FROBENIUS Q-GROUPS

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Summary : An important problem in **Q**-group theory is to classify particular classes of **Q**-groups. In this note we shall completely classify the Frobenius **Q**-groups.

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Özet : Bu çalışmada Frobenius **Q**-grupları tamamen sınıflandırılmaktadır.

Definition. A **Q**-group is a finite group all whose characters are rational valued.

Proposition 1 (see [3]). Let G be a finite group. Then G is a **Q**-group if and only if for every $x \in G$, $N_G(\langle x \rangle)/C_G(\langle x \rangle) = \mathrm{Aut}(\langle x \rangle)$.

Theorem 2 (see [2], chap. V and 8). Suppose G is a Frobenius group having Frobenius kernel F and Frobenius subgroup H. Then:

(1) Let x be a nonidentity element of H. If a conjugate x^y is contained in H, then y is an element of H.

(2) If $x \in H - \{1\}$, then $C_G(x) \subseteq H$.

(3) If $|H|$ is even, then

a) H contains only one involution.

b) Let i be the involution of H. Then, for every $f \in F$, $f^i = f^{-1}$.

Proposition 3. Suppose G is a **Q**-group and H a subgroup without fusion in G (this means that if $x, y \in H$ and there is an element z of G such that $x^z = y$, then there is an element t of H such that $x^t = y$). Then, H is also a **Q**-group.

Proof. Clearly by Proposition 1.

Theorem 4. Suppose G is a Frobenius $\mathbf{Q}$-group having Frobenius kernel F and Frobenius subgroup H. Then:

(1) H has even order, contains a Sylow 2-subgroup of G, is without fusion in G and is also a $\mathbf{Q}$-group.

(2) A Sylow 2-subgroup of G is isomorphic either to $\mathbf{Z}_2$ or to the quaternion group Q_8 of order 8.

(3) F is an odd order normal subgroup of G.

Proof. By Theorem 2 part (1) H is without fusion in G, so that H is also a $\mathbf{Q}$-group and has even order. By Theorem 2 part (3) a, H contains only one involution so that a Sylow 2-subgroup of H is isomorphic either to a cyclic group or to a generalized quaternion group. By Proposition 31, pg. 31 of [3] then a Sylow 2-subgroup of H is either $\mathbf{Z}_2$ or Q_8.

Let $f \in F$ be an involution and i be the involution of H. By Theorem 2 part (3) b $f^i = f^{-1} = f$ so that $f \in C_G(i)$. By part (2) of Theorem 2, $C_G(i) \subseteq H$, therefore $f \in H$, contradiction. Hence F has odd order, and H contains a Sylow 2-subgroup of G. Therefore, a Sylow 2-subgroup of G is either $\mathbf{Z}_2$ or Q_8.

Theorem 5. Let G be a Frobenius $\mathbf{Q}$-group. Then, $G = E_3 \mathbf{Z}_2$ where E_3 is an elementary abelian 3-subgroup and $\mathbf{Z}_2$ inverts all elements of E_3.

Proof. By Theorem 4, a Sylow 2-subgroup of G is either $\mathbf{Z}_2$ or Q_8. If a Sylow 2-subgroup of G is $\mathbf{Z}_2$ by Proposition 33, pg. 33 of [3] the statement is true.

If a Sylow 2-subgroup of G is Q_8, then by Glauberman Z^* theorem (see [4]) $\mathbf{Z}(G)$ is not trivial. Hence the involution of H can't inverse the elements of F, contradiction to Theorem 2 part (3) b.

Corollary 6. The only Frobenius groups which can be embedded without fusion in a symmetric group are those of the form $E_3 \mathbf{Z}_2$ of the previous theorem.

Proof. By [1], a finite group can be embedded without fusion in a symmetric group if and only if it is a $\mathbf{Q}$-group.

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