is available. Precisely, denote $k = \text{card}(A)$ and let m be another cardinal with

$$k < m = \text{aleph number}$$

(Note that each of these conditions requires the Axiom of Choice. Because, the former is not, in general valid without the trichotomy law; and the latter may fail, in general, without the aleph hypothesis, cf. Sierpinski [10, Ch. 16, § 1]). Denote also by μ the initial ordinal associated to m. Assume the transfinite sequence $\mathscr{S}$ of all $(f^{\xi}(a))$ is constructible over $W(\mu)$, in the sense of (2.5′). Then, $\mathscr{S}$ is order isomorphic with $W(\mu)$; and consequently, it has the cardinality m. This, hovever, is in contradiction with

$$m = \text{card}(\mathscr{S}) \leqslant \text{card}(A) = k.$$

Hence, the transfinite sequence in question must stop for a certain ordinal $\beta < \mu$; and the proof of Proposition 1 is complete.

As a consequence of these facts, the mapping (from A to itself)

(D$_4^2$) $f^{\infty}(a) = \sup_{(\xi)} \{f^{\xi}(a)\}, \; a \in A$

is well defined. The basic properties of it are collected in

Proposition 2. The following are valid:

$$f^{\xi}(a) \leqslant f^{\infty}(a), \text{ for all } \xi \text{ and all } a \in A\text{; and, consequently, } f^{\infty} \text{ is progressive over } A; \tag{2.7}$$

$$f^{\xi}(f^{\infty}(a)) = f^{\infty}(f^{\xi}(a)) = f^{\infty}(a), \text{ for all } \xi \text{ and all } a \in A\text{; hence, in particular, } f^{\infty}(a) \text{ is an element of } \mathrm{Fix}(f), \text{ for each } a \text{ in } A; \tag{2.8}$$

$$\text{if } f \text{ is increasing over } A, \text{ then so is } f^{\infty}. \tag{2.9}$$

Proof. The first part is obvious. For the second part, it suffices to note the stationarity property (2.6) as well as

$$f^{\xi+\eta}(a) = f^{\eta}(f^{\xi}(a)), \text{ for all } \xi, \eta \text{ and all } a \in A. \tag{2.10}$$

The last part is a consequence of the fact that, under the precised assumption about f,

$$f^{\xi} \text{ is increasing over } A, \text{ for all } \xi. \tag{2.11}$$

Hence the result. q.e.d.

Now, as f^{∞} is progressive too, the transfinite sequence of all its iterates $((f^{\infty})^{\xi}(a))$ is again stationary beyond a certain ordinal $\gamma = \gamma(a)$ (which also depends on f and A), for each a in A. Hence, the mapping (from A to itself)

(D_5^2) $(f^{\infty})^{\infty}(a) = \sup_{(\xi)} \{(f^{\infty})^{\xi}(a)\},\ a \in A$

is well defined, etc. This procedure may continuate indefinitely and seems to generate interesting problems. But, for the developments below, these are not effectively needed; and so, we do not discuss them.

§ 3. ADMISSIBLE CONES IN TOPOLOGICAL VECTOR SPACES

Let E be a (real) vector space and $\mathscr{T}$, *a* linear topology over it (introduced in the usual way). Denote

(D_1^3) $\mathscr{C}(E) =$ the class of all cones in E;

it is easily shown to be semi-complete with respect to the converse inclusion ($\supseteq$) over the family $\mathscr{P}(E)$ (of all subsets in E). Moreover, $\mathscr{C}(E)$ is invariant to the closure operator "cl" induced by the ambient linear topology ($\mathscr{T}$). As a consequence of this, the selfmap of $\mathscr{C}(E)$

(D_2^3) $T(X) = X \cap \mathrm{lin\,cl}(X),\ X \in \mathscr{C}(E)$

is well defined; it is clearly shown to be progressive (with respect to the converse inclusion). Hence, for any cone C in E, the transfinite sequence of iterates $(T^{\xi}(C))$ defined as

(D_3^3) $T^0(C) = C$

$T^{\lambda}(C) = T(T^{\lambda-1}(C))$, if λ is a first kind ordinal

$= \cap\{T^{\xi}(C);\ \xi < \lambda\}$, otherwise is effectively constructable (Here, the ordinals ξ are in fact admissible ordinals with respect to a sufficiently large Grothendieck universe $\mathscr{G}$ containing/including E. But, in the following, the word "admissible" will be deleted). Now, clearly,

$$T^{\xi}(C) \supseteq T^{\eta}(C), \text{ whenever } \xi \leqslant \eta. \tag{3.1}$$

So, the question arises of to what extent is the property

$$T^{\xi}(C) \supset T^{\eta}(C), \text{ for } \xi < \eta \tag{3.1'}$$

avoidable (where, as usually, $\supset$ stands for the strict converse inclusion). Moreover -supposing this would be true- it is natural to ask of which supplementary properties has the associated self-mapping T^{∞} of $\mathscr{C}(E)$, introduced as

(D_4^3) $T^{\infty}(X) = \bigcap_{(\xi)} \{T^{\xi}(X)\}$, $X \in \mathscr{C}(E)$.

An appropriate answer to these is concentrated in

Proposition 3. Let the notations above be accepted. Then,

A) For any cone $\mathscr{C}$ in E there may be determined an ordinal $\beta = \beta(C)$ (which also depends on E and T) such that the transfinite sequence $(T^{\xi}(C))$ becomes stationary beyond β, in the sense

$$T^{\beta}(C) = T^{\xi}(C), \text{ for all } \xi \geqslant \beta\,. \tag{3.2}$$

B) The selfmap T^{∞} of $C(E)$ (introduced as above) is well defined and increasing with respect to the converse inclusion.

Proof. The first part follows by Proposition 1 and the remarks following it. The second part is immediate, via Proposition 2, because T is increasing with respect to the converse inclusion. Hence the result. q.e.d.

With these informations at hand, let us now return to the notion of admissible cone introduced in Section 1. A useful characterization of it may be given along the lines below:

Proposition 4. The following are equivalent:

(i) C is an admissible cone (in the sense of (D_5^1))

(ii) $\left\{\begin{array}{l} D = \text{subcone of } C \text{ and } \mathrm{cl}(D) = \text{linear subspace imply} \\ C \cap \mathrm{cl}(D) = \text{linear subspace} \end{array}\right.$

(iii) $T^{\infty}(C) = \mathrm{lin}(C)$.

Proof. (iii) $\Longrightarrow$ (i): Let L be a closed subspace of lin cl(C) with cl$(L \cap C)$ = linear subspace. It is not hard to see, via transfinite induction, that

$$L \cap C = L \cap T^{\xi}(C), \text{ for all } \xi \tag{3.3}$$

(Actually, this relation holds for any subspace L of E with $\mathrm{cl}(L\cap C)$ = subspace of E). The deep part of the induction argument is the verification for $\xi = 1$. This, in turn, may be obtained, by means of

$$L\cap C = L\cap C\cap \mathrm{lin}\ \mathrm{cl}(L\cap C) \subseteq L\cap C\cap \mathrm{lin}\ \mathrm{cl}(L)\cap \mathrm{cl}(C)).$$

As a direct consequence,

$$L\cap C = L\cap T^{\infty}(C) = \text{linear subspace}$$

and the implication follows.

(i) $\Longrightarrow$ (ii): Let D be a subcone of C with $\mathrm{cl}(D)$ = linear subspace, and put $L = \mathrm{cl}(D)$. We have (by some elementary devices)

$$\mathrm{cl}(L\cap C) = L\,(= \text{closed linear subspace of } \mathrm{lin}\ \mathrm{cl}(C)).$$

Hence, by (i), $L\cap C = \mathrm{cl}(D)\cap C$ is a linear subspace.

(ii) $\Longrightarrow$ (iii): Denote for simplicity $D = T^{\infty}(C)$. It is simply to verify, via $D=T(D)$, that $\mathrm{cl}(D)$ is a linear subspace of $\mathrm{lin}\ \mathrm{cl}(C)$. Hence, by (ii), $C\cap \mathrm{cl}(D)$ is a linear subspace. On the other hand, we have, by (3.3) (with $L=\mathrm{cl}(D)$), that

$$C\cap \mathrm{cl}(D) = T^{\xi}(C)\cap \mathrm{cl}(D)\,,\ \text{for all } \xi\,.$$

This immediately gives (by the adopted notation)

$$C\cap \mathrm{cl}(D) = D\cap \mathrm{cl}(D) = D\,;$$

or, in other words, D is a linear subspace (of $\mathrm{lin}\,(C)$). Now, it is easy to verify, via transfinite induction, that

$$\mathrm{lin}\,(C) = \mathrm{lin}\ T^{\xi}(C),\ \text{for all } \xi \tag{3.4}$$

(As before, the deep part of the induction argument is one concerning the case $\xi = 1$; and this, by the definition of T, is immediate). Consequently,

$$\mathrm{lin}\,(C) = \mathrm{lin}\,(D) = D$$

and the assertion is proved. q.e.d.

It is useful to note that, in view of (3.4), a sufficient condition for the admissibility of C is

(K_7) $\quad T^{\gamma}(C) = \mathrm{lin}\,(C)$, for some ordinal γ.

On the other hand, the properties (i)-(iii) above are, respectively, equivalent with their counterparts

(i*) The property ($D_5^!$), with L, an arbitrary subspace of E

(ii*) $\left\{\begin{array}{l}\text{The property (ii), with the subcone } D \text{ of } C \text{ being, in addition,}\\ \text{linearly compatible with } C \text{ (i.e., } \mathrm{lin}\,(D) = \mathrm{hn}\,(C))\end{array}\right.$

(iii*) $T^{\infty}(C)$ is a linear subspace (of E).

Some related aspects were discussed in Sterna-Karwat [13].

Denote in the following

(D_5^3) $\mathscr{A}(E)$ = the class of all admissible cones in E.

A basic property of this family is precised in

Proposition 5. The class $\mathscr{A}(E)$ is semi-complete with respect to the converse inclusion.

Proof. Let $\mathscr{M}$ be a family of admissible cones in E and put $C = \cap \mathscr{M}$. We have (by part B) of Proposition 3

$$T^{\infty}(C) \subseteq \cap \{T^{\infty}(K);\ K \in \mathscr{M}\} = \cap \{\operatorname{lin}(K);\ K \in \mathscr{M}\} = \operatorname{lin}(C).$$

On the other hand, it follows from (3.4) that $T^{\infty}(C) \supseteq \operatorname{lin}(C)$. Hence $T^{\infty}(C) = \operatorname{lin}(C)$; and this, by the statament above, proves that C is an admissible cone, as desired. q.e.d.

We complete these facts with another one, of algebraic nature.

Proposition 6. The class of all admissible cones has the invariance property

$$C \in \mathscr{A}(E) \Longrightarrow -C \in \mathscr{A}(E). \tag{3.5}$$

Proof. It is easy to see, using the transfinite induction, that

$$T^{\xi}(-C) = -T^{\xi}(C), \text{ for all } \xi \tag{3.6}$$

(And, to do this, it will suffice noting that the validity of (3.6) for $\xi = 1$ is assured). This, combined with (iii) of Proposition 4 ends the argument. q.e.d.

It is not without importance to specify that, in all these developments, the (non-integer) scalar multiplication operation in E were not effectively used. So, these results remain valid in case of E being a *topological* (*additive*) *group* and C, a *semigroup* in E.

§ 4. Some relative comp-max statements

Let $(E, \mathscr{T})$ be a (real) topological vector space. It is our aim in the following to show that the comp-max problem formulated in Section 1 may be put in a more general setting. This, coupled with the developments of the preceding section will enable us getting a local version of Theorem 2, with a different proof than the original one due to Sterna-Karwat [11] (See the next section for technical details).

We start our considerations with the following "diagonal" version of Theorem 1:

Theorem 4. Let $\alpha > 0$ be an ordinal number and $(C_\xi\ ;\ \xi < \alpha)$, a family of closed cones in E, descending with respect to the usual inclusion in $\mathscr{P}(E)$; that is

(DF) $C_\xi \supseteq C_\eta$, whenever $\xi \leqslant \eta < \alpha$.

Then,

$$\cap\{\max(H, C_\xi)\ ;\ \xi < \alpha\} \text{ is nonempty and cofinal (modulo } C_0) \text{ in } H, \text{ for each (nonempty) compact part } H \text{ of } E. \tag{4.1}$$

Proof. Let x be an arbitrary element of H. By Theorem 1, we find a $y_0 \in \max(H, C_0)$ with $x \leqslant y_0 \pmod{C_0}$. Further, starting from this y_0, there exists, again by Theorem 1, a $y_1 \in \max(H, C_1)$ with $y_0 \leqslant y_1 \pmod{C_1}$; note that, as $C_0 \supseteq C_1$, one gets

$$y_0 \leqslant y_1 \pmod{C_0} \text{ (hence } x \leqslant y_1 \pmod{C_0}).$$

Generally, suppose that, for the ordinal number $\lambda < \alpha$ we constructed a net $(y_\xi)_{\xi<\lambda}$ in H, with the properties

$$y_\xi \leqslant y_\eta \pmod{C_\xi}, \text{ when } \xi < \eta < \lambda \tag{4.2}$$

$$y_\zeta \in \max(H, C_\xi), \text{ for each } \zeta < \lambda. \tag{4.3}$$

If λ is a first kind ordinal, put $\lambda - 1 = \mu$. We thus have

$$y_\xi \leqslant y_\mu \pmod{C_\xi}, \text{ when } \xi \leqslant \mu.$$

Now, again by Theorem 1, choose a $y_\lambda \in \max(H, C_\lambda)$ with $y_\mu \leqslant y_\lambda \pmod{C_\lambda}$. From (DF) + (4.2), one has

$$y_\xi \leqslant y_\lambda \pmod{C_\xi}, \text{ for } \xi < \lambda. \tag{4.2'}$$

That is, (4.2) holds with $\eta = \lambda$ (Note, formally, that (4.3) also holds for $\xi = \lambda$). If λ is a second kind ordinal, the net $(y_\xi)_{\xi<\lambda}$ has, by the compactness of H, an accumulation point (in H) say t. In view of

$$\{y_\xi\ ;\ \zeta \leqslant \xi < \lambda\} \subseteq y_\zeta + C_\zeta,\ \zeta < \lambda$$

plus the closedness of C_ζ, it is clear that

$$y_\zeta \leqslant t \pmod{C_\zeta}, \text{ for each } \zeta < \lambda.$$

Again by Theorem 1, choose $y_\lambda \in \max(H, C_\lambda)$ with $t \leqslant y_\lambda \pmod{C_\lambda}$. Hence, (4.2') is fulfilled; and, from this, (4.2) is valid for $\eta = \lambda$ (That (4.3) also holds for $\zeta = \lambda$ is trivial). Summing up, the net (y_ξ) is constructible over $W(\alpha)$ so that (4.2) + (4,3) be fulfilled (with $\lambda = \alpha$). But, in this case, the procedure we just described may be used as well to produce a point y in H, with

$$y_\xi \leqslant y \pmod{C_\xi}, \text{ for all } \xi < \alpha.$$

The obtained point is an element of $\cap\{\max(H, C_\xi);\ \xi < \alpha\}$; and, moreover, $x \leqslant y \pmod{C_0}$. Hence the conclusion. q.e.d.

We now introduce a useful convention. Let C, D be a couple of cones in E with $C \supseteq D$. For any (nonempty) part Y of E, denote by $\max(Y; C, D)$ the subset of all $z \in Y$ with the property

(D_1^4) $w \in Y$, $z \leqslant w \pmod{C} \Longrightarrow z \leqslant w \pmod{D}$.

We note the trivial implication

$$D = \operatorname{lin}(C) \Longrightarrow \max(Y; C, D) = \max(Y, C). \tag{4.4}$$

Under the model of Section 1, we say that the pair (C, D) has the *comp-max property*, when

(D_2^4) $\max(Y; C, D)$ is nonempty, for each (nonempty) compact part H of E.

As an application of the statement above, one has

Theorem 5. Let C be a cone in E. Then, the pair $(C, T^\infty(C))$ has the comp-max property, in the sense

$$\max(H; C, T^\infty(C)) \text{ is nonempty and cofinal } (\operatorname{mod} \operatorname{cl}(C)) \text{ in } H, \text{ for each (nonempty) compact part } H \text{ of } E. \tag{4.5}$$

Proof. Denote for simplicity

$$C_\xi = T^\xi(C), \text{ for all } \xi\,; \; D = T^\infty(C)$$

(Here, the selfmap T of $\mathscr{C}(E)$ is one introduced in Section 3). It follows by Proposition 3 that, an ordinal $\beta = \beta(C)$ may be found so that the (descending) net (C_ξ) becomes stationary beyond β; that is,

$$C_\beta = C_\xi\,, \text{ for all } \xi \geqslant \beta \text{ (hence } C_\beta = D).$$

Note that, as a consequence of this, the net $(\operatorname{cl}(C_\xi);\ \xi < \beta+1)$ is again descending. Let H be a (nonempty) compact part of E. By the statement above,

$$H_C = \cap\{\max(H, \operatorname{cl}(C_\xi));\ \xi < \beta+1\}$$

is nonempty and cofinal (mod cl(C)) in H. We now claim that $H_C \subseteq \max(H; C, D)$ (and this will complete the argument). Let x be arbitrary fixed in H and suppose $y \in H$ fulfils

$$x \leqslant y \pmod{C} \text{ (that is, } x \leqslant y \pmod{C_0}).$$

We thus have $x \leqslant y$ (mod cl(C_0)); this, plus $x \in \max(H, \operatorname{cl}(C_0))$ gives $x \leqslant y$ (mod hn cl(C_0)) wherefrom (again by the information above)

$$x \leqslant y \pmod{C_0 \cap \operatorname{lin}\operatorname{cl}(C_0) = C_1}.$$

Generally, assume that, for an ordinal number $\lambda < \beta+1$ one has an information like

$$x \leqslant y \pmod{C_\xi}, \text{ for all } \xi < \lambda\,. \tag{4.6}$$

If λ is a first kind ordinal, the argument above (with $C_{\lambda-1}$ in place of C_0) gives

$$x \leqslant y \ (\text{mod } C_\lambda) \ (\text{i.e., (4.6) holds for } \xi = \lambda).$$

If λ is a second kind ordinal then (4.6) plus the construction of C_λ yields the same conclusion. Hence, (4.6) is anyway true over $W(\beta+1)$. In particular, this must be the case for $\xi = \beta$; that is, $x \leqslant y$ (mod D). Summing up, x is an element of max $(H; C, D)$. As x was arbitrary chosen in H_C, the claim follows. q.e.d.

§ 5. Local versions of Theorems 2 and 3

Now, with these informations at hand, we are able to make precise the considerations developed in the introductory section. Let E be a linear space over the reals, endowed with a linear topology $\mathscr{T}$. Under the conventions precised in Section 1, call the cone C in E, admissible when (D^1_5) holds. Remember that, in view of Proposition 3, this property may be also expressed as

$$D = \text{lin}(C) \ (\text{where } D = T^\infty(C)).$$

But, in such a case, the comp-max property of the pair (C, D) is equivalent with the comp-max property of C. So, combining with Theorem 5, we get the following local version of Theorem 2:

Theorem 2′. Let the cone C in E be admissible. Then, C is a comp-max cone, in the sense

$$\text{max}(H, C) \text{ is nonempty and cofinal (mod cl}(C)) \text{ in } H, \text{ for each (nonempty) compact part } H \text{ of } E. \tag{5.1}$$

The following particular aspect of this result is to be noted. Let the ambient linear topology $\mathscr{T}$ over E be Hausdorff separated and locally convex. Denote by E^* the *topological dual* of E (that is, the class of all continuous linear functionals over E). The pair (E, E^*) is, with respect to the bilinear form

(D^5_1) $\langle x, x^* \rangle = x^*(x), \ x \in E, \ x^* \in E^*$,

a *dual couple*; and the initial topology over E is compatible with duality. On the other hand, the weak topology $\mathscr{T}_w$ over E is also compatible with duality; hence (denoting by wcl the closure operator attached to $\mathscr{T}_w$)

$$\text{wcl}(X) = \text{cl}(X), \text{ for each convex part } X \text{ of } E \tag{5.2}$$

(See, for instance, Cristescu [4, Ch. 3, § 3]). As an immediate consequence, the selfmap T of $\mathscr{C}(E)$ introduced as in Section 3 has the same form in both $\mathscr{T}$ and $\mathscr{T}_w$; and this, combined with Proposition 4, shows the class of admissible cones is the same in either of these topologies. This immediately gives the following improvement of the statement above (for such linear topologies):

Theorem 2*. Let the ambient (Hausdorff separated) topological vector space E be locally convex. Then, conclusion of Theorem 2′ is retainable with "compact" replaced by "weakly compact", i.e.,

$$\begin{aligned}&\max(H, C) \text{ is nonempty and cofinal (mod cl}(C)) \text{ in}\\ &E, \text{ for each (nonempty) weakly compact part } H \text{ of } E.\end{aligned} \tag{5.3}$$

This result may be also viewed as a local variant of one in Sterna-Karwat [12]. For other aspects of the problem we refer to the paper of Borwein [1].

Returning to the general setting, it is the moment to specify that the comp-max property of a cone C (in E) appears as a global one; because, all compact parts H (of E) are involved (in (CM)). So, it is natural to ask of whether or not is this removable. The answer is positive and may be given along the following lines: Let L be a closed subspace of E. We say that the cone C in E has the comp-max property over L, when

(D_1^5) $\max(H, C)$ is nonempty, for aech (nonempty) compact part H of L.

Clearly, the comp-max property over E (of the underlying cone) may be transferred upon any closed subspace L (of E). The reciprocal is also valid, in some circumstances. Precisely, we have

Theorem 6. Suppose the closed subspace L (of E) includes C. Then, the comp-max property (over E) of C is equivalent with its comp-max property over L.

Proof. Suppose C is a comp-max cone over L, and let H be an arbitrary fixed (nonempty) compact part of E. By Theorem 1, $\mathrm{cl}(C)$ is anyway a comp-max cone in E; so, $\max(H, \mathrm{cl}(C))$ is nonempty. Without loss (making a translation if necessary) one may suppose $0 \in H$. Denote $K = H \cap \mathrm{cl}(C)$. By hypothesis, $\max(K, C)$ is nonempty; and, to complete the argument, it will suffice proving that

$$\max(K, C) \subseteq \max(H, C). \tag{5.4}$$

To do this, let z be arbitrary fixed in $\max(K, C)$ and w in H be such that $z \leqslant w$ (mod C). We thus have $z \leqslant w$ (mod $\mathrm{cl}(C)$); and, in view of $z \in \mathrm{cl}(C)$, one gets $w \in H \cap \mathrm{cl}\, C) = K$. This, combined with the choice of z, yields $z \leqslant w$ (mod $\mathrm{lin}(C)$); that is, $z \in \max(H, C)$ and the conclusion follows. q.e.d.

As a consequence, the comp-max property of a cone C in E is equivalent with one relative to any closed linear subspace L including C; for instance, $L = \mathrm{cl}\,(C - C)$. This may be useful in some concrete situations, such as the one characterized by

(K_5) $\left\{\begin{array}{l}\mathscr{T}\, L \text{ is stronger or equal to } \mathscr{T}_f/L, \text{ for some}\\ \text{closed subspace } L \text{ (of } E) \text{ including } C\end{array}\right.$

(Here, $\mathscr{T}/L$ is he *trace* of the linear topology $\mathscr{T}$ over the (closed subspace) L). In particular, the global condition of Theorem 3 implies (K_5); hence, the corresponding (modulo (K_5)) form of Theorem 6 may be viewed as a local variant of the quoted result. Concerning this last aspect, the following facts are of interest: Denote

$$LT(E) = \text{the family of all linear topologies over } E.$$

We call the member $\mathscr{T}$ of $LT(E)$, *normal*, when

(D_2^5) each cone C in E has the comp-max property.

The class of all such linear topologies is nonempty, by Theorem 3. We also have the immed ate implication

$$\begin{array}{c}\text{each member of } LT(E) \text{ which is stronger} \\ \text{than a normal one s also normal.}\end{array} \tag{5.5}$$

So, it is natural to define

(D_3^5) $\mathscr{T}_{cm} = \inf \{\mathscr{T} \in LT(E);\ \mathscr{T} \text{ is normal}\}$.

Clearly, $\mathscr{T}_{cm}$ is a member of $LT(E)$; it will be called the *comp-max topology* of E. Moreover, one has by Theorem 3, that

$$\mathscr{T}_{cm} \text{ is coarser or equal with } \mathscr{T}_f. \tag{5.6}$$

An open problem to be solved in this context is that of classifying $\mathscr{T}_{cm}$ from a normality viewpoint; i.e., of whether or not is this (linear) topology (over E) normal. Of course, the natural setting for this problem is the infinite-dimensional one; but (cf. the concrete example in Section 1) even the finite dimensional case must be treated with care. For other aspects of the problem we refer to the 1989 monograph by Luc [7].

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