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STRONG A_σ-SUMMABILITY DEFINED BY A MODULUS

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Abstract : In he present note we introduce some new sequence spaces by using a modulus function f and examine some properties of these sequence spaces.

BİR MODÜL YARDIMIYLA TANIMLANAN KUVVETLİ A_σ-TOPLANABİLİRLİK

Özet : Bu çalışmada bir f modül fonksiyonu kullanılarak bazı yeni dizi uzayları tanımlanmakta ve bunların bazı özelikleri incelenmektedir.

INTRODUCTION

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on m, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(2) $\phi(e) = 1$ where $e = (1, 1, 1, \ldots)$, and

(3) $\phi((x_{\sigma(n)})) = \phi(x)$ for all $x \in m$.

The mappings σ are assumed one-to-one and such that $\sigma^k(n) \neq n$ for all positive integers n and k, where $\sigma^k(n)$ denotes the k th iterate of the mapping σ at n.

For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space c of real convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. Consequently, $c \subset V_\sigma$, where V_σ is the set of bounded sequences all of whose σ-means are equal [3].

When $\sigma(n) = n+1$, the σ-mean are the classical Banach limits on m and V_σ is the set of almost convergent sequences [1].

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If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [3] that

$$V_\sigma = \{ x = (x_n) : \lim_m t_{mn}(x) = Le, \text{ uniformly in } n, \ L = \sigma\text{-}\lim x\}$$

where

$$t_{mn}(x) = (x_n + Tx_n + \dots + T^m x_n) / (m + 1).$$

Recently, Mursaleen [4] defined strongly σ-convergent sequences replacing the Banach limits by σ-means, in the following manner:

A bounded sequence $x = \{x_k\}$ is said to be strongly σ-convergent to a number L if and only if

$$\lim_m 1/m \sum_{k=1}^{m} | x_{\sigma k(n)} - L | \to 0, \text{ uniformly in } n.$$

The following inequality will be used frequently throughout the paper:

$$| a_K + b_K |^{p_K} \leqslant C (| a_K |^{p_K} + | b_K |^{p_K}) \qquad (*)$$

where $a_K, b_K \in \mathbf{C}, 0 < p_K \leqslant \sup p_K = H, C = \max (1, 2^{H-1})$ [2].

Definition 1 [7]. A function $f: [0, \infty] \to [0, \infty]$ is called a modulus if

a) $f(x) = 0$ if and only if $x = 0$,

b) $f(x + y) \leqslant f(x) + f(y)$,

c) f is increasing and

d) f is continuous from the right at 0.

Several authers including Maddox [3], Öztürk and Bilgin [6] and some others have studied some new sequence spaces defined by a modulus function.

By using a modulus function f and a nonnegative regular matrix A, we defined the sequence space $w(A_\sigma f)$ as follows [5]:

$$w(A_\sigma f) = \left\{ x \in w: \lim_m \sum_k a_{mk} f(| x_{\sigma k(n)} - L |) = 0, \text{ for some } L \text{ uniformy in } n \right\}.$$

Σ_k denotes the summation $k = 1$ to ∞ and w denotes all complex valued sequences.

Definition 2. Let $p = (p_K)$ be a sequence of strictly positive real numbers, f be a modulus, and A be an infinite matrix of nonnegative real numbers. We write

$$[A_\sigma f, p] = \left\{x \in w : \lim_m \sum_k a_{mk} f(|x_{\sigma k(n)} - L|)^{p_k} = 0, \text{ for some } L \text{ uniformly in } n\right\}$$

$$[A_\sigma f, p]_0 = \left\{x \in w : \lim_m \sum_k a_{mk} f(|x_{\sigma k(n)}|)^{p_k} = 0, \text{ uniformly in } n\right\}$$

$$[A_\sigma f, p]_\infty = \left\{x \in w : \sup_{m,n} \sum_k a_{mk} f(|x_{\sigma k(n)}|)^{p_k} < \infty\right\}.$$

If $x - Le \equiv [A_\sigma f, p]_0$, we say that x is strongly A_σ-summable to L with respect to the modulus f. If x is strongly A_σ-summable to L with respect to the modulus we write $x_K \to L\,[A_\sigma f, p]$.

Note that if $p_K = 1$ for all K and A is a nonnegative regular matrix summability method, then

$$[A_\sigma f, p| = w(A_\sigma f) \text{ and } [A_\sigma f, p]_0 = w(A_\sigma f)_0 .$$

The spaces $w(A_\sigma f)$ and $w(A_\sigma f)_0$ were introduced and discussed in [5].

If $f(x) = x$, the spaces $[A_\sigma f, p]$, $[A_\sigma f, p]_0$, and $[A_\sigma f, p]_\infty$ reduce to $[A_\sigma p]$, $[A_\sigma p]_0$, and $[A_\sigma p]_\infty$ respectively.

We first prove

Lemma 1. $[A_\sigma f, p]$, $[A_\sigma f, p]_0$ and $[A_\sigma f, p]_\infty$ are linear spaces over the complex field $\mathbf{C}$.

Proof. We consider only $[A_\sigma f, p]_0$. Others can be treated similarly. Let $x, y \in [A_\sigma f, p]_0$. For $\lambda, \mu \in \mathbf{C}$, there exist integers M_λ and N_μ such that $|\lambda| \leqslant M_\lambda$ and $|\mu| \leqslant N_\mu$. From Definition 1 (*b*) and (*) we have

$$\sum_k a_{mk} f(|x_{\sigma k(n)} + \mu\, y_{\sigma k(n)}|)^{p_K} \leqslant C M_\lambda^H \sum_k a_{mk} f(|x_{\sigma k(n)}|)^{p_K} + $$

$$+ C N_\mu^H \sum_k a_{mk} f(|y_{\sigma k(n)}|).$$

For $m \to \infty$, since $x, y \in [A_\sigma f, p]_0$, we have $\lambda x + \mu y \in [A_\sigma f, p]_0$. Thus $[A_\sigma f, p]_0$ is linear space over $\mathbf{C}$.

Theorem 1. $[A_\sigma f, p]_0$ and $[A_\sigma f, p]$ ($\inf p_K > 0$) are complete linear topological spaces paranormed by h defined by

$$h(x) = \sup_{m,n} \left\{\sum_k a_{mk} f(|x_{\sigma k(n)}|)^{p_K}\right\}^{1/M}, \quad M = \max\{1, H\}.$$

Proof. Just consider $[A_\sigma f, p]_0$; the other is similarly.

Clearly $h(\theta) = 0$ and $h(x) = h(-x)$. Take any $x, y \equiv [A_\sigma f, p]_0$. Since $\frac{p_K}{M} \leqslant 1$ and $M \geqslant 1$, using the Minkowski's inequality and the definition of f, for all m, n we have

$$\left\{\sum a_{mk} f(|x_{\sigma k(n)} + y_{\sigma^k(n)}|)^{p_K}\right\}^{1/M} \leqslant \left\{\sum_k a_{mk} f(|\lambda x_{\sigma k(n)}|)^{p_K}\right\}^{1/M} + \\ + \left\{\sum_k a_{mk} f(|y_{\sigma k(n)}|)^{p_K}\right\}^{1/M}.$$

Now it follows that h is subadditive. Finally, to check the continuity of multiplication, let us take any complex λ. By definition of f we have

$$h(\lambda x) = \sup_{m,n}\left\{\sum_k a_{mk} f(|x_{\sigma k(n)} \cdot \lambda|)^{p_K}\right\}^{1/M} \leqslant \{1 + [|\lambda|]\}^{H/M} h(x),$$

where $[t]$ denotes the integer spart of t, whence $\lambda \to 0$, $x \to \theta$ imply $h(\lambda x) \to 0$ and also $x \to \theta$, λ fixed imply $h(\lambda x) \to 0$. We show that $\lambda \to 0$, x fixed imply $h(\lambda x) \to 0$.

Let $x \in [A_\sigma f, p]$ then as $m \to \infty$,

$$s_{nm} = \sum_k a_{mk} f(|x_{\sigma k(n)} - L|)^{p_K} \to 0 \text{ uniformly in } n.$$

For $|\lambda| < 1$, we have (by (*))

$$\sum_k a_{mk} f(|\lambda x_{\sigma k(n)}|)^{p_K} \leqslant C \sum_k a_{mk} f(|\lambda x_{\sigma k(n)} - \lambda L|)^{p_K} + \\ + C\sum_k a_{mk} f(|\lambda L|)^{p_K} \leqslant C \sum_{k \leq M} a_{mk} f(|\lambda x_{\sigma k(n)} - \lambda L|)^{p_K} + \\ + C \sum_{k > M} a_{mk} f(|x_{\sigma k(n)} - L|)^{p_K} + C \sum_k a_{mk} f(|\lambda L|)^{p_K}.$$

Let $\varepsilon > 0$ and choose M such that for each n, m and $k > M$ implies $s_{nm} < \varepsilon/2C$. For each M, by continuity of f, as $\lambda \to 0$ (inf $p_K > 0$)

$$\sum_{k \leq M} a_{mk} f(|\lambda x_{\sigma k(n)} - \lambda L|)^{p_K} + \sum_k a_{mk} f(|\lambda L|)^{p_K} \to 0.$$

Then choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\sum_{k \leq M} a_{mk} f([\lambda x_{\sigma k(n)} - \lambda L|)^{p_K} + \sum_k a_{mk} f(|\lambda L[)^{p_K} < \varepsilon/2C.$$

Hence we have

$$\sum_k a_{mk} f(|\lambda x_{\sigma k(n)}|)^{p_K} < \varepsilon$$

and $h(\lambda x) \to 0$ $(\lambda \to 0)$. Thus $[A_\sigma f, p]_0$ is paranormed linear topological space by h.

Now, we show that $[A_\sigma f, p]_0$ is complete with respect to its paranorm topologies. Let (x^s) be a Cauchy sequence in $[A_\sigma f, p]_0$. Then, we write $h(x^s - x^t) \to 0$, $s, t \to \infty$, i.e., as $s, t \to \infty$ for all n and m, we write

$$\sum_k a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - \overset{(t)}{x}_{\sigma k(n)}|)^{p_K} \to 0. \tag{1}$$

Hence, for each fixed n and k, as $s, t \to \infty$, we have $f(|\overset{(s)}{x}_{\sigma k(n)} - \overset{(t)}{x}_{\sigma k(n)}|)^{p_K} \to 0$ and for each fixed n and k, $(\overset{(s)}{x}_{\sigma k(n)})_s$ be a Cauchy sequence in C. Since C is complete, as $s \to \infty$, $(\overset{(s)}{x}_{\sigma k(n)})_s \to (x_{\sigma k(n)})$ say. Now from (1), we have for $\varepsilon > 0$, there exists a natural number N such that

$$\sum_k a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - \overset{(t)}{x}_{\sigma k(n)}|)^{p_K} < \varepsilon \tag{2}$$

for all n, m and $s, t > N$. Hence for any fixed natural number M, we have from (2),

$$\sum_{k \leq M} a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - \overset{(t)}{x}_{\sigma k(n)}|)^{p_K} < \varepsilon \tag{3}$$

for all n, m and $s, t > N$. By taking $t \to \infty$ in the above expression we obtain

$$\sum_{k \leq M} a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - x_{\sigma k(n)}|)^{p_K} < \varepsilon$$

for all n, m and $s < N$. Since M is arbitrary, by taking $M \to \infty$ we obtain

$$\sum_k a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - x_{\sigma k(n)}|)^{p_K} < \varepsilon$$

for all n, m and $s > N$, that is, $h(x^{(s)} - x) \to 0$ as $s \to \infty$ and thus $x^s \to x$ as $s \to \infty$.

Also, for each s, there exists $L^{(s)}$ with

$$\sum_k a_{mk} f(|\overset{(s)}{x}_{\sigma k(n)} - L^{(s)}|)^{p_K} \to 0 \ (m \to \infty) \tag{4}$$

uniformly in n. From regularity of A, Definition 1(b) and (4), we have $f(|L^{(s)} - L^{(t)}|)^{p_K} \to 0$ $(s, t \to \infty)$ and $(L^{(s)})$ is a Cauchy sequence in $\mathbf{C}$, so $(L^{(s)})$ converges to L, say. Consequently we get

$$\sum_k a_{mk} f(|x_{\sigma k(n)} - L|)^{p_K} \to 0 \ (m \to \infty)$$

uniformly in n. So that $x \in [A_\sigma f, p]$ and the space is complete.

Theorem 2. i) If f is a modulus f and x is strongly A_σ-summable to L, then x is strongly summable to L with respect to the modulus.

ii) If $\lim\limits_{t\to\infty} \dfrac{f(t)}{t} = \beta > 0$ then $[A_\sigma f, p] = [A_\sigma p]$.

Proof. i) Let $x \in [A_\sigma p]$, so that

$$\lim_m \sum_k a_{mk} \mid x_{\sigma k(n)} \mid^{p_K} = 0, \text{ uniformly in } n.$$

Let $1 > \varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leqslant t \leqslant \delta$. We write $t_K = \mid x_{\sigma k(n)} - L \mid^{p_K}$ and consider

$$\sum_k a_{mk} f(\mid x_{\sigma k(n)} - L \mid)^{p_K} = \Sigma_1 + \Sigma_2 .$$

The remainder of the claim can be proved by using the techniques similar to those used in Theorem 5 of Öztürk and Bilgin [6].

ii) In viev of Theorem we need only show that $[A_\sigma f, p] \subseteq [A_\sigma p]$. For any modulus function, the existance of positive limit given with β was given in [5]. Now $\beta > 0$ and let $x \in [A_\sigma f, p]$. Since $\beta > 0$, for every $t > 0$, we write $f(t) \geq \beta t$. From this inequality, it is easy to see that $x \in [A_\sigma p]$. This completes the proof.

Theorem 3. Let A be a nonnegative regular matrix and f be a modulus, then $[A_\sigma f, p] \subseteq [A_\sigma f, p]_\infty$.

Proof. Let $x \in [A_\sigma f, p]$. From Definition 1 (b) and (*) we have

$$\sum_k a_{mk} f(\mid x_{\sigma k(n)} \mid)^{p_K} \leqslant C \sum_k a_{mk} f(\mid x_{\sigma k(n)} - L \mid)^{p_K} + C \sum_k a_{mk} f(\mid L \mid)^{p_K}.$$

There exists an integer M_L such that $\mid L \mid \leqslant M_L$. Hence we have

$$\sum_k a_{mk} f(\mid x_{\sigma k(n)} \mid)^{p_K} \leqslant C \sum_k a_{mk} f(\mid x_{\sigma k(n)} - L \mid)^{p_K} + C \{M_L f(1)\}^H \sum a_{mk} .$$

Since A is regular and $x \in [A_\sigma f, p]$, we get $x \in [A_\sigma f, p]_\infty$ and this completes the proof.

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