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$\alpha\beta$-Statistical e-Convergence for Double Sequences

Yurdal SEVER

Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, 03200 Afyonkarahisar.

e-posta: ysever@aku.edu.tr ORCID ID: http://orcid.org/0000-0002-5102-1384

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Abstract

In this article, we define the concept of αβ natural density which is a generalization of the natural density concept given for pairs of integer. The concept of $\alpha\beta$-statistical e-convergence is introduced with the help of this density. After that some elementary properties of this type of convergence are examined. Also, we define the notions of $\alpha\beta$-statistical limit inferior and superior in e-sense. Finally we give some theorems related to them.

Çift Diziler için $\alpha\beta$-İstatistiksel e-Yakınsaklık

Anahtar kelimeler
Çift dizi; e-yakınsaklık; $\alpha\beta$-istatistiksel e-yakınsaklık; $\alpha\beta$-istatistiksel e-alt limit ve üst limit.

Öz

Bu makalede, tam sayı ikilileri için verilen yoğunluk kavramının bir genelleştirilmesi olan $\alpha\beta$ doğal yoğunluk kavramını tanımladık. Bu yoğunluk kavramı yardımıyla çift diziler için $\alpha\beta$-istatistiksel e-yakınsaklık kavramı tanıtıldı. Daha sonra bu tip yakınsaklığın temel özellikleri incelendi. Ayrıca, e-anlamında $\alpha\beta$-istatistiksel alt limit ve üst limit kavramlarını tanımladık. Son olarak bu kavramlarla ilgili teoremler verdik.

1. Introduction

Throughout the paper the symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{N}^2$ are used for the positive integers, the real numbers, and the pairs of positive integers respectively. We will use the symbol Ω for the vector space, coordinate-wise addition and scalar multiplication, of all real or complex double sequences. In double sequences, there exist more than one types of convergence due to order of elements of $\mathbb{N}^2$. One of them is Pringsheim (1898) convergence which is the best known and well-studied. In this type convergence, a double sequence $y=(y_{ij})$ converges to the number p, written $P-\lim_{ij} y_{ij}=p$, if for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $|y_{ij}-p|<\varepsilon$ for all $i,j>n_0$. In Pringsheim convergence the row-index i and the column-index j tend to infinity independently from each other.

The essential deficiency of this type of convergence is that a convergent sequence does not require to be bounded. Hardy (1917) defined the concept of regular sense, does not have this shortcoming, for double sequence. In regular convergence, both the row-index and the column-index of the double sequence need to be convergent besides the convergent in Pringsheim's sense.

The notion of e-convergence of double sequences, which is substantially weaker than the Pringsheim convergence, was defined by Boos et al. in (1997).

A double sequence $y=(y_{ij})$ is called e-convergent to ρ, written $e-\lim_{ij} y_{ij}=\rho$, if

$$\forall\varepsilon>0\ \exists j_0\in\mathbb{N}\ \forall j\geq j_0\ \exists i_j\in\mathbb{N}\ \forall i\geq i_j\colon |y_{ij}-\rho|<\varepsilon.$$

In contrast to Pringsheim convergence, e-convergence declares that the row-index i linked to the column-index j whenever it goes to infinity. A real double sequence $y=(y_{ij})$ is called e-bounded if there exists positive real number M such that (Zeltser 2001)

$$\exists j_0 \in \mathbb{N}\, \forall j \geq j_0\, \exists i_j \in \mathbb{N}\, \forall i \geq i_j : |y_{ij}| < M.$$

Moreover e-convergence of double sequences has been studied by Zeltser (2001, 2002) and Sever and Talo (2014, 2018).

Fast (1951) presented the concept of statistical convergence. Let H be a subset of natural numbers. The natural density of the set H is defined by

$$\delta(H) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in H\}|$$

when the limit exists, where the symbol $|S|$ is used for the number of elements in S. if we consider the definition of natural density, $\delta(H) \neq 0$ means that either $\delta(H)$ is greater than 0 or the set H does not have natural density.

A sequence (y_n) of numbers is called statistically convergent to s, written $st - \lim_{n\to\infty} y_n = s$, if for every $\varepsilon > 0$ we have

$$\lim_{k\to\infty} \frac{1}{k} |\{n \leq k : |y_n - s| \geq \varepsilon\}| = 0.$$

The concepts of statistical limit inferior and superior were introduced by Fridy and Orhan (1997). Many researchers contribute the statistical analogues of different types of convergence for double sequences (see; Mursaleen and Edely 2003, Móricz 2003, Çakan and Altay 2006, Edely and Mursaleen 2006). Recently statistical e-convergence for double sequence was introduced by Sever and Talo (2017). A double sequence $y = (y_{ij})$ is called statistically e-convergent to the number ρ, written $st_e - \lim_{ij} y_{ij} = \rho$, if for all positive number ε the natural density of the set

$$\{j : \delta(\{i : |y_{ij} - \rho| \geq \varepsilon\}) = 0\}$$

is equal to 1. In other words

$$\delta\big(\{j : \delta(\{i : |y_{ij} - \rho| \geq \varepsilon\}) = 0\}\big) = 1.$$

Aktuğlu (2014) introduced $\alpha\beta$-statistical convergence of ordinary sequences as follows:

Let us take $\alpha(k)$ and $\beta(k)$ which are non-decreasing sequences of natural numbers such that $\alpha(k) \leq \beta(k)$ and $\beta(k) - \alpha(k) \to \infty$ when $k \to \infty$. The set of pairs (β, α) are symbolized by Γ. For each pair $(\beta, \alpha) \in \Gamma$, $0 < \eta \leq 1$ and $H \subseteq \mathbb{N}$. Define

$$\delta^{\alpha\beta}(H, \eta) = \lim_{k\to\infty} \frac{|H \cap I_k^{\alpha\beta}|}{(\beta(k) - \alpha(k) + 1)^\eta} \qquad (1.1)$$

where $I_k^{\alpha\beta}$ is used for the closed interval $[\alpha(k), \beta(k)]$. It is called $\alpha\beta$ natural density order η.

A sequence y is said to be $\alpha\beta$-statistically convergent of order η to s, written $st^{\alpha\beta-\eta} - \lim_{k\to\infty} y = s$, if for every $\varepsilon > 0$,

$$\delta^{\alpha\beta}(\{n : |y_n - s| \geq \varepsilon\}, \eta) = \lim_{k\to\infty} \frac{|\{n \in I_k^{\alpha\beta} : |y_n - s| \geq \varepsilon\}|}{(\beta(k) - \alpha(k) + 1)^\eta} = 0.$$

For $\eta = 1$, we get that y is $\alpha\beta$-statistically convergent to s, and written $st^{\alpha\beta} - \lim_{k\to\infty} y = s$. Also, Karaisa (2016) studied statistical $\alpha\beta$-summability.

We can easily derive the following lemma from (1.1), similarly to Lemma 1 given by Aktuğlu (2014).

Lemma 1.1 Let H_1 and H_2 be two subsets of $\mathbb{N}$ and $0 < \eta \leq 1$; then for each pair $(\beta, \alpha) \in \Gamma$,

a) if $\delta^{\alpha\beta}(H_1, \eta) = 1$ and $\delta^{\alpha\beta}(H_2, \eta) = 1$ then $\delta^{\alpha\beta}(H_1 \cap H_2, \eta) = 1$,

b) $\delta^{\alpha\beta}(H_1 \cup H_2, \eta) \leq \delta^{\alpha\beta}(H_1, \eta) + \delta^{\alpha\beta}(H_2, \eta)$.

We only interested in real double sequences in the present study.

2. Main Result

In this section we extend the notion of statistical e-convergence of double sequences to the notion of $\alpha\beta$-statistical e-convergence of order η of double sequences as follow:

In contrast to ordinary sequence, double sequence has two indices. So, we need four non-decreasing sequences of positive integer. For this reason we take $\alpha_t(k)$ and $\beta_t(k)$ for $t = 1, 2$. By choosing $\alpha_t(k)$ and $\beta_t(k)$, we get new kind of convergence of double sequence, defined before or not, on e-sense.

Definition 2.1 Let $(\beta_t,\alpha_t)\in\Gamma$, $0<\eta_t\le 1$ and $H\subseteq\mathbb{N}$. For $t=1,2$;

$$\delta^{\alpha_t\beta_t}(H,\eta_t)=\lim_{k\to\infty}\frac{|H\cap I_n^{\alpha_t\beta_t}|}{(\beta_t(k)-\alpha_t(k)+1)^{\eta_t}}$$

where the closed interval $[\alpha_t(k),\beta_t(k)]$ is represented by $I_k^{\alpha_t\beta_t}$. It is called $\alpha_t\beta_t$ natural density order η_t.

A double sequence $x=(x_{ij})$ is called $\alpha\beta$-statistically e-convergent of order η to ξ if for all positive number ε the set

$$\{j:\delta^{\alpha_2\beta_2}(\{i:|x_{ij}-\xi|\ge\varepsilon\},\eta_2)=0\}$$

has $\alpha_1\beta_1$ natural density 1 order η_1. In this case, we denote this as $st_e^{\alpha\beta-\eta}-\lim_{ij}x_{ij}=\xi$. If we take $\eta_{1,2}=1$, then we have $\alpha\beta$-statistical e-convergence of the double sequence x to ξ and it is abbreviated as $st_e^{\alpha\beta}-\lim_{ij}x=\xi$.

We will see at the Example 2.4 that this definition is significant generalization of both e-convergence and statistical e-convergence of double sequences.

Remark 2.2 It is obvious that if $0<\eta_t\le\gamma_t\le 1$, for $t=1,2$ and $st_e^{\alpha\beta-\eta}-\lim_{ij}x_{ij}=\xi$ then

$$st_e^{\alpha\beta-\gamma}-\lim_{ij}x_{ij}=\xi.$$

Lemma 2.3 Let $(\beta_t,\alpha_t)\in\Gamma$ for $t=1,2$ and let x be a double sequence. If $e-\lim_{ij}x_{ij}=\xi$ then

$$st_e^{\alpha\beta}-\lim_{ij}x_{ij}=\xi.$$

Proof: The proof of the lemma is easily obtained from the fact that $\alpha\beta$-natural density order η of finite set is zero.

Example 2.4 Let $x=(x_{ij})$ be defined as

$$x_{ij}:=\begin{cases} i+j, & j\ge i,\\ i\cdot j, & j<i \text{ and } j \text{ or } i \text{ are square},\\ 0, & j<i \text{ and } j \text{ and } i \text{ are not square}.\end{cases}\quad (2.1)$$

Then, it is easy to see that $st_e-\lim_{ij}x_{ij}=0$. But, take $\alpha_t(k)=1$, $\beta_t(k)=k^{\frac{1}{\eta_t}}$, and $\eta_t=\frac{1}{2}$ for $t=1,2$ then $st_e^{\alpha\beta-\frac{1}{2}}-\lim_{ij}x_{ij}=0$ does not hold.

The rest of the paper $(\beta_t,\alpha_t)\in\Gamma$ and $0<\eta_t\le 1$ for $t=1,2$.

Theorem 2.5 Let $\lambda\in\mathbb{R}$ and let (x_{ij}) and (y_{ij}) be two double sequences. If $st_e^{\alpha\beta-\eta}-\lim_{ij}x_{ij}=\xi_1$ and $st_e^{\alpha\beta-\eta}-\lim_{ij}y_{ij}=\xi_2$. Then, the followings hold

a) $st_e^{\alpha\beta-\eta}-\lim_{ij}\lambda\cdot x_{ij}=\lambda\cdot\xi_1$,

b) $st_e^{\alpha\beta-\eta}-\lim_{ij}(x_{ij}+y_{ij})=\xi_1+\xi_2$.

Proof:

a) The equality is trivially true if $\lambda=0$. Let $\lambda\ne 0$. Then we have

$$|\lambda\cdot x_{ij}-\lambda\cdot\xi_1|\ge\varepsilon\Leftrightarrow|x_{ij}-\xi_1|\ge\frac{\varepsilon}{|\lambda|},$$

and for $(\beta_2,\alpha_2)\in\Gamma$, $0<\eta_2\le 1$,

$$\delta^{\alpha_2\beta_2}(\{i:|x_{ij}-\xi_1|\ge\frac{\varepsilon}{|\lambda|}\},\eta_2)=0,$$

and for $(\beta_1,\alpha_1)\in\Gamma$, $0<\eta_1\le 1$, for every $\varepsilon>0$

$$\delta^{\alpha_1\beta_1}(\{j:\delta^{\alpha_2\beta_2}(\{i:|x_{ij}-\xi_1|\ge\frac{\varepsilon}{|\lambda|}\},\eta_2)=0\},\eta_1)=1.$$

This implies $st_e^{\alpha\beta-\eta}-\lim_{ij}\lambda\cdot x_{ij}=\lambda\cdot\xi_1$.

b) Take positive number ε. Since (x_{ij}) and (y_{ij}) are $\alpha\beta$-statistically e-convergent of order η to the numbers ξ_1 and ξ_2, respectively, then for given positive number ε

$$H_1=\{j:\delta^{\alpha_2\beta_2}(\{i:|x_{ij}-\xi_1|\ge\frac{\varepsilon}{2}\},\eta_2)=0\}$$

and

$$H_2=\{j:\delta^{\alpha_2\beta_2}(\{i:|y_{ij}-\xi_2|\ge\frac{\varepsilon}{2}\},\eta_2)=0\}$$

with $\delta^{\alpha_1\beta_1}(H_1,\eta_1)=1$ and $\delta^{\alpha_1\beta_1}(H_2,\eta_1)=1$. If we take $H=H_1\cap H_2$, then we have $\delta^{\alpha_1\beta_1}(H,\eta_1)=1$. Since

$$|x_{ij}+y_{ij}-(\xi_1+\xi_2)|\le|x_{ij}-\xi_1|+|y_{ij}-\xi_2|.$$

For each $j\in H$ we have

$$\{i:|x_{ij}+y_{ij}-(\xi_1+\xi_2)|\ge\varepsilon\}\subseteq\{i:|x_{ij}-\xi_1|\ge\frac{\varepsilon}{2}\}\cup\{i:|y_{ij}-\xi_2|\ge\frac{\varepsilon}{2}\}$$

and

$$\delta^{\alpha_2\beta_2}(\{i:|x_{ij}+y_{ij}-(\xi_1+\xi_2)|\ge\varepsilon\},\eta_2)\le\delta^{\alpha_2\beta_2}\left(\left\{i:|x_{ij}-\xi_1|\ge\frac{\varepsilon}{2}\right\},\eta_2\right)+\delta^{\alpha_2\beta_2}(\{i:|y_{ij}-\xi_2|\ge\frac{\varepsilon}{2}\},\eta_2).$$

Therefore

$$\delta^{\alpha_2\beta_2}\big(\{i: |x_{ij} + y_{ij} - (\xi_1 + \xi_2)| \geq \varepsilon\}, \eta_2\big) = 0$$

holds and we have

$$H \subseteq \big\{j: \big(\{i: |x_{ij} + y_{ij} - (\xi_1 + \xi_2)| \geq \varepsilon\}\big) = 0\big\}.$$

Hence, we get

$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}\big(\{i: \left|x_{ij} + y_{ij} - (\xi_1 + \xi_2)\right| \geq \varepsilon\}, \eta_2\big) = 0\}, \eta_1\big) = 1.$$

In the sequel we can give the definition of $\alpha\beta$-statistical e-bounded for double sequences order η. After that we introduce the notions of $\alpha\beta$-statistical e-limit superior and inferior order η for double ones and show some theorems which characterize new concepts.

Definition 2.6 A double sequences $x = (x_{ij})$ is called $st_e^{\alpha\beta-\eta}$-bounded below if there exists a real number M_1 such that

$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < M_1\}, \eta_2) = 0\}, \eta_1\big) = 1.$$

Also, $x = (x_{ij})$ is called $st_e^{\alpha\beta-\eta}$-bounded above if there exists a real number M_2 such that

$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} > M_2\}, \eta_2) = 0\}, \eta_1\big) = 1.$$

If the sequence $x = (x_{ij})$ is both $st_e^{\alpha\beta-\eta}$-bounded below and $st_e^{\alpha\beta-\eta}$-bounded above then it is called $st_e^{\alpha\beta-\eta}$-bounded.

Definition 2.7 Let $x = (x_{ij})$ be a double sequences. Let us define

$$K_x := \{k \in \mathbb{R}: \delta^{\alpha_1\beta_1}(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} > k\}, \eta_2) = 1\}, \eta_1) = 1\},$$

and

$$L_x := \{l \in \mathbb{R}: \delta^{\alpha_1\beta_1}(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < l,\}, \eta_2) = 1\}, \eta_1) = 1\}.$$

Then

$$st_e^{\alpha\beta-\eta} - \limsup x := \begin{cases} \inf L_x, & L_x \neq \emptyset, \\ \infty, & \text{otherwise} \end{cases}$$

is called $st_e^{\alpha\beta-\eta}$ limit superior of x and

$$st_e^{\alpha\beta-\eta} - \liminf x := \begin{cases} \sup K_x, & K_x \neq \emptyset, \\ -\infty, & \text{otherwise} \end{cases}$$

is called $st_e^{\alpha\beta-\eta}$ limit inferior of x.

Obviously, if $x = (x_{ij})$ is $st_e^{\alpha\beta-\eta}$-bounded, then the sets K_x and L_x are not empty set. Hence, both of $st_e^{\alpha\beta-\eta}$ limit inferior and $st_e^{\alpha\beta-\eta}$ limit superior of x are finite numbers.

Theorem 2.8 If $st_e^{\alpha\beta-\eta}$ limit superior of x is finite number ψ, then for all positive number ε

$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < \psi + \varepsilon\}, \eta_2) = 1\}, \eta_1\big) = 1,$$
$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} > \psi - \varepsilon\}, \eta_2) \neq 0\}, \eta_1\big) \neq 0. \tag{2.1}$$

On the contrary, if for all positive number ε the condition (2.1) holds then $\psi = st_e^{\alpha\beta-\eta} - \limsup x$.

Proof:

Assume that $st_e^{\alpha\beta-\eta} - \limsup x = \psi$. In this case $\psi = \inf L_x$. According to the definition of infimum of a set, for $\varepsilon > 0$, there exists $\psi_\varepsilon \in L_x$ such that $\psi_\varepsilon \leq \psi + \varepsilon$. Since $\psi_\varepsilon \in L_x$ and considering the definition of the set L_x, we have $\delta^{\alpha_1\beta_1}(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < \psi_\varepsilon\}, \eta_2) = 1\}, \eta_1) = 1$.

Since

$$\big\{j: \delta^{\alpha_2\beta_2}\big(\{i: x_{ij} < \psi_\varepsilon\}, \eta_2\big) = 1\big\} \subseteq \{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < \psi + \varepsilon\}, \eta_2) = 1\},$$

we get that

$$\delta^{\alpha_1\beta_1}\big(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < \psi + \varepsilon\}, \eta_2) = 1\}, \eta_1\big) = 1.$$

We now illustrate the second formula of (2.1). Define $H_1 = \{l: \delta^{\alpha_2\beta_2}(\{i: x_{ij} > \psi - \varepsilon\}, \eta_2) \neq 0\}$ and assume that $\delta^{\alpha_1\beta_1}(H_1, \eta_1) = 0$. In this case for each $l \in H_1^c$, we get $\delta^{\alpha_2\beta_2}(\{i: x_{ij} > \psi - \varepsilon\}, \eta_2) = 0$. In other words, $\delta^{\alpha_2\beta_2}(\{i: x_{ij} \leq \psi - \varepsilon\}, \eta_2) = 1$. Thus,

$$H_1^c \subseteq \{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} \leq \psi - \varepsilon\}, \eta_2) = 1\}.$$

So that $\psi - \varepsilon \in L_x$. Hence, $\psi - \varepsilon \geq \inf L_x = \psi$ which is a contradiction. Then, $\delta^{\alpha_1\beta_1}(H_1, \eta_1) \neq 0$.

On the contrary, assume that the condition (2.1) holds for a real number ψ. This implies that for given positive number ε we get $\psi + \varepsilon \in L_x$.

$$st_e^{\alpha\beta-\eta} - \limsup x = \inf B_x \leq \psi + \varepsilon. \tag{2.2}$$

On the other hand for each $l \in L_x$ we have $H_2 = \{j: \delta^{\alpha_2\beta_2}\big(\{i: x_{ij} < l\}, \eta_2\big) = 1\}$ with $\delta^{\alpha_1\beta_1}(H_2, \eta_1) = 1$. So $\delta^{\alpha_1\beta_1}(H_1, \eta_1) \neq 0$, there exists $j_1 \in H_1 \cap H_2$ such that

$$\delta^{\alpha_2\beta_2}\left(\{i: x_{ij_1} < l\}, \eta_2\right) = 1$$

and

$$\delta^{\alpha_2\beta_2}(\{i: x_{ij_1} > \psi - \varepsilon\}, \eta_2) \neq 0.$$

Thus there exists i_1 such that $\psi - \varepsilon < x_{i_1 j_1} < l$. Since this holds for each $l \in L_x$ we get

$$\psi - \varepsilon \leq \inf L_x = st_e^{\alpha\beta-\eta} - \limsup x. \quad (2.3)$$

Considering the conditions (2.2) and (2.3), since ε is arbitrary we obtain $\psi = st_e^{\alpha\beta-\eta} - \limsup x$ which is desired.

By duality we easily obtain the following theorem for $st_e^{\alpha\beta-\eta}$ limit infimum of x without proof.

Theorem 2.9 If $st_e^{\alpha\beta-\eta}$ limit inferior of x is finite real number φ, then for all positive number ε

$$\delta^{\alpha_1\beta_1}\left(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} < \varphi + \varepsilon\}, \eta_2) \neq 0\}, \eta_1\right) \neq 0,$$
$$\delta^{\alpha_1\beta_1}\left(\{j: \delta^{\alpha_2\beta_2}(\{i: x_{ij} > \varphi - \varepsilon\}, \eta_2) = 1\}, \eta_1\right) = 1. \quad (2.4)$$

On the contrary, if for all positive number ε the condition (2.4) is satisfied then

$$\varphi = st_e^{\alpha\beta-\eta} - \liminf x.$$

Theorem 2.10 Let ξ be a finite real number.

$$st_e^{\alpha\beta-\eta} - \liminf x = st_e^{\alpha\beta-\eta} - \limsup x = \xi$$
$$\Leftrightarrow st_e^{\alpha\beta-\eta} - \lim\ x = \xi.$$

Proof:

Let us assume that $st_e^{\alpha\beta-\eta} - \lim\ x = \xi$. Then for all positive number ε, the set

$$H = \left\{j: \delta^{\alpha_2\beta_2}\left(\{i: |x_{ij} - \xi| \geq \varepsilon\}, \eta_2\right) = 0\right\}$$

with $\delta^{\alpha_1\beta_1}(H, \eta_1) = 1$. So, we have for $j \in H$,

$$\delta^{\alpha_2\beta_2}\left(\{i: x_{ij} \geq \xi + \varepsilon\}, \eta_2\right) = 0$$

and

$$\delta^{\alpha_2\beta_2}\left(\{i: x_{ij} \leq \xi - \varepsilon\}, \eta_2\right) = 0$$

i.e.,

$$\delta^{\alpha_2\beta_2}\left(\{i: x_{ij} < \xi + \varepsilon\}, \eta_2\right) = 1$$

and

$$\delta^{\alpha_2\beta_2}\left(\{i: x_{ij} > \xi - \varepsilon\}, \eta_2\right) = 1.$$

This implies that $\xi + \varepsilon$ belongs to L_x and $\xi - \varepsilon$ belongs to K_x. Consequently, the inequality

$$\xi - \varepsilon \leq st_e^{\alpha\beta-\eta} - \liminf x = \sup K_x$$
$$\leq st_e^{\alpha\beta-\eta} - \limsup x = \inf L_x \leq \xi + \varepsilon$$

holds. Since ε is arbitrary,

$$st_e^{\alpha\beta-\eta} - \liminf x = st_e^{\alpha\beta-\eta} - \limsup x = \xi$$

is obtained.

On the contrary, let us take

$$st_e^{\alpha\beta-\eta} - \liminf x = st_e^{\alpha\beta-\eta} - \limsup x = \xi.$$

Therefore, for all $\varepsilon > 0$ there exist the sets H_1 and H_2,

$$H_1 := \left\{j: \delta^{\alpha_2\beta_2}\left(\{i: x_{ij} < \xi + \varepsilon\}, \eta_2\right) = 1\right\},$$
$$H_2 := \{j: \delta^{\alpha_2\beta_2}\left(\{i: x_{ij} > \xi - \varepsilon\}, \eta_2\right) = 1\}$$

with $\delta^{\alpha_1\beta_1}(H_1, \eta_1) = 1$ and $\delta^{\alpha_1\beta_1}(H_2, \eta_1) = 1$. If we take $H = H_1 \cap H_2$, then we have $\delta^{\alpha_1\beta_1}(H, \eta_1) = 1$. For $j \in H$ we have

$$\delta^{\alpha_2\beta_2}\left(\{i: |x_{ij} - \xi| < \varepsilon\}, \eta_2\right) = 1$$

or

$$\delta^{\alpha_2\beta_2}\left(\{i: |x_{ij} - \xi| \geq \varepsilon\}, \eta_2\right) = 0.$$

Since

$$H \subseteq \left\{j: \delta^{\alpha_2\beta_2}\left(\{i: |x_{ij} - \xi| \geq \varepsilon\}, \eta_2\right) = 0\right\},$$

we get

$$\delta^{\alpha_1\beta_1}\left(\{j: \delta^{\alpha_2\beta_2}\left(\{i: |x_{ij} - \xi| \geq \varepsilon\}, \eta_2\right) = 0\}, \eta_1\right) = 1.$$

As a result, $st_e^{\alpha\beta-\eta} - \lim\ x = \xi$.

Finally, we state the following theorem that can be easily showed similar to the argument used by Sever and Talo (2014).

Theorem 2.11 We have the following statements for double sequences $y = (y_{ij})$ and $z = (z_{ij})$.

a) $st_e^{\alpha\beta-\eta} - \liminf y \leq st_e^{\alpha\beta-\eta} - \limsup y$,

b) $st_e^{\alpha\beta-\eta} - \limsup (-y) = -(st_e^{\alpha\beta-\eta} - \liminf y)$,

c) $st_e^{\alpha\beta-\eta} - \limsup (y + z) \leq st_e^{\alpha\beta-\eta} - \limsup y + st_e^{\alpha\beta-\eta} - \limsup z$,

d) $st_e^{\alpha\beta-\eta} - \liminf (y + z) \geq st_e^{\alpha\beta-\eta} - \liminf y + st_e^{\alpha\beta-\eta} - \liminf z$.

4. Kaynaklar

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