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INVESTIGATION OF NONLINEAR SINGULAR INTEGRAL EQUATIONS WITH HILBERT KERNEL BY THE METHOD OF MECHANICAL QUADRATURE

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Abstract : The paper concerns the investigation of a class of non-linear singular integral equations with Hilbert kernel by means of mechanical quadrature method.

HILBERT ÇEKİRDEKLİ, LİNEER OLMAYAN SİNGÜLER İNTEGRAL DENKLEMLERİN MEKANİK KARELEME YÖNTEMİYLE İNCELENMESİ

Özet : Bu çalışmada Hilbert çekirdekli, lineer olmayan bir singüler integral denklem sınıfı, mekanik kareleme yöntemiyle incelenmektedir.

INTRODUCTION

In the present paper we investigate the approximate solution of the following class of nonlinear singular integral equations (NSIE) in Holder spaces [2, 6, 7], by mechanical quadrature method [1, 2, 4, 7].

$$u(\sigma) = f(\sigma) + \frac{\lambda}{2\pi}\int_0^{2\pi} F[\tau, u(\tau)] \cot\frac{\tau-\sigma}{2}\, d\tau + \frac{\lambda}{2\pi}\int_0^{2\pi} Q[\sigma, \tau, u(\tau)]\, d\tau. \quad (1)$$

Definition 1 [2, 3].

a) We denote by Φ the class of all continuous monotonic increasing functions $\varphi(\delta)$ defined on $[0, 2\pi]$ such that $\varphi(\delta) \neq 0$ at $\delta > 0$ and $\varphi(0) = 0$.

b) Let E_n be an n-dimensional space with norm

$$\| u \|_{E_n} = \max_i | u_i |.$$

c) We denote by $H_{\varphi,n}$ the space of 2π-periodic continuous vector functions $u = \{u_1, \ldots, u_n\}$ such that

$$\omega_u(\delta) = O(\varphi(\delta)),$$

with norm :

$$\| u \|_{\varphi, n} = \max \left\{ \max_{\sigma} \| u(\sigma) \|_{E_n}, \ \sup_{\delta} \frac{\omega_u(\delta)}{\varphi(\delta)} \right\}$$

where

$$\omega_u(\delta) = \sup_{\substack{|\sigma_1 - \sigma_2| \leq \delta \\ \delta > 0}} \| u(\sigma_1) - u(\sigma_2) \|_{E_n} .$$

d) We define

$$H_{\varphi, n}(M) = \{ u \in H_{\varphi, n} : \| u \|_{\varphi, n} \leq M, M > 0 \}.$$

e) The space $H^{(N)}_{\varphi, n}$ $(\varphi \in \Phi)$ - $2N$-dimensional space of vectors

$z = (z_0, z_1, \ldots, z_{2N-1})$ with the norm

$$\| z \|^{(N)}_{\varphi, n} = \left\{ \max_{k=\overline{0, 2N-1}} \| z_k \|_{E_n}, \ \max \frac{\omega_z(\delta)}{\varphi\left(\frac{\delta \pi}{N}\right)} \right\}.$$

f) Let $\nu(\varphi)$ be the set of numbers $\beta : 0 < \beta < 1$ for which

$$\lim_{t \to 0} \frac{\varphi(t)}{t^{\beta}} \ln^3 \frac{1}{t} = 0.$$

Note that $\nu(\varphi)$ is nonempty, if $\varphi(\delta) \in \Phi$ then

$$\varphi(t) = O(t^{\beta_0}) \text{ for some } \beta_0 > 0.$$

For NSIE (1) we have the following:

i) The vector functions:

$$u = \{u_1, \ldots, u_n\}, \ f = \{f_1, \ldots, f_n\},$$

$$F[\tau, u] = \{F_1[\tau, u_1, \ldots, u_n], \ldots, F_n[\tau, u_1, \ldots, u_n]\},$$

$$Q[\sigma, \tau, u] = \{Q_1[\sigma, \tau, u_1, \ldots, u_n], \ldots, Q_n[\sigma, \tau, u_1, \ldots, u_n]\}$$

and the components of the vector $Q[\sigma, \tau, u]$ has the form

$$Q_i = \sum_{j=1}^{n} Q_{ij}[\sigma, \tau, u_1, \ldots, u_n] .$$

ii) The function $F[\tau, u(\tau)]$ defined at $0 \leq \tau \leq 2\pi$, $\| u \|_{E_n} \leq M$, 2π-periodic in τ and has partial derivaties up to second order and satisfies the following condition:

$$\left| \frac{\partial^p F[\tau_1, \overline{u}]}{\partial \tau^{\alpha_0} \partial u_1^{\alpha_1} \ldots \partial u_n^{\alpha_n}} - \frac{\partial^p F[\tau_2, \overline{\overline{u}}]}{\partial \tau^{\alpha_0} \partial u_1^{\alpha_1} \ldots \partial u_n^{\alpha_n}} \right| \leq \qquad (2)$$

$$\leq \mu(p) \{ \varphi(|\tau_1 - \tau_2|) + \| \overline{u} - \overline{\overline{u}} \|_{E_n} \}$$

where $\alpha_0 + \alpha_1 + \ldots + \alpha_n = p$, $p = 0, 1, 2$ and $\mu(p)$ is a constant.

iii) The function $Q_{ij}[\sigma, \tau, u]$ defined at $0 \leq \sigma \leq 2\pi$, $\| u \|_{E_n} \leq M$, 2π-periodic by σ and τ and has derivatives up to the second order by σ moreover it satisfies the following condition:

$$\left| \frac{d^p Q_{ij}[\sigma_1, \tau, \bar{u}]}{\partial \sigma^p} - \frac{d^p Q_{ij}[\sigma_2, \tau, \bar{\bar{u}}]}{d\sigma^p} \right| \leq \tag{3}$$

$$\leqslant \mu'(p) \{ \varphi_1(|\sigma_1 - \sigma_2|) + \| \bar{u} - \bar{\bar{u}} \|_{E_n} \},$$

where $\mu'(p)$ is a constant.

iv) The 2π-periodic vector function $f \in H_{\varphi, n}(M)$.

From Definition 1 and conditions (2) and (3), it is clear that the vector functions $F[\tau, u]$ and $Q[\sigma, \tau, u]$ belong to the space $H_{\varphi, n}(M)$.

This work consists of the following three sections:

1. The solution in the Holder space $H_{\varphi, n}$

Theorem 1.1. Let the functions $F[\tau, u]$ and $Q[\sigma, \tau, u]$ satisfy the conditions (2) and (3), then for $|\lambda| < |\lambda_0|$ (λ_0 arbitrary small) equation (1) has a unique solution in $H_{\varphi, n}(M)$. This solution can be obtained by the method of successive approximations and uniformly converges in the space $L_2^{(N)}$.

Proof. Let $u \in H_{\varphi, n}(M)$, then from [1, 4] the operator

$$(\mathrm{Au})(\sigma) = f(\sigma) + \frac{\lambda}{2\pi} \int_0^{2\pi} F[\tau, u(\tau)] \cot \frac{\tau - \delta}{2} d\tau + \frac{\lambda}{2\pi} \int_0^{2\pi} Q[\sigma, \tau, u(\tau)] \, d\tau \tag{1.1}$$

transforms $H_{\varphi, n}(M)$ into $H_{\varphi, n}(\lambda M')$. Therefore, if $|\lambda| M' < M$, hence the operator (1.1) transforms $H_{\varphi, n}(M)$ into itself. From conditions (2), (3) and using M. Riesz's theorem [5], for arbitrary $u_1, u_2 \in H_{\varphi, n}(M)$, we have:

$$\| \mathrm{Au}_1 - \mathrm{Au}_2 \|_{L_2} \leqslant \left\{ \int_0^{2\pi} \left| \frac{\lambda}{2\pi} \int_0^{2\pi} [F[\tau, u_1(\tau)] - F[\tau, u_2(\tau)]] \times \right.\right.$$

$$\left.\left. \times \cot \frac{\tau - \sigma}{2} d\tau \right|^2 d\delta \right\}^{1/2} + \left\{ \int_0^{2\pi} \left| \frac{\lambda}{2\pi} \int_0^{2\pi} [Q[\sigma, \tau, u_1(\tau)] - \right.\right.$$

$$\left.\left. - Q[\sigma, \tau, u_2(\tau)]] \right|^2 \right\}^{1/2} \leqslant |\lambda| \mu(0) (1 + c(2)) \| u_1 - u_2 \|_{L_2},$$

where $c(2)$ is a constant.

If $|\lambda|\mu(0)(1+c(2))<1$, then the operator (1.1) is contraction mapping. Taking

$$\lambda_0 = \min\left\{\frac{M}{M'}, \frac{1}{\mu(0)(1+c(2))}\right\}$$

then the equation (1) has a unique solution in $H_{\varphi,n}$ and this solution can be found by the method of successive approximations.

Theorem 1.2. Let u_q, $u \in H_{\varphi,n}(M)$ and $\lim_{q\to\infty} \| u_q - u \|_{L_2} = 0$ then $\lim_{q\to\infty} \| u_q - u \|_C = 0$.

Proof. Let $G \in H_{\varphi,n}(M)$ can be written in the form

$$G(s) = \frac{1}{h}\int_s^{s+h} G(x)\,dx - \frac{1}{h}\int_s^{s+h} [G(x) - G(s)]\,dx,$$

then we have

$$|G(s)| \leqslant \frac{1}{h}\int_s^{s+h} |G(x)|\,dx + \frac{1}{h}\int_s^{s+h} |G(x) - G(s)|\,dx.$$

Using Holder inequality on the first term in the right part of the last inequality we have

$$\| G(s) \|_C \leqslant \frac{1}{h}\left(\int_s^{s+h} |G(x)|^2\,dx\right)^{1/2} h^{1/2} + \frac{1}{h}\int_s^{s+h} \| G \|_{\varphi,n}\,\varphi(|x-s|)\,dx \leqslant$$

$$\leqslant h^{-1/2}\left(\int_{-\pi}^{\pi} |G(x)|^2\,dx\right)^{1/2} + \frac{\| G \|_{\varphi,n}}{h}\,\varphi(h)\,h.$$

Putting $G(s) = u_q(s) - u(s)$, $h = \| u_q - u \|_{L_2}$ we obtain

$$\| u_q - u \|_C \leqslant \| u_q - u \|_{L_2}^{1/2} + 2M\,\varphi(\| u_q - u \|_{L_2}),$$

then $\| u_q - u \|_C \to 0$ as $q \to \infty$, that is the successive approximations converge.

2. The solution in the discrete Holder space $H_{\varphi,n}^{(N)}$

In [1, 4], for singular integral

$$(\mathrm{Ju})(s) = \frac{1}{2\pi} \int_0^{2\pi} u(\sigma) \cot \frac{\sigma - s}{2} \, d\sigma, \quad u(\sigma) \in H_{\varphi, n}$$

construct the quadratic formula:

$$(\mathrm{Ju})(s) = \frac{1}{N} \sum_{k=0}^{2N-1} u_k \operatorname{Sin}^2 \frac{s - s_k}{2} \cot \frac{s_k - s}{2}, \tag{2.1}$$

where

$$u_k = u(s_k), \quad s_k = \frac{k\pi}{N}.$$

Formula (2.1) at node points s_k takes the form:

$$(\mathrm{Ju})(s_t) = \frac{1}{2N} \sum_{\substack{k=0 \\ k \neq t}}^{2N-1} u_k [1 - (-1)^{k-t}] \cot \frac{s_k - s_t}{2}. \tag{2.2}$$

Applying the quadratic formula (2.2) to first and second integral of equation (1) we get

$$u(t_l) = f(t_l) + \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} F[t_k, u(t_k)] [1 - (-1)^{k-l}] \cot \frac{t_k - t_l}{2} +$$

$$+ \frac{\lambda}{2N} \sum_{k=0}^{2N-1} Q[t_l, t_k, u(t_k)] + R_N[F] + \lambda R_N[Q], \; l = \overline{0, 2N-1},$$

neglect the remainder terms, hence we arrive to the following system of algebraic equations

$$z_l = \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} F[t_k, z_k] [1 - (-1)^{k-l}] \cot \frac{t_k - t_l}{2} + \frac{\lambda}{2N} \sum_{k=0}^{2N-1} Q[t_l, t_k, z_k], \tag{2.3}$$

where $z_l = u(t_l)$.

Take $E^{(N)} z = (E_0^{(N)} z, \ldots, E_{2N-1}^{(N)} z)$,

where

$$E_l^{(N)} z = \frac{\lambda}{2N} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} F[\tau_l, z_k] [1 - (-1)^{k-l}] \cot \frac{\tau_k - \tau_l}{2} + \frac{\lambda}{2N} \sum_{k=0}^{2N-1} Q[t_l, t_k, z_k].$$

In other words, $F^{(N)} z = \lambda A^{(N)} Gz$ where

$$A^{(N)} z = (A_0^{(N)} z, \ldots, A_{2N-1}^{(N)} z),$$
$$G z = (G z_0, \ldots, G z_{2N-1}) \in H_{\varphi, n}^{(N)} (R)$$

and $A^{(N)}$ is bounded [3, 7], that is $\| A^{(N)} \|_{H_{\varphi, n}^{(N)}} \leqslant c, \ c > 0$. Thus we have

$$\| E^{(N)} z \|_{H_{\varphi, n}^{(N)}} \leqslant | \lambda | R c.$$

Now, let

$$z^{(1)}, z^{(2)} \in H_{\varphi, n}^{(N)} (M), \text{ then}$$

$$\| E^{(N)} z^{(1)} - E^{(N)} z^{(2)} \|_{L_2^{(N)}} =$$

$$= \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left| \frac{\lambda}{2N} \sum_{k=0}^{2N-1} [F[\tau_l, z_k^{(1)}] - F[\tau_l, z_k^{(2)}] \, [1 - (-1)^{k-l}] \cot \frac{\tau_k - \tau_l}{2} \right|^2 \right\}^{1/2} +$$

$$+ \left\{ \frac{\pi}{N} \sum_{l=0}^{2N-1} \left| \frac{\lambda}{2N} \sum_{k=0}^{2N-1} [Q [t_l, t_k, z_k^{(1)}] - Q [t_l, t_k, z_k^{(2)}]] \right|^2 \right\}^{1/2},$$

since

$$\| E^{(N)} \|_{L_2^{(N)}} \leqslant c (2), \ [1],$$

and by using the conditions (2) and (3) and from [4] we have

$$\| E^{(N)} z^{(1)} - E^{(N)} z^{(2)} \|_{L_2^{(N)}} \leqslant | \lambda | \{ c (2) \mu (0) + \mu' (1) \psi (N) \} \| z^{(1)} - z^{(2)} \|_{L_2^{(N)}}, \tag{2.4}$$

where

$$\psi^2 (N) = \frac{\pi}{N} \sum_{l=0}^{2N-1} \left[\frac{1}{N} \sum_{k=0}^{2N-1} | t_l - t_k | \right]^2.$$

By means of contraction mapping principle at

$$\lambda < \min \left\{ \frac{M}{Rc}, \frac{1}{c (2) \mu (0) + \mu' (1) \psi (N)} \right\}$$

the system (2.3) for arbitrary $N \geq 2$ has a uniqe solution in $H_{\varphi, n}^{(N)} (M)$ and the following theorem will be valid:

Theorem 2.1. Let the function $F[\tau, u]$ and $Q[\delta, \tau, u]$ satisfy the conditions (2) and (3) respectively, then the system of nonlinear algebraic equations (2.3), for arbitrary $N \geq 2$, has a unique solution in $H^{(N)}_{\varphi, n}$ and this solution can be found by the method of successive approximations.

3. The rate of convergence of the approximate solution

On using the solution $z^* \in H^{(N)}_{\varphi, n}(M)$ of system (2.3), for arbitrary $N \geq 2$, the approximate solution $u^{(N)}(\sigma)$ of integral equation (1) is given by the following formula:

$$u^{(N)}(\sigma) = f(\sigma) + \frac{\lambda}{N} \sum_{k=0}^{2N-1} F[t_k, z_k^*] \sin^2 \frac{\sigma - t_k}{2} \cot \frac{t_k - \sigma}{2} + \frac{\lambda}{2N} \sum_{k=0}^{2N-1} Q[\sigma, t_k, z_k^*] \tag{3.1}$$

where, at $\sigma = t_l$ the summation taking by all k different from l. Also, if

$$|\lambda| < \min \left\{ \frac{M}{Rc}, \frac{1}{c(2)\,\mu(0) + \mu'(1)\,\psi(N)} \right\} \tag{3.2}$$

then the equation (1) has a unique solution $u^*(\sigma) \in H_{\varphi, n}(M)$. Applying the quadratic formula (2.2) to equation (1) at node points σ_l, we obtain

$$u^*(\sigma_l) = f(\sigma_l) + \frac{\lambda}{2\pi} \sum_{\substack{k=0 \\ k \neq l}}^{2N-1} F[\sigma_k, u^*(\sigma_k)]\,[1 - (-1)^{k-l}] + \cot \frac{\sigma_k - \sigma_l}{2} + \lambda R_N[R] + \lambda R_N[Q], \quad l = \overline{0, 2N-1}.$$

Put $z^{(1)} = u^*$ and $z^{(2)} = z^*$ in (2.4) and from (3.2) we obtain

$$\| u^* - z^* \|_{L_2^{(n)}} \leqslant |\lambda| (\| R_N[F] \|_C + \| R_N[Q] \|_C) \times \{1 - |\lambda| [c(2)\,\mu(0) + \mu'(1)\,\psi(N)]\}^{-1}. \tag{3.3}$$

From [1, 4] we have

$$\| u^*(\sigma) - u^{(N)}(\sigma) \|_C = \max_i \{ \| u_i(\sigma) - u^{(N)}(\sigma) \|_C \} \leqslant$$

$$\leqslant 2 |\lambda| \mu(0) (1 + \pi)(1 + \ln 2N) \max_l \| u^*(\sigma_l) - z_l^* \|_C +$$

$$+ |\lambda| (\| R_N[F] \|_C + \| R_N[Q] \|_C),$$

since

$$\| R_N[F] \|_C + \| R_N[Q] \|_C \leqslant \text{const.} \left[\varphi\left(\frac{\pi}{N^{2\beta}}\right) \ln N \right], [3], \qquad (3.4)$$

then from (3.3) and (3.4), finally the following inequality will be valid:

$$\| u^*(\sigma) - u^{(N)}(\sigma) \|_C \leqslant \text{const.} \left[\varphi\left(\frac{\pi}{N^{2\beta}}\right) \ln N + \varphi\left(\frac{1}{N}\right) \frac{\ln^3 N}{N^{-\beta}} \right], \quad \beta \in \nu(\varphi).$$

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