THE BASIS NUMBER OF THE COMPOSITION OF GRAPHS*,**

Mohammad Q. HAILAT - Ma'ared Y. AL-ZOUBI

Department of Mathematics and Computer Science, United Arab Emirates University, Al-Ain, P.O. Box: 17551, United Arab Emirates

Department of Mathematics, Yarmouk University, Irbid-Jordan

Summary: The basis number of a graph G is defined to be the least integer k such that G has a k-fold basis for its cycle space. We investigate the basis number of the composition of two paths, two cycles, a path and a cycle, a path and a wheel, a cycle and a wheel, a star and a wheel, a star and a path, a star and a cycle, a wheel and a path, a wheel and a cycle, a star and a wheel, and a star and a star.

GRAFLARIN BİLEŞKESİNİN BAZ SAYISI

Özet : Bu çalışmada iki yol, iki devre, bir yol ve bir devre, bir yol ve bir tekerlek, bir devre ve bir tekerlek, bir yıldız ve bir tekerlek, bir yıldız ve bir yol, bir yıldız ve bir devre, bir tekerlek ve bir yol, bir tekerlek ve bir devre, bir yıldız ve bir tekerlek, son olarak ta bir yıldız ile diğer bir yıldızın bileşkesinin baz sayısı araştırılmaktadır.

1. INTRODUCTION

In [4] S. Hulsurkar studied the graph structure on Weyl groups. He showed that with some exceptions the graph $\Gamma(W)$ is non-planar where $\Gamma(W)$ is the graph defined for Weyl groups which is compatible with the partial order introduced earlier for the proof of Verma's conjecture on Weyl's dimension polynomial [5]. The importance of Hulsurkar study lies in the fact that it will ultimately shed some light on the modular representations of semi-simple Lie algebras and Chevalley groups [8].

In 1937 S. Maclane [6] proved that a graph G is planar if and only if $b(G) \leq 2$ where $b(G)$ is the basis number of G (defined below). Thus the basis number of

* This research was supported in part by Yarmouk University, and it was revised at the UAE University.

** AMS (MOS) Subject Classification (1991)
Primary : 05C38, 05C75
Secondary : 05B25

certain classes of non-planar graphs will play an important role in studying the graphs $\Gamma(W)$. In 1981 E. Schemeichel [7] investigated the basis number of certain classes of non-planar graphs, namely, complete graphs and complete bipartite graphs. Then, J. Banks and E. Schemeichel [2] proved that for $n \geq 7$, $b(Q_n) = 4$, where Q_n is the n-cube. Also A. A. Ali in [1] investigated the basis number of the strong product of two paths, two cycles, a path and a cycle, and a star and a cycle. The purpose of this paper is to investigate the basis number of the **composition** of two paths, two cycles, a star and a path, a path and a cycle, a path and a wheel, a cycle and a wheel, a star and a wheel, a star and a path, a star and a cycle, a wheel and a path, a wheel and a cycle, a star and a wheel and a star and a star. It happens that the strong product of two graphs G_1 and G_2 is a subgraph of the composition of G_1 and G_2.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we consider only finite connected graphs without loops and multiple edges. Our terminology and notation will be standard except as indicated. For undefined terms, see [3].

Let G be a connected graph, and let $e_1, e_2, \ldots, e_q$ be an ordering of the edges in G. Then any subset of edges F corresponds to a (0,1)-vector $(a_1, a_2, \ldots, a_q)$ in the usual way, with $a_i = 1$ (resp., $a_i = 0$) if and only if $e_i \in F$ (resp., $e_i \notin F$). These vectors form a q-dimensional vector space over the field $\mathbf{Z}_2$. The vectors corresponding to the cycles in G generate a subspace of $(\mathbf{Z}_2)\, q$ called the **cycle space of** G, and denoted by $\zeta(G)$ (For brevity in the sequal, we will say that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\zeta(G)$). It is known that $\dim(\zeta(G)) = \gamma(G) = q - p + 1$ (Cor. 4.5 (a) of [3]) where p, q denote, respectively, the number of vertices and edges in G. Each vector in $\zeta(G)$ represents either a cycle or an edge-disjoint union of cycles.

2.1 Definition. The **composition** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1[G_2]$, is a graph with a vertex-set $V_1 \times V_2$ and an edge-set $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) \mid \text{either } u_1 u_2 \in E_1 \text{ or } [u_1 = u_2 \text{ and } v_1 v_2 \in E_2]\}$.

2.2 Definition. The **strong product** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \circ G_2$, is a graph with a vertex-set $V_1 \times V_2$ and an edge-set $E(G_1 \circ G_2) = \{(u_1, v_1)(u_2, v_2) \mid \text{either } [u_1 = u_2 \text{ and } v_1 v_2 \in E_2], \text{ or } [u_1 u_2 \in E_1 \text{ and } v_1 = v_2], \text{ or } [u_1 u_2 \in E_1 \text{ and } v_1 v_2 \in E_2]\}$. Note that $G_1 \circ G_2$ is isomorphic to $G_2 \circ G_1$ and $|E(G_1 \circ G_2)| = p_1 q_2 + p_2 q_1 + 2 q_1 q_2$ where $p_i = |V_i|$ and $m_i = |E_i|$ for $i = 1, 2$. But $G_1[G_2]$ is not isomorphic to $G_2[G_1]$ since $|E(G_1[G_2])| = p_1 q_2 + p_2^2 q_1$ while $|E(G_2[G_1])| = p_2 q_1 + p_1^2 q_2$.

2.3 **Definition.** A basis of $\zeta(G)$ is a **k-fold** basis if each edge of G occurs in at most k of the cycles in the basis. The **basis number of G** (denoted by $b(G)$) is the smallest k such that $\zeta(G)$ has a k-fold basis.

In this paper we use the notations P_n, C_n, S_n and W_n to denote the vertex-set of these graphs, the addition group $\mathbf{Z}_n$ of positive integers residue module n and the edge-set as follows:

$$E(P_n)=\{i(i+1) \mid 0 \le i \le n-2\},\ E(C_n)=\{i(i+1) \mid 0 \le i \le n-1\},$$

$$E(S_n)=\{0i \mid 1 \le i \le n-1\} \text{ and } E(W_n)=E(S_n) \cup \{i(i+1) \mid 1 \le i \le n-1\}.$$

We denote P_n by $012 \ldots (n-1)$ and C_n by $012 \ldots (n-1)\,0$.

3. MAIN RESULTS

In this section we compute the basis number of $C_m[P_2]$, $C_m[C_2]$, $P_m[C_3]$, $C_m[P_3]$ and $P_m[P_3]$ and we show that it is equal to 3. We also show that the basis number of $C_m[C_n]$, $P_m[P_n]$, $C_m[P_n]$, $P_m[C_n]$, $P_m[S_n]$, $P_m[W_n]$, $S_m[P_n]$, $C_m[S_n]$, $C_m[W_n]$, $S_m[C_n]$, $S_m[W_n]$ and $W_m[P_n]$ is either 3 or 4 for some restrictions on m and n.

One may easily see that $P_m[P_2]$ and $P_m[C_2]$ are planar graphs, and therefore, $b(P_m[P_2]) = b(P_m[C_2]) = 2$.

Proposition 3.1. For every integer $m \ge 3$, the graphs $C_m[P_2]$ and $C_m[C_2]$ are nonplanar.

Proof. It is easy to see that $C_3[P_2]=K_6$, so it is nonplanar. Now we consider the graph $C_m[P_2]$ such that $m \ge 4$. Contract the edges $\{(i,1)(i+1,1) \mid 2 \le i \le m-2\}$ to a new vertex v. Also, contract the edges $\{(i,0)(i+1,0) \mid 2 \le i \le m-2\}$ to a new vertex w. Then the resulting graph contains K_6 as a subgraph. That is $C_m[P_2]$ is contractible to a nonplanar graph, so that it is nonplanar. Since $C_m[P_2]$ is a subgraph of $C_m[C_2]$ then $C_m[C_2]$ is nonplanar.

Theorem 3.2. For every integer $m \ge 3$, we have $b(C_m[P_2])=b(C_m[C_2])=3$.

Proof. Since the graphs $C_m[P_2]$ and $C_m[C_2]$ are nonplanar for $m \ge 3$, it follows, by the Theorem of Maclane mentioned in the introduction, that $b(C_m[P_2]) \ge 3$ and $b(C_m[C_2]) \ge 3$ for all $m \ge 3$. Consider $C_m[P_2]$. To prove that $b(C_m[P_2]) \le 3$, we exhibit a 3-fold basis for $\zeta(C_m[P_2])$. Let $Q = (0,0)\,(1,0)\,(2,0) \ldots (m-1,0)\,(0,0)$; $A = \{(i,0)\,(i+1,0)\,(i,1)\,(i,0),$ $(i,1)\,(i+1,1)\,(i+1,0)\,(i,1),\ (i,0)\,(i+1,0)\,(i,1)\,(i+1,1)\,(i,0) \mid i \in \mathbf{Z}_m\}$. Let $B(C_m[P_2]) = A \cup \{Q\}$. Since the graph $G=C_m[P_2]-\{(i,0)\,(i+1,1) \mid i \in \mathbf{Z}_m\}$ is a planar graph and the 3-cycles of A represent the boundaries of the interior faces, so that A is a basis for $\zeta(G)$. Since each 4-cycle in A contains an edge, namely, of the form $(i, 0)\,(i+1, 1)$ that does not occur in any other cycle of A, A is an independent set of cycles in $\zeta(C_m[P_2])$. But it is clear that Q cannot be generated from A. Thus $B(C_m[P_2])$ is an independent set of cycles in $\zeta(C_m[P_2])$. Since $|B(C_m[P_2])| = 3m+1 = \dim\ \zeta(C_m[P_2])$, it follows that $B(C_m[P_2])$ is a basis of $\zeta(C_m[P_2])$. It is easy to see that $B(C_m[P_2])$ is a 3-fold basis for $\zeta(C_m[P_2])$. Hence, $b(C_m[P_2]) = 3$ for all $m \geq 3$. Next, we consider $C_m[C_2]$. The graph $C_m[C_2]$ can be obtained from the graph $C_m[P_2]$ by joining any two vertices $(i, 0)$, $(i, 1)$ by another edge, for all $i \in \mathbf{Z}_m$. Let $B(C_m[C_2]) = B(C_m[P_2]) \cup \{(i, 0), (i, 1)\,(i, 0) \mid i \in \mathbf{Z}_m\}$. Then $B(C_m[C_2])$ is an independent set of cycles in $(C_m[C_2])$ since each of the new cycles has an edge not occuring in any other cycle of $B(C_m[C_2])$. Since $|B(C_m[C_2])| = = \dim \zeta(C_m[C_2])$, then $B(C_m[C_2])$ is a basis for $\zeta(C_m[C_2])$. It is easy to verify that $B(C_m[C_2])$ is a 3-fold basis for $\zeta(C_m[C_2])$. Hence $b(C_m[C_2]) = 3$. This completes the proof of Theorem 3.2.

Theorem 3.3. For each $m \geq 2$, we have $b(P_m[C_3]) = b(C_m[P_3]) = 3$.

Proof. It is easy to see that $P_m[C_3]$ contains $m-1$ copies of the complete bipartite graph $K_{3,3}$, each of which is denoted by $K_{(i,3),\,(i+1,3)}$, $i \in \mathbf{Z}_{m-1}$. In addition to that it contains the set of edges $S = \{(i, 0)\,(i, 1), (i, 1)\,(i, 2), (i, 2)\,(i, 0) \mid i \in \mathbf{Z}_m\}$. Since $K_{3,3}$ is a nonplanar subgraph of $P_m[C_3]$ then $P_m[C_3]$ is a nonplanar graph, so that, by Maclane Theorem, $b(P_m[C_3]) \geq 3$. To prove that $b(P_m[C_3]) \leq 3$, it is enough to exhibit a 3-fold basis for $(P_m[C_3])$. For each $i \in \mathbf{Z}_{m-1}$, let

$$
\begin{aligned}
A_i &= \{(i, 0)\,(i+1, 0)\,(i, 1)\,(i+1, 1)\,(i, 0),\ (i, 0)\,(i+1, 1)\,(i, 1)\,(i+1, 2)\,(i, 0),\\
&\quad\ (i, 0)\,(i+1, 0)\,(i, 2)\,(i+1, 1)\,(i, 0),\ (i, 1)\,(i+1, 1)\,(i, 2)\,(i+1, 2)\,(i, 1)\},\\
A_i' &= \{(i+1, 0)\,(i, 1)\,(i, 2)\,(i+1, 0),\ (i+1, 2)\,(i, 0)\,(i, 1)\,(i+1, 2),\\
&\quad\ (i, 2)\,(i+1, 1)\,(i+1, 2)\,(i, 2),\ (i, 2)\,(i+1, 0)\,(i+1, 1)\,(i+1, 2)\,(i, 2)\},\\
&\quad\ B_i = A_i \cup A_i'
\end{aligned}
$$

Let $F = \{f_i = (i, 0)\,(i-1, 0)\,(i, 2)\,(i, 0) \mid 1 \leq i \leq m-1\} \cup \{(0, 0)\,(0, 1)\,(0, 2)\,(0, 0)\}$ and $B(P_m[C_3]) = F \cup \left(\bigcup_{i=0}^{m-2} B_i\right)$. It is already proved in Theorem 2.3 of Schemeichel [7] that each A_i is a basis for the subspace $\zeta(K_{(i,3),\,(i+1,3)})$ for all $i \in \mathbf{Z}_{m-1}$. Then each A_i is an independent set of cycles in $\zeta(P_m[C_3])$. Also for each $i \in \mathbf{Z}_{m-1}$, every cycle in A_i' contains an edge from the set $H = \{(i, 0)\,(i, 1), (i, 1)\,(i, 2), (i+1, 0)\,(i+1, 1), (i+1, 1)\,(i+1, 2)\}$ that does not appear in any other cycle of $A_i \cup A_i'$. Thus each B_i is an independent set of cycles in $\zeta(P_m[C_3])$. Moreover, since the edge-sets of $K_{(i,3),\,(i+1,3)}$ are pairwise-disjoint sets where $i \in \mathbf{Z}_{m-1}$, then the cycles of B_j are independent from the cycles of B_k for all $j, k \in \mathbf{Z}_{m-1}$, $j \neq k$. Thus $\bigcup_{i=0}^{m-1} B_i$ is an independent set of cycles in $\zeta(P_m[C_3])$.

Since each $f_i \in F$ contains the edge $(i, 2)(i, 0)$; $i \in \mathbf{Z}_m$, that does not occur in any other cycle of $B(P_m[C_3])$, therefore, $B(P_m[C_3])$ is an independent set of cycles in $\zeta(P_m[C_3])$. But $|B(P_m[C_3])| = 9m - 8 = \dim \zeta(P_m[C_3])$, hence $B(P_m[C_3])$ is a basis for $\zeta(P_m[C_3])$. It is a simple matter to verify that $B(P_m[C_3])$ is a 3-fold basis for $\zeta(P_m[C_3])$. Hence $b(P_m[C_3]) = 3$. Next, we consider $C_m[P_3]$. It contains m copies of $K_{3,3}$; denote each one by $K_{(i,3),(i+1,3)}$ where, $i \in \mathbf{Z}_m$, noting that the last copy is $K_{(m-1,3),(0,3)}$. Also $C_m[P_3]$ contains the set $H' = \{(i, 0)(i, 1), (i, 1)(i, 2) \mid i \in \mathbf{Z}_m\}$. Since $C_m[P_3]$ contains some copies of $K_{3,3}$ then $b(C_m[P_3]) \geq 3$, for all $m \geq 3$. To find a 3-fold basis for $\zeta(C_m[P_3])$ let A_i and A_i' be the same sets above and let $Q = (0, 0)(1, 0)(2, 0)\ldots(m-1, 0)(0, 0)$. Let $B_i = A_i \cup A_i'$. Then $\bigcup_{i=0}^{m-1} B_i$ is an independent set of cycles as seen above and it is clear that Q is independent from all the cycles of $\bigcup_{i=0}^{m-1} B_i$. Thus the set $B(C_m[P_3]) = \{Q\} \cup \left(\bigcup_{i=0}^{m-1} B_i\right)$ is an independent set of cycles in $\zeta(C_m[P_3])$. Since $|B(C_m[P_3])| = 8m + 1 = \dim \zeta(C_m[P_3])$ then the set $B(C_m[P_3])$ is a basis for $\zeta(C_m[P_3])$. One can easily verify that $B(C_m[P_3])$ is a 3-fold basis for $\zeta(C_m[P_3])$. Hence $b(C_m[P_3]) = 3$ for all $m \geq 3$.

Corollary 3.4. For every $m \geq 2$, we have $b(P_m[P_3]) = 3$.

Proof. Note that $P_m[P_3]$ is a nonplanar subgraph of $P_m[C_3]$ consisting of $m-1$ copies of $K_{3,3}$ and the set $\{(i, 0)(i, 1), (i, 1)(i, 2) \mid i \in \mathbf{Z}_m\}$. Then the set $B(P_m[P_3]) = B(P_m[C_3]) - P$ is a 3-fold basis for $\zeta(P_m[P_3])$.

Lemma 3.5. Let G be a graph with p vertices and q edges. If $|C|$ denotes the length of the cycles C, and $\beta = \{C_1, \ldots, C_d : |C_i| \geq r\}$ be a k-fold basis of $\zeta(G)$, then $rd \leq \sum_{i=1}^{d} |C_i| \leq kq$, where $d = \dim \zeta(G)$.

Proof. Since $|C_i| \geq r$ for all $1 \leq i \leq d$, we have $\sum_{i=1}^{d} |C_i| \geq \sum_{i=1}^{d} r = rd$. Also, since β is a k-fold basis, we have $\sum_{i=1}^{d} |C_i| \leq kq$. Therefore, the lemma holds.

Lemma 3.6. Let m, n be two positive integers such that
$3mn + \left[\dfrac{3n^2 m + 3mn - 3}{4}\right] \geq mn^2 + 1$, where $[x]$ is the greatest integer less than or equal to x. Then $n \leq 14$.

Proof. Since $3mn+\left[\frac{3n^2 m+3mn-3}{4}\right]\leq 3mn+\frac{3n^2 m+3nm-3}{4}$ we have $mn^2+1\leq 3mn+\frac{3n^2 m+3mn-3}{4}$. This implies that $4mn^2+4\leq 12mn+3n^2 m+$ $+3mn-3$, so that $mn^2+7\leq 15mn$. This implies that $mn^2<15mn$, so that $n\leq 14$.

Theorem 3.7. If $m\geq 2$, $n\geq 3$ then $3\leq b(C_m[C_n])\leq 4$. Moreover, $b(C_m[C_n])=4$ for all $m\geq 2$, $n\geq 15$.

Proof. One may easily see that $C_m[C_n]$ contains m copies of $K_{n,n}$, each of which is of the form $K_{(r,n),(r+1,n)}$ where $r\in \mathbf{Z}_m$. In addition, $C_m[C_n]$ contains the set of edges $S=\{(r,i)(r,i+1)\mid r\in \mathbf{Z}_m,\ i\in \mathbf{Z}_n\}$. It is clear that $C_m[C_n]$ is a nonplanar graph for all $m\geq 2$, $n\geq 3$. Thus $b(C_m[C_n])\geq 3$. To prove that $b(C_m[C_n])\leq 4$, we have to exhibit a 4-fold basis for $\zeta(C_m[C_n])$. For each $r\in \mathbf{Z}_m$, define the following sets:

$A_r=\{(r,i)(r+1,j)(r,i+1)(r+1,j+1)(r,i)\mid 0\leq i,j\leq n-2\}$,

$A'_r=\{(r,0)(r+1,i)(r+1,i+1)(r,0),\ (r+1,n-1)(r,i)(r,i+1)(r+1,n-1)\mid 0\leq i\leq n-2)\}$, $B_r=A_r\cup A'_r$,

$D=\{d_i=(i,0)(i,1)(i,2)\ldots(i,n-1)(i,0)\mid i\in \mathbf{Z}_m\}$ and

$Q=(0,0)(1,0)(2,0)\ldots(m-1,0)(0,0)$.

Let $B(C_m[C_n])=\left(\bigcup_{r=0}^{m-1} B_r\right)\cup D\cup\{Q\}$. For every $r\in \mathbf{Z}_m$, A_r is a 4-fold basis of the subspace $\zeta(K_{(r,n),(r+1,n)})$, as proved in Theorem 2.4 of Schemeichel [7]. Using the same argument as of Theorem 3.3 after replacing H by $H'=(r,i)(r,i+1)$, $(r+1,i)(r+1,i+1)\mid\ i\in \mathbf{Z}_{n-i}\}$ we then have $\bigcup_{r=0}^{m-1} B_r$ is an independent set of cycles in $\zeta(C_m[C_n])$. Clearly Q cannot be generated from $\bigcup_{r=0}^{m-1} B_r$, so it is independent from the cycles of $\bigcup_{r=0}^{m-1} B_r$. Moreover, each cycle $d_i\in D$ contains the edge $(i,n-1)(i,0)$ which does not occur in any other cycle of $B(C_m[C_n])$. Thus $B(C_m[C_n])$ is an independent set of cycles in $\zeta(C_m[C_n])$. Since $|B(C_m[C_n])|=$ $=n^2 m+1=\dim\zeta(C_m[C_n])$, it follows that $B(C_m[C_n])$ is a basis for $\zeta(C_m[C_n])$. It is easy to verify that $B(C_m[C_n])$ is a 4-fold basis. Hence $b(C_m[C_n])\leq 4$ for all $m\geq 2$, $n\geq 3$, so that $3\leq b(C_m[C_n])\leq 4$ for all $m\geq 2$, $n\geq 3$.

On the other hand, suppose that $\zeta(C_m[C_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta| \leq 3mn$, since every 3-cycle in β must contain an edge from the set $S = \{(r, i)(r, i+1) \mid r \in \mathbf{Z}_m$ and $i \in \mathbf{Z}_n\}$, $|S| = mn$ and the fold of every edge of S is at most 3. But $|\beta| \leq 3mn < mn^2 + 1 = \dim \zeta(C_m[C_n])$ for $m \geq 2$ and $n \geq 3$. Hence β is not a basis for $\zeta(C_m[C_n])$, a contradiction.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4(mn^2+1) \leq 3(mn^2+mn)$, since $\dim \zeta(C_m[C_n]) = mn^2+1$, $|E(C_m[C_n])| = mn^2+mn$ and $|C_i| \geq 4$ for every $C_i \in \beta$. But this inequality cannot hold for any positive integers m and n. Hence this case cannot happen.

Case 3. Suppose that β consists of s 3-cycles, and t cycles of length greater than or equal to 4. Then $s \leq 3mn$ since we have at most $3mn$ 3-cycles in β as we explained in Case 1. Since $|E(C_m[C_n])| = mn^2+mn$, the fold of every edge of $C_m[C_n]$ is at most 3 in β and $3s$ edges are joined to make the s 3-cycles, we have $t \leq \left[\dfrac{3mn^2+3mn-3s}{4}\right]$. Then $mn^2+1 = \dim \zeta(C_m[C_n]) = |\beta| = s+t \leq 3mn+$ $+\left[\dfrac{3n^2m+3mn-3s}{4}\right]$, so that $mn^2+1 \leq 3mn + \left[\dfrac{3n^2m+3mn-3}{4}\right]$ since $s \geq 1$. This implies that $n \geq 14$, by Lemma 3.6, a contradiction to the assumption that $n \geq 15$. From the above 3 cases we deduce that $\zeta(C_m[C_n])$ has no 3-fold basis for all $m \geq 2$ and $n \geq 15$. Hence $b(C_m[C_n]) = 4$ for all $m \geq 2$ and $n \geq 15$.

Lemma 3.8. Let m, n be two positive integers $(m \geq 2)$ such that $4(m-1)(n^2-1) \leq 3(m(n-1)+n^2(m-1))$. Then $n \leq 5$.

Proof. Since $4(m-1)(n^2-1) \leq 3(m(n-1)+n^2(m-1))$ then $4n^2(m-1) -$ $-4m+4 \leq 3mn-3m+3n^2(m-1)$, so that $n^2(m-1)+4 \leq 3mn+m = m(3n+1)$. That is $\dfrac{n^2}{3n+1} + \dfrac{4}{(3n+1)(m-1)} \leq \dfrac{m}{m-1} \leq 2$, so that $\dfrac{n^2}{3n+1} < 2$. This implies that $n \leq 6$. But of $n=6$ the inequality in the statement of the Lemma does not hold, therefore, $n \leq 5$.

Lemma 3.9. Let m, n be two positive integers $(m \geq 2)$ such that $(m-1)(n^2-1) \leq 3m(n-1) + \left[\dfrac{3n^2(m-1)+3m(n-1)-3}{4}\right]$. Then $n \leq 29$.

Proof. Suppose that $(m-1)(n^2-1) \leq 3m(n-1) + \left[\dfrac{3n^2(m-1)+3m(n-1)-3}{4}\right]$.

Since $\left[\dfrac{3n^2(m-1)+3m(n-1)-3}{4}\right] \leq \dfrac{3n^2(m-1)+3m(n-1)-3}{4}$ then

$4(m-1)(n^2-1) \leq 12mn-12m+3n^2(m-1)+3mn-3m-3$, so that $4n^2(m-1)-4m+4 \leq 15mn-15m+3n^2(m-1)-3$. This implies that $n^2(m-1)+7 \leq 15mn-11m$, so that $\frac{n^2}{15n-11} \leq \frac{m}{m-1} \leq 2$. That is $n^2+22 \leq 30n$, so that $n \leq 29$.

Following the same idea as in the previous two lemmas we have the following two lemmas without proof:

Lemma 3.10. Let m, n be two positive integers $(m \geq 2)$ such that

$$m(n^2-1)+1 \leq 3m(n-1)+\left[\frac{3mn^2+3m(n-1)-3}{4}\right].$$

Then $n \leq 14$.

Lemma 3.11. Let m, n be two positive integers $(m \geq 2)$ such that

$$n^2(m-1)+1 \leq 3mn+\left[\frac{3mn+3n^2(m-1)-3}{4}\right].$$

Then $n \leq 29$.

Theorem 3.12. For every $m \geq 2$, $n \geq 4$, we have $3 \leq b(P_m[P_n])$, $b(P_m[C_n])$, $b(C_m[P_n]) \leq 4$. Moreover, $b(P_m[P_n])=4$ for $m \geq 2$, $n \geq 30$; $b(C_m[P_n])=4$ for $m \geq 2$, $n \geq 15$ and $b(P_m[C_n])=4$ for $m \geq 2$, $n \geq 30$.

Proof. It is easy to see that $P_m[P_n]$, $C_m[P_n]$, and $P_m[C_n]$ are nonplanar subgraphs of $C_m[C_n]$. And $B(C_m[P_n])=B(C_m[C_n])-D$, $B(P_m[C_n])=B(C_m[C_n])-(B_{m-1} \cup \{Q\})$ and $B(P_m[P_n])=B(P_m[C_n] \cap B(C_m[P_n])$ are 4-fold subbasis of $B(C_m[C_n])$ for the subspaces $\zeta(C_m[P_n])$, $\zeta(P_m[C_n])$ and $\zeta(P_m[P_n])$ respectively, where $B(C_m[C_n])$ is the basis of $\zeta(C_m[C_n])$ which is obtained in Theorem 3.7.

On the other hand, suppose that $\zeta(P_m[P_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta| \leq 3m(n-1)$ since every 3-cycle in β must contain an edge from the set $H=\{(r,i)(r,i+1) \mid r \in \mathbf{Z}_m$ and $i \in \mathbf{Z}_{n-1}\}$, $|H|=m(n-1)$ and the fold of every edge of S is at most 3. But $|\beta| \leq 3m(n-1) < (m-1)(n^2-1) = \dim \zeta(P_m[P_n])$ for $m \geq 2$, $n \geq 30$. Hence β cannot be a basis of $\zeta(P_m[P_n])$, a contradiction.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4(m-1)(n^2-1) \leq 3(m(n-1)+n^2(m-1))$ since $P_m[P_n]$ has $m(n-1)+n^2(m-1)$ edges, and $|C| \geq 4$ for every $C \in \beta$. It follows that $n \geq 5$ by Lemma 3.8, a contradiction to the assumption

Case 3. Suppose that β consists of s 3-cycles and t cycles of length greater than or equal to 4. Then $s \leq 3m(n-1)$ since we have at most $3m(n-1)$ 3-cycles

in β as we explained in Case 1. Since the fold of every edge of $E(P_m[P_n])$ is at most 3, and $3s$ edges are joined to make the s 3-cycles, then $t \le \left[\frac{3((m-1)n^2+m(n-1))-3s}{4}\right]$. Thus we have $(m-1)(n^2-1) = \dim \zeta(P_m[P_n]) =$
$= |\beta| = s + t \le 3m(n-1) + \left[\frac{3((m-1)n^2+m(n-1))-3s}{4}\right] \le 3m(n-1) +$
$+ \left[\frac{3(n^2(m-1)+m(n-1))-3}{4}\right]$. This implies that $n \le 29$ by Lemma 3.9, a contradiction to the assumption. It follows that $\zeta(P_m[P_n])$ has no 3-fold basis for $m \ge 2$, $n \ge 30$. Therefore, $b(P_m[P_n]) = 4$ for $m \ge 2$, $n \ge 30$. For $\zeta(C_m[P_n])$ and $\zeta(P_m[C_n])$ we can assume that each of them has a 3-fold basis. Then we can argue following the same line as in the case of $\zeta(P_m[P_n])$, and using Lemmas 3.5, 3.10 and 3.11 to get a contradiction. Then the proof of Theorem 3.12 is complete.

Theorem 3.13. For every $m \ge 2$, $n \ge 4$, we have $3 \le b(P_m[S_n]) \le 4$. Moreover, $b(P_m[S_n]) = 4$ for $m \ge 2$, $n \ge 30$.

Proof. For each $m \ge 2$, $n \ge 4$, $P_m[S_n]$ contains $m-1$ copies of the nonplanar graph $K_{n,n}$, each one is denoted by $K_{(r,n),(r+1,n)}$; $r \in \mathbf{Z}_{m-1}$. Also $P_m[S_n]$ contains the set of edges $H = \{(r,0)(r,j) : 1 \le j \le n-1,\ r \in \mathbf{Z}_m\}$. It is clear that $b(P_m[S_n]) \ge 3$ by Maclane Theorem. To prove that $b(P_m[S_n]) \le 4$, it is enough to find a 4-fold basis for $\zeta(P_m[S_n])$. For each $r \in \mathbf{Z}_{m-1}$, define the following sets:

$A_r = \{(r,i)(r+1,j)(r,i+1)(r+1,j+1)(r,i) \mid i,j \in \mathbf{Z}_{n-1}\}$,

$D_r = \{(r+1,0)(r,i)(r,0)(r,i+1)(r+1,0) \mid 1 \le i \le n-2\} \cup$
$\{(r+1,0)(r,1)(r,0)(r+1,0)\}$,

$D'_r = \{(r,n-1)(r+1,i)(r+1,0)(r+1,i+1)(r,n-1) \mid 1 \le i \le n-2\} \cup$
$\{(r,n-1)(r+1,0)(r+1,1)(r,n-1)\}$,

$A'_r = D_r \cup D'_r$ and $B_r = A_r \cup A'_r$.

We claim that $\bigcup_{r=0}^{m-2} B_r$ is a basis for $\zeta(P_m[S_n])$. To show that, note that for every $r \in \mathbf{Z}_{m-1}$, A_r is a basis for $K_{(r,n),(r+1,n)}$ as proved in Theorem 2.4 of Schemeichel [7]. Thus A_r is an independent set of cycles in $\zeta(P_m[S_n])$. Also A'_r is an independent set of cycles because any linear combination of cycles from A'_r modulo 2 is either a cycle or an edge-disjoint union of cycles, moreover, each cycle in A'_r contains one or two edges from the set H and these edges occur in no cycle of A_r. Then the cycles of A'_r are independent with the cycles of A_r. Thus $B_r = A_r \cup A'_r$ is an independent set of cycles in $\zeta(P_m[S_n])$. Since the edge-sets of $K_{(r,n)(r+1,n)}$ are pairwise disjoint and any cycle that can be generated from

B_r cannot be generated from B_k for all $k, r \in \mathbf{Z}_{m-1}$, $k \neq r$, thus $B(P_m[S_n]) = \bigcup_{r=0}^{m-2} B_r$ is an independent set of cycles in $\zeta(P_m[S_n])$. But $|B(P_m[S_n])| = (m-1)(n^2-1) = \dim \zeta(P_m[S_n])$. Hence $B(P_m[S_n])$ is a basis for $\zeta(P_m[S_n])$. One may easily see that $B(P_m[S_n])$ is a 4-fold basis. Hence $3 \leq b(P_m[S_n]) \leq 4$ for all $m \geq 2$, $n \geq 4$.

On the other hand, using the same arguments as of Theorem 3.12, considering the set H as it is defined in this theorem, we can prove that $\zeta(P_m[S_n])$ cannot have any 3-fold basis for all $m \geq 2$, $n \geq 30$. Hence $b(P_m[S_n]) = 4$ for all $m \geq 2$, $n \geq 30$.

Theorem 3.14. For every $m \geq 3$, $n \geq 4$, we have $3 \leq b(C_m[S_n]) \leq 4$. Moreover, $b(C_m[S_n]) = 4$ for all $m \geq 3$, $n \geq 15$.

Proof. The graph $C_m[S_n]$ contains $P_m[S_n]$ with a new copy of $K_{n,n}$ which is $K_{(m-1,n),(0,n)}$. For every $r \in \mathbf{Z}_m$, let B_r be as in the proof of Theorem 3.7. Using the same arguments of Theorem 3.13, we can show that $\bigcup_{r=0}^{m-1} B_r$ is an independent set of cycles in $\zeta(C_m[S_n])$. Note that the cycle

$$Q = (0, 0)(1, 0)(2, 0)\ldots(m-1, 0)(0, 0)$$

is independent from the cycles of $\bigcup_{r=0}^{m-1} B_r$. Thus $B(C_m[S_n]) = \left(\bigcup_{r=0}^{m-1} B_r\right) \cup \{Q\}$ is an independent set of cycles in $\zeta(C_m[S_n])$. But $|B(C_m[S_n])| = (n^2-1)m + 1 = \dim \zeta(C_m[S_n])$. Hence, $B(C_m[S_n])$ is a basis for $\zeta(C_m[S_n])$. It is easy to verify that $B(C_m[S_n])$ is a 4-fold basis. Hence $3 \leq b(C_m[S_n]) \leq 4$ for all $m \geq 3$, $n \geq 4$.

On the other hand, using the same arguments as of Theorem 3.12, replacing the set S of Theorem 3.7 by the set H of Theorem 3.13, we deduce that $\zeta(C_m[S_n])$ cannot have any 3-fold basis for all $m \geq 2$, $n \geq 15$. Hence $b(C_m[S_n]) = 4$ for all $m \geq 2$ and $n \geq 15$.

The following two Lemmas are needed in the proof of Theorem 3.17. We mention them without proof since their proof is on the same line as of the previous Lemmas.

Lemma 3.15. Let m, n be two positive integers such that $(m-1)(n^2-1) \leq 5m(n-1)$. Then $n \leq 11$.

Lemma 3.16. Let m, n be two positive integers ($m \geq 2$), such that $(m-1)(n^2-1)+m(n-1) \leq 6m(n-1)+\left[\frac{6m(n-1)+3n^2(m-1)-3}{4}\right]$. Then $n \leq 51$.

Theorem 3.17. For every $m \geq 2$, $n \geq 4$, we have $3 \leq b(P_m[W_n]) \leq 4$. Moreover, $b(P_m[W_n])=4$ for all $m \geq 2$ and $n \geq 52$.

Proof. The graph $P_m[W_n]$ consists of the nonplanar graph $P_m[P_n]$ with the set $S=\bigcup_{r=0}^{m-1} S_r$; $S_r=\{(r, 0)(r, i) \mid 2 \leq i \leq n-1\} \cup \{(r, 1)(r, n-1)\}$. Thus $b(P_m[W_n]) \geq 3$. To show that $b(P_m[W_n]) \leq 4$, we exhibit a 4-fold basis for $\zeta(P_m[W_n])$. For each $r \in \mathbf{Z}_{m-1}$, define the following sets of cycles in $\zeta(P_m[W_n])$:

$$A_r=\{(r, n-1)(r+1, i)(r+1, 0)(r+1, i+1)(r, n-1) \mid 1 \leq i \leq n-2\} \cup \{(r, 1)(r, 2) \ldots (r, n-1)(r, 1)\} \text{ and}$$

$$A_{m-1}=\{(1, 0)(0, i)(0, 0)(0, i+1)(1, 0) \mid 1 \leq i \leq n-2\} \cup \{(0, 1)(0, 2) \ldots (0, n-1)(0, 1)\}.$$

Let $A=\bigcup_{r=0}^{m-1} A_r$ and $B(P_m[W_n])=B(P_m[P_n]) \cup A$, where $B(P_m[P_n])$ is the 4-fold basis of $\zeta(P_m[P_n])$ that was obtained in Theorem 3.12. Then $B(P_m[P_n])$ is an independent set of cycles in $\zeta(P_m[W_n])$. For each $r \in \mathbf{Z}_m$, A_r is an independent set of cycles in $\zeta(P_m[W_n])$ since any linear combination of cycles from A_r modulo 2 is either a cycle or an edge-disjoint union of cycles. Moreover, each cycle from A_r contains one or two edges from the set S_r and these edges do not occur in any other cycle of the set $B(P_m[W_n])-A_r$. Thus $B(P_m[W_n])$ is an independent set of cycles in $\zeta(P_m[W_n])$. But $|B(P_m[W_n])|=(m-1)(n^2-1)+$ $+m(n-1)=\dim \zeta(P_m[W_n])$, therefore, $B(P_m[W_n])$ is a basis for $\zeta(P_m[W_n])$. It is easy to verify that $B(P_m[W_n])$ is a 4-fold basis for $\zeta(P_m[W_n])$. Hence $3 \leq b(P_m[W_n]) \leq 4$ for all $m \geq 2$, $n \geq 4$.

On the other hand, suppose that $\zeta(P_m[W_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta| \leq 6m(n-1)$, since every 3-cycle in β must contain at least one edge from the set $S=\{(i, j)(i, j+1) \mid i \in \mathbf{Z}_m, j \in \mathbf{Z}_{n-1}\} \cup \{(i, 0)(i, j) \mid i \in \mathbf{Z}_m, 2 \leq j \leq n-1\} \cup \{(i, 1)(i, n-1) \mid i \in \mathbf{Z}_m\}$, $|S|=2m(n-1)$ and the fold of every edge of S is at most 3. But $|\beta| \leq$ $\leq 6m(n-1)<(m-1)(n^2-1)+m(n-1)=\dim \zeta(P_m[W_n])$ for all $m \geq 2$, $n \geq 52$ by Lemma 3.15. Hence β cannot be a basis of $\zeta(P_m[W_n])$, a contradiction.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4((m-1)(n^2-1)+m(n-1)) \leq$ $\leq 3(2m(n-1)+n^2(m-1))$. That is $n^2(m-1) \leq 4(m-1)+2m(n-1)$, so that $n^2(m-1)+4 \leq 2mn$. But this inequality cannot hold for all $m \geq 2$ and $n \geq 52$, a contradiction.

Case 3. Suppose that β consists of s 3-cycles and t cycles of length greater than or equal to 4. Then $s \leq 6m(n-1)$ since we have at most $6m(n-1)$ 3-cycles in β as we explained in Case 1. Since the fold of every edge of $P_m[W_n]$ is at most 3 and $3s$ edges are joined to make the s 3-cycles, then

$$t \leq \left[\frac{3(2m(n-1)+n^2(m-1))-3s}{4}\right] \leq \left[\frac{6m(n-1)+3n^2(m-1)-3}{4}\right] \text{ being } s \geq 1.$$

Then we have $(m-1)(n^2-1)+m(n-1) = \dim \zeta(P_m[W_n]) = |\beta| = s+t \leq$ $\leq 6m(n-1)+\left[\dfrac{6m(n-1)+3n^2(m-1)-3}{4}\right]$, which contradicts Lemma 3.16.

This finishes the proof of Theorem 3.17.

We need the following Lemma without proof:

Lemma 3.18. Let m, n be two positive integers $(m \geq 3)$, such that $m(n^2-1)+m(n-1)+1 \leq 6m(n-1)+\left[\dfrac{3mn^2+3m(n-1)-3}{4}\right]$. Then $n \leq 22$.

Theorem 3.19. For every $m \geq 3$, $n \geq 4$, we have $3 \leq b(C_m[W_n]) \leq 4$. Moreover, $b(C_m[W_n])=4$ for all $m \geq 3$ and $n \geq 23$.

Proof. It is clear that $E(C_m[W_n])=E(C_m[P_n]) \cup S$, where

$$S=\{(r,0)(r,i)(r,1)(r,n+1) \mid r \in \mathbf{Z}_m,\ 2 \leq i \leq n-1\}.$$

Let $A_r = \{(r,n-1)(r+1,i)(r+1,0)(r+1,i+1)(r,n-1) \mid 1 \leq i \leq n-2\} \cup$ $\cup \{(r,1)(r,2)\dots(r,n-1)(r,1)\}$, for $r \in \mathbf{Z}_m$, $A = \bigcup_{r=0}^{m-1} A_r$ and $B(C_m[W_n]) =$ $=B(C_m[P_n]) \cup A$ where $B(C_m[P_n])$ is the basis of $\zeta(C_m[P_n])$, which is obtained in Theorem 3.12. Using the same arguments as in Theorem 3.17 we can show that $B(C_m[P_n]) \cup A$ is an independent set of cycles in $\zeta(C_m[W_n])$. Thus $B(C_m[W_n])$ is an independent set of cycles in $\zeta(C_m[W_n])$. But $|B(C_m[W_n])| =$ $= m(n^2+n-2)+1 = \dim \zeta(C_m[W_n])$ then $B(C_m[W_n])$ is a basis for $\zeta(C_m[W_n])$. It is easy to see that $B(C_m[W_n])$ is a 4-fold basis for $\zeta(C_m[W_n])$. Hence $3 \leq b(C_m[W_n]) \leq 4$ for all $m \geq 3$, $n \geq 4$.

On the other hand, suppose $\zeta(C_m[W_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta| \leq 6m(n-1)$ since every 3-cycle in β must contain an edge from the set S and the fold of every edge of S is at most 3. But $|\beta| \leq 6m(n-1) < m(n^2-1) + m(n-1) + 1 = \dim \zeta(C_m[W_n])$ for all $m \geq 3$ and $n \geq 23$. Hence β cannot be a basis of $\zeta(C_m[W_n])$, a contradiction.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4(m(n^2-1) + m(n-1) + 1) \leq 3(n^2m + 2m(n-1))$. That is $n^2m + 4 \leq 4m + 2mn - 2m$, so that $n^2m + 4 \leq 2m(n-1)$. But this inequality cannot hold for all $m \geq 3$ and $n \geq 23$.

Case 3. Suppose that β consists of s 3-cycles, and t cycles of length greater than or equal to 4. Then $s \leq 6m(n-1)$ since we have at most $6m(n-1)$ 3-cycles in β as we explained in Case 1. Since the fold of every edge of $C_m[W_n]$ is at most 3 and $3s$ edges are joined to make the s 3-cycles, we have

$$t \leq \left[\frac{3n^2m + 3m(n-1) - 3s}{4}\right] \leq \left[\frac{3n^2m + 3m(n-1) - 3}{4}\right]$$

being $s \geq 1$. Therefore, $m(n^2-1) + m(n-1) + 1 = \dim \zeta(C_m[W_n]) = |\beta| = s + t \leq 6m(n-1) + \left[\frac{3n^2m + 3m(n-1) - 3}{4}\right]$. This contradicts Lemma 3.18 for all $m \geq 3$ and $n \geq 23$. Therefore, $\zeta(C_m[W_n])$ cannot have any 3-fold basis for all $m \geq 3$ and $n \geq 23$. Hence $b(C_m[W_n]) = 4$ for all $m \geq 3$ and $n \geq 23$.

Theorem 3.20. For every $m \geq 4$, $n \geq 3$, we have $3 \leq b(S_m[P_n]) \leq 4$. Moreover, $b(S_m[P_n]) = 4$ for all $m \geq 4$ and $n \geq 20$.

Proof. The graph $S_m[P_n]$ contains $m-1$ copies of $K_{n,n}$. We denote these copies by $K_{(0,n),(r,n)}$; $1 \leq r \leq m-1$. Note that each of these copies are joined to the vertices $(0, i)$ where $i \in \mathbf{Z}_n$. Also $S_m[P_n]$ contains the set of edges $S = \{(i,j)(i,j+1) \mid i \in \mathbf{Z}_m, j \in \mathbf{Z}_n\}$. It is clear that $b(S_m[P_n]) \geq 3$. We now exhibit a 4-fold basis for $\zeta(S_m[P_n])$. For each $1 \leq r \leq m-1$, let

$A_r = \{(0,i)(r,j)(0,j+1)(r,j+1)(0,i) \mid i,j \in \mathbf{Z}_n\}$,

$A'_r = \{a'_i = (0,0)(r,i)(r,i+1)(0,0)] \, i \in \mathbf{Z}_{n-1}\}$.

Also define the following sets:

$A''_r = \{(r,0)\ (0,i)\ (r-1,0)\ (0,i+1)\ (r,0) \mid i \in \mathbf{Z}_{n-1}\}$ if r is even and $2 \leq r \leq m-1$,

and

$A'''_r - \{(r,n-1)(0,n-i-1)(r-1,n-1)(0,n-i-2)(r,n-1) \mid i \in \mathbf{Z}_{n-1}\}$ if r is odd and $3 \leq r \leq m-1$.

$$\text{Let } A = \bigcup_{r=0}^{m-1} A_r,\ A' = \left(\bigcup_{r=1}^{m-1} A_r\right) \cup \{(1,n-1)(0,i)(0,i+1)(1,n-1) \mid i \in \mathbf{Z}_{n-1}\},$$

$A'' \quad \cup \{A''_r \mid r \text{ is even and } 2 \leq r \leq m-1\}$,

$A''' = \cup \{A'''_r \mid r \text{ is odd and } 3 \leq r \leq m-1\}$.

Define the set $B(S_m[P_n]) = A \cup A' \cup A'' \cup A'''$. For each $1 \leq r \leq m-1$, A_r is the 4-fold basis constructed in Theorem 2.4 of Schemeichel [7] for the subspace $\zeta(K_{(0,n),(r,n)})$. Since the edge-sets of the graphs $K_{(0,n),(r,n)}$ are pairwise-disjoint then A is an independent set of cycles in $\zeta(S_m[P_n])$. From the definition of $A'' \cup A'''$, any linear combination of cycles in $A'' \cup A'''$ (mod 2) is either a cycle or an edge-disjoint union of cycles. Thus $A'' \cup A'''$ is an independent set of cycles in $\zeta(S_m[P_n])$. Moreover, every 4-cycle $C_r \in A'' \cup A'''$ cannot be generated from the cycles of A because it consists of two edges from $K_{(0,n),(r,n)}$ and the others from $K_{(0,n),(r+1,n)}$ where $1 \leq r \leq m-2$. Therefore $A \cup A'' \cup A'''$ is an independent set of cycles in $\zeta(S_m[P_n])$. It follows that $B(S_m[P_n])$ is an independent set of cycles in $\zeta(S_m[P_n])$ since every cycle $a'_i \in A'$ contains the edge $(r, i)(r, i+1)$ which does not occur in any other cycle of $B(S_m[P_n])$. Therefore $B(S_m[P_n])$ is a basis for $\zeta(S_m[P_n])$. One may easily see that $B(S_m[P_n])$ is a 4-fold basis of $\zeta(S_m[P_n])$. Hence $3 \leq b(S_m[P_n]) \leq 4$ for all $m \geq 4$, $n \geq 3$.

On the other hand, using the same arguments as of Theorem 3.12, we can prove that $\zeta(S_m[P_n])$ cannot have a 3-fold basis for all $m \geq 4$ and $n \geq 20$.

Theorem 3.21. For every $m \geq 4$, $n \geq 3$, we have $3 \leq b(S_m[C_n]) \leq 4$. Moreover, $b(S_m[C_n]) = 4$ for all $m \geq 4$ and $n \geq 20$.

Proof. It is clear that $E(S_m[C_n]) = E(S_m[P_n]) \cup H$ where $H = \{(i, n-1)(0, n-1) \mid i \in \mathbf{Z}_m\}$. Let

$$D = \{d_i = (i, 0)(i, 1)(i, 2) \dots (i, n-1)(i, 0) \mid i \in \mathbf{Z}_m\}$$

and $B(S_m[C_n]) = B(S_m[P_n]) \cup D$ where $B(S_m[P_n])$ is the basis of $\zeta(S_m[P_n])$ exhibited in Theorem 3.20. Since $B(S_m[P_n])$ is an independent set of cycles in $\zeta(S_m[C_n])$ and each $d_i \in D$ contains the edge $(i, n-1)(i, 0)$ which does not occur in any other cycle of $B(S_m[C_n])$, then $B(S_m[C_n])$ is an independent set of cycles in $\zeta(S_m[C_n])$. But $|B(S_m[C_n])| = n^2(m-1)+1 = \dim \zeta(S_m[C_n])$, hence $B(S_m[C_n])$ is a basis for $\zeta(S_m[C_n])$. One may easily verify that $B(S_m[C_n])$ is a 4-fold basis for $\zeta(S_m[C_n])$. Therefore, $3 \leq b(S_m[C_n]) \leq 4$ for all $m \geq 4$, $n \geq 3$.

On the other hand, using the same arguments as of Theorem 3.12 in the case $P_m[C_n]$, we can prove that $\zeta(S_m[C_n])$ has no 3-fold basis for all $m \geq 4$ and $n \geq 20$.

Lemma 3.22. Let m, n be two positive integers $(m \geq 4)$ such that $(m-1)(2n^2-1) \leq 6mn - 3m + \left[\dfrac{6n^2(m-1)+3m(n-1)-3}{4}\right]$. Then $n \leq 17$.

Proof. Using the same idea as in the proofs of the previous Lemmas we have $\frac{2n^2(m-1)}{27m}<n$. Therefore, $n<\frac{27m}{2(m-1)}$, so that $n<\frac{27}{2}\left(\frac{4}{3}\right)=18$. That is $n\leq 17$.

Theorem 3.23. For every $m\geq 4$, $n\geq 3$, we have $3\leq b(W_m[P_n])\leq 4$. Moreover, $b(W_m[P_n])=4$ for all $m\geq 4$ and $n\geq 18$.

Proof. We consider $W_m[P_n]$ as the graph constructed from joining the nonplanar graphs $S_m[P_n]$ and $C^*_{m-1}[P_n]$ at the set of edges $H=\{(i,j)(i,j+1) \mid 1\leq i\leq m-1,\ j\in \mathbf{Z}_{n-1}\}$, where C^*_{m-1} denotes the cycle $123\ldots(m-1)1$. It is clear that $W_m[P_n]$ is a nonplanar graph and by Maclane Theorem, $b(W_m[P_n])\geq 3$. To prove that $b(W_m[P_n])\leq 4$, we exhibit a 4-fold basis for $\zeta(W_m[P_n])$ as follows:

Let $D=\{d_i=(0,n-1)(i,n-1)(i+1,n-1)(0,n-1) : 1\leq i\leq m-2\}$, $B(W_m[P_n])=B(S_m[P_n])\cup B(C^*_{m-1}[P_n])\cup D$, where $B(S_m[P_n])$ and $B(C^*_{m-1}[P_n])$ are the bases of the subspaces $\zeta(S_m[P_n])$ and $\zeta(C^*_{m-1}[P_n])$ that are obtained in Theorem 2.20 and Theorem 3.12 respectively. Clearly $B(S_m[P_n])$ and $B(C^*_{m-1}[P_n])$ are two independent sets of cycles in $\zeta(W_m[P_n])$. Since $E(S_m[P_n])\cap E(C^*_{m-1}[P_n])=H$, then it is clear that non of the cycles of $B(S_m[P_n])$ can be generated from $B(C^*_{m-1}[P_n])$ and vice-versa. That is $B(S_m[P_n])\cup B(C^*_{m-1}[P_n])$ is an independent set of cycles in $\zeta(W_m[P_n])$. But $d_i\in D$ contains the edge $(i,n-1)(i+1,n-1)$ that does not occur in any other cycle of $D\cup B(S_m[P_n])$. Thus $D\cup B(S_m[P_n])$ is an independent set of cycles in $\zeta(W_m[P_n])$. Also every d_i contains an edge of the form $(0,n-1)(i,n-1)$ that does not appear in any other cycle of $B(C^*_{m-1}[P_n])$. Thus $D\cup B(C^*_{m-1}[P_n])$ is an independent set of cycles in $\zeta(W_m[P_n])$. Therefore, $B(W_m[P_n])$ is an independent set of cycles in $\zeta(W_m[P_n])$ and $|B(W_m[P_n])|=(m-1)(2n^2-1)=\dim\zeta(W_m[P_n])$. It follows that $B(W_m[P_n])$ is a basis for $\zeta(W_m[P_n])$. It is easy to show that $B(W_m[P_n])$ is a 4-fold basis for $\zeta(W_m[P_n])$. Hence $3\leq b(W_m[P_n])\leq 4$ for all $m\geq 4$, $n\geq 3$.

On the other hand, suppose $\zeta(W_m[P_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta|\leq 6mn-3m$, since every 3-cycle in β must contain an edge from the set $M=\{(i,j)(i,j+1) \mid i\in\mathbf{Z}_m,\ j\in\mathbf{Z}_{n-1}\}\cup\{(0,i)(j,i)\mid i\in\mathbf{Z}_n,\ 1\leq j\leq m-1\}\cup\{(1,i)(m-1,i)\mid i\in\mathbf{Z}_n\}$, $|M|=2mn-m$ and the fold of every edge of S is at most 3. But $|\beta|\leq 6mn-3m<(m-1)(2n^2-1)=\dim\zeta(W_m[P_n])$ for all $m,n\geq 4$, a contradiction to the fact that β is a basis for $\zeta(W_m[P_n])$.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4(m-1)(2n^2-1)\leq 3(2n^2(m-1)+m(n-1))$, so that $3n^2(m-1)-4m+4\leq 6n^2(m-1)+3m(n-1)$. That is

$2n^2(m-1)+4 \leq 3mn+m$. But this inequality cannot hold for all $m, n \geq 4$. Therefore, this case cannot happen.

Case 3. Suppose that β consists of s 3-cycles, and t cycles of length greater than or equal to 4. Then $s \leq 6mn-3m$ since we have at most $6mn-3m$ 3-cycles in β as explained in Case 1. Since the fold of every edge of $W_m[P_n]$ is at most 3 and $3s$ edges are joined to make the s 3-cycles then

$$t \leq \left[\frac{6n^2(m-1)+3m(n-1)-3s}{4}\right] \leq \left[\frac{6n^2(m-1)+3m(n-1)-3}{4}\right],$$

where $s \geq 1$. Therefore, $(m-1)(2n^2-1) = \dim \zeta(W_m[P_n]) = |\beta| = s+t \leq 6mn - -3m + \left[\frac{6n^2(m-1)+3m(n-1)-3}{4}\right]$, so that $n \leq 17$ by Lemma 3.22, a contradiction to the fact that $n \geq 18$.

It follows that $\zeta(W_m[P_n])$ has no 3-fold basis for all $m \geq 4$ and $n \geq 18$. That is $b(W_m[P_n])=4$ for $m \geq 4$ and $n \geq 18$.

Finaly, we study the basis number of the composition of a wheel and a cycle, and a star and a wheel in the following two theorems. But first we need the following Lemma:

Lemma 3.24. Let m, n be two positive integers $(m \geq 4)$, such that $(m-1)(2n^2-1)+m \leq 6mn + \dfrac{6n^2(m-1)+3mn-3}{4}$. Then $n \leq 17$.

Theorem 3.25. For every $m \geq 4$, $n \geq 3$, we have $3 \leq b(W_m[C_n]) \leq 4$. Moreover, $b(W_m[C_n])=4$ for all $m \geq 4$ and $n \geq 18$.

Proof. It is easy to see that $E(W_m[C_n]) = E(W_m[P_n]) \cup S$ where $S = \{(i, n-1)(i, 0) \mid i \in \mathbf{Z}_m\}$. Let $F = \{f_i = (i, 0)(i, 1)(i, 2)\ldots(i, n-1)(i, 0) \mid i \in \mathbf{Z}_m\}$, $B(W_m[C_n]) = B(W_m[P_n]) \cup F$ where $B(W_m[P_n])$ is the basis of the subspace $\zeta(W_m[P_n])$. Then $B(W_m[P_n])$ is an independent set of cycles in $\zeta(W_m[C_n])$. Since $f_i \in F$ contains the edge $(i, n-1)(i, 0)$ which does not occur in any other cycle of $B(W_m[C_n])$ then $B(W_m[C_n])$ is an independent set of cycles in $\zeta(W_m[C_n])$. But $|B(W_m[C_n])| = (m-1)(2n^2-1)+m = \dim \zeta(W_m[C_n])$. Then $B(W_m[C_n])$ is a basis for $\zeta(W_m[C_n])$. It is easy to see that $B(W_m[C_n])$ is a 4-fold basis of $\zeta(W_m[C_n])$. Hence $3 \leq b(W_m[C_n]) \leq 4$ for all $m \geq 4$, $n \geq 3$.

On the other hand, suppose that $\zeta(W_m[C_n])$ has a 3-fold basis β. Then we have three cases:

Case 1. Suppose that β consists only of 3-cycles. Then $|\beta| \leq 6mn$ since every 3-cycle in β must contain at least one edge from the set

$$N = M \cup \{(i, n-1)(i, 0) \mid i \in \mathbf{Z}_m\},$$

$|N| = 2mn$ and the fold of every edge of S is at most 3, where M

is the set of edges which is defined in Case 1 of Theorem 3.23. But $|\beta| \leq 6mn < (m-1)(2n^2-1)+m = \dim \zeta(W_m[C_n])$ for all $m, n \geq 4$. Hence β cannot be a basis of $\zeta(W_m[C_n])$, a contradiction.

Case 2. Suppose that β consists only of cycles of length greater than or equal to 4. Then Lemma 3.5 implies that $4((m-1)(2n^2-1)+m) \leq 3(2n^2(m-1)+mn)$, so that $8n^2(m-1)-3m+4 \leq 6n^2(m-1)+3mn$. That is $2n^2(m-1)+4 \leq 3m(n+1)$. But this inequality cannot hold for all $m, n \geq 4$. Hence this case cannot happen.

Case 3. Suppose that β consists of s 3-cycles and t cycles of length greater than or equal to 4. Then $s \leq 6mn$ since we have at most $6mn$ 3-cycles in β as explained in Case 1. Since the fold of every edge of $W_m[C_n]$ is at most 3 and $3s$ edges are joined to make the s 3-cycles, then $t \leq \left[\dfrac{6n^2(m-1)+3mn-3s}{4}\right] \leq$ $\leq \left[\dfrac{6n^2(m-1)+3mn-3}{4}\right] \leq \dfrac{6n^2(m-1)+3mn-3}{4}$, where $s \geq 1$. Therefore,

$$(m-1)(2n^2-1)+m = \dim \zeta(W_m[C_n]) = |\beta| = s+t \leq 6mn + \frac{6n^2(m-1)+3mn-3}{4},$$

so that $n \leq 17$ by Lemma 3.24, a contradiction to the fact that $n \geq 18$. Therefore, $\zeta(W_m[C_n])$ has no 3-fold basis for all $m \geq 4$ and $n \geq 18$. It follows that $b(W_m[P_n]) = 4$ for $m \geq 4$ and $n \geq 18$.

Theorem 3.26. For every $m, n \geq 4$, we have $3 \leq b(S_m[W_n]) \leq 4$. Moreover, $b(S_m[W_n]) = 4$ for all $m \geq 4$ and $n \geq 52$.

Proof. The graph $S_m[W_n]$ consists of $m-1$ copies of $K_{n,n}$ in the form $K_{(0,n)(r,n)}$ where $1 \leq r \leq m-1$, with the sets of edges $H = \{(r,0)(r,i) \mid r \in Z_m, 1 \leq i \leq n-1\}$ and $K = \{(r,i)(r,i+1) \mid r \in Z_m, 1 \leq i \leq n-2\} \cup \{(r,1)(r,n-1) \mid r \in Z_m\}$. It is clear that $b(S_m[W_n]) \geq 3$. To prove that $b(S_m[W_n]) \leq 4$, we exhibit a 4-fold basis for $\zeta(S_m W_n])$. For each $r = 1, 2, \ldots, m-1$, we define the following sets:

$A'_r = \{(0, n-1)(r, i)(r, 0)(r, i+1)(0, n-1) \mid 1 \leq i \leq n-2)\} \cup$
$\{(0, n-1)(r, 0)(r, 1)(0, n-1)\}$,

$A'_0 = \{(1, n-1)(0, i)(0, 0)(0, i+1)(1, n-1) \mid 1 \leq i \leq n-2\} \cup$
$\{(1, n-1)(0, 0)(0, 1)(1, n-1)\}$

and $A - \bigcup_{r=0}^{m-1} A_r$.

Let $D = \{d_r = (r, 0)(r, i)(r, i+1)(r, 0) \mid r \in \mathbb{Z}_m, 1 \leq i \leq n-2\} \cup \{d_{r_0} = (r, 0)(r, 1)(r, n-1)(r, 0) \mid r \in \mathbb{Z}_m\}$, and $B(S_m[W_n]) = A \cup A' \cup A'' \cup A''' \cup D$, where A', A'' and A''' are the same sets that are defined in Theorem 3.20. It is clear that $A \cup A'' \cup A'''$ is an independent set of cycles in $\zeta(S_m[W_n])$. Since any linear combination of cycles from A' is either a cycle or an edge-disjoint union of cycles then A' is an independent set of cycles in $\zeta(S_m[W_n])$. Moreover, each cycle of A' contains one or two edges from the set H and these edges occur in no cycle of $A \cup A'' \cup A'''$. Thus non of the cycles of A' can be generated from $A \cup A'' \cup A'''$, so that $A \cup A' \cup A'' \cup A'''$ is an independent set of cycles in $\zeta(S_m[W_n])$. Also, since every cycle in D contains one edge from K and this edge does not occur in any other cycle of $B(S_m[W_n])$ then $B(S_m[W_n])$ is an independent set of cycles in $\zeta(S_m[W_n])$. But $|B(S_m[W_n])| = (m-1)(n^2-1) + m(n-1) = \dim \zeta(S_m[W_n])$. Hence $B(S_m[W_n])$ is a basis for $\zeta(S_m[W_n])$. One may easily see that $B(S_m[W_n])$ is a 4-fold basis for $\zeta(S_m[W_n])$. Hence $3 \leq b(S_m[W_n]) \leq 4$, for all $m, n \geq 4$.

On the other hand, using the same arguments of Theorem 3.17, just replace the set S of Theorem 3.17 by the set N of this theorem, we can prove that $\zeta(S_m[W_n])$ has no 3-fold basis for all $m \geq 4$ and $n \geq 52$.

Corollary 3.27. For every $m, n \geq 4$, we have $3 \leq b(S_m[S_n]) \leq 4$. Moreover, $b(S_m[S_n]) = 4$ for all $m \geq 4$ and $n \geq 52$.

Proof. It is enough to note that $E(S_m[S_n]) = E(S_m[W_n]) - K$ and $S_m[S_n]$ is a nonplanar graph where K is the set of edges defined in Theorem 3.26. Thus $B(S_m[S_n]) = B(S_m[W_n]) - D$ is a 4-fold subbasis of $B(S_m[W_n])$ for the subspace $\zeta(S_m[S_n])$.

On the other hand, using the same arguments of Theorem 3.12, considering the set H as it is defined in Theorem 3.26, we can prove that $\zeta(S_m[S_n])$ has no 3-fold basis for all $m \geq 4$ and $n \geq 52$.

REFERENCES

[1] ALI, A.A. : *The basis number of the strong product of graphs* (to appear in Mu'tah Lil-Buhooth Wa Al-Dirasat).

[2] BANKS, J.A. - SCHEMEICHEL, E.F. : *The basis number of the n-cube*, J. Combin. Theory, Ser. B., 33 (1982), No. 2, 95-100.

[3] HARARY, F. : Graph Theory, 2nd ed., Addison-Wesley, Reading, Massachusetts, 1972.

[4] HULSURKAR, S.G. : *Non-planarity of graphs on Weyl groups*, Jour. Math. Phy. Sci., 24 (1990), No. 6.

[5] HULSURKAR, S.G. : *Proof of Verma's conjecture on Weyl's dimension polynomial*, Inventiones Math., 27 (1974), 45-52.

[6] MACLANE, S. : *A combinatorial condition for planar graphs*, Fund. Math., 28 (1937), 22-32.

[7] SCHEMEICHEL, E.F. : *The basis number of a graph*, J. Combin. Theory, Ser. B., 30 (1981), No. 2, 123-129.

[8] VERMA, D.N. : Role of affine Weyl groups in the representation theory of algebraic Chevalley groups and their Lie algebras, "Lie groups and their representations", Proceedings of the 1971 Summer School on Representation Theory (Ed. I. M. Gel'fand). Budapest: Akademiai Kiado.