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ON UMBILICAL HYPERSURFACE OF A GENERALIZED RECURRENT RIEMANNIAN SPACE

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Summary : We consider umbilical hypersurface of a generalized recurrent Riemannian space and find the conditions for such hypersurface to be conformally recurrent.

GENELLEŞTİRİLMİŞ BİR "RECURRENT" RIEMANN UZAYININ AMBİLİK HİPERYÜZEYİ HAKKINDA

Özet : Bu çalışmada genelleştirilmiş bir "recurrent" Riemann uzayının ambilik hiperyüzeyi gözönüne alınmakta ve böyle bir hiperyüzeyin konform "recurrent" olabilmesi için gerekli koşullar bulunmaktadır.

INTRODUCTION

In a recent paper [1] the author and H.A. Biswas introduced and studied a type of non-flat Riemannian space whose curvature tensor R_{hijk} satisfies the condition

$$\nabla_l R_{hijk} = \lambda_l R_{hijk} + \mu_l (g_{hk} g_{ij} - g_{hj} g_{ik}) \tag{1}$$

where λ_l and μ_l are two vectors; μ_l is non-zero and ∇ denotes covariant differentiation with respect to the metric tensor. Such a space has been called a generalized recurrent space. μ_l is called its associated vector and an n-space of this kind has been denoted by GK_n. If μ_l bacomes zero in (1), then the space reduces to a recurrent space introduced by Walker [2].

Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional $(n > 3)$ Riemannian space covered by a system of coordinate neighbourhoods (U, y^α). Let (M, g) be a hypersurface of $(\overline{M}, \overline{g})$ defined in a local coordinate system by means of the system of parametric equations $y^\alpha = y^\alpha (x^i)$, where g is the induced metric. Here and in the sequel, Greek indices take the values $1, 2, \ldots, n+1$ and Latin indices take the values $1, 2, \ldots, n$. Let N^α be a local unit normal to (M, g) and let $B_i^\alpha = \partial y^\alpha / \partial x^i$.

Then

$$g_{ij} = \overline{g}_{\alpha\beta} B_i^\alpha B_j^\beta, \tag{2}$$

$$\bar{g}_{\alpha\beta} N^{\alpha} B_j^{\beta} = 0, \ \bar{g}_{\alpha\beta} N^{\alpha} N^{\beta} = \varepsilon, \ \varepsilon = \pm 1 \tag{3}$$

$$B_i^{\alpha} B_j^{\beta} g^{ij} = \bar{g}^{\alpha\beta} - \varepsilon N^{\alpha} N^{\beta} . \tag{4}$$

We denote by $\bar{R}_{\alpha\beta\gamma\delta}$, $\bar{R}_{\alpha\beta}$ and $\bar{R}$ the curvature tensor, the Ricci tensor and the scalar curvature of $(\bar{M}, \bar{g})$ respectively and by R_{ijkl} , R_{ij} and R the corresponding object of the hypersurface. Let h be the second fundamental form of the hypersurface and let ∇ be the operator of covariant differentiation with respect to the metric tensor. Then the Gauss and Codazzi equations for (M, g) of $(\bar{M}, \bar{g})$ can be written in the form ([3], p. 149)

$$\bar{R}_{\alpha\beta\gamma\delta} B_i^{\alpha} B_j^{\beta} B_k^{\gamma} B_l^{\delta} = R_{ijkl} - \varepsilon (h_{il} h_{jk} - h_{ik} h_{jl}),$$

$$\bar{R}_{\alpha\beta\gamma\delta} N^{\alpha} B_j^{\beta} B_k^{\gamma} B_l^{\delta} = \nabla_l h_{jk} - \nabla_k h_{jl} .$$

Also ([2], pp. 147-148)

$$\nabla_r B_j^{\beta} = \varepsilon h_{rj} N^{\beta}, \ \nabla_r N^{\alpha} = - h_{ra} g^{at} B_t^{\alpha}. \tag{5}$$

If there exist on (M, g) two functions α, β and a covariant vector v_i such that

$$h_{ij} = \alpha g_{ij} + \beta v_i v_j , \tag{6}$$

(M, g) is said to be quasiumbilical ([4], p. 147). If $\beta = 0$, (M, g) is an umbilical hypersurface. Miyazawa and Chuman [5] investigated totally umbilical subspaces of recurrent Riemannian space. Among others, they proved that such subspace is conformally recurrent [6].

The aim of this paper is to find the necessary and sufficient conditions for such hypersurface to be conformally recurrent.

Using (6) we can rewrite the Gauss and Codazzi equations as follows:

$$\bar{R}_{\alpha\beta\gamma\delta} B_i^{\alpha} B_j^{\beta} B_k^{\gamma} B_l^{\delta} = R_{ijkl} - \varepsilon \alpha^2 (g_{il} g_{jk} - g_{ik} g_{jl}) \tag{7}$$

$$\bar{R}_{\alpha\beta\gamma\delta} N^{\alpha} B_j^{\beta} B_k^{\gamma} B_l^{\delta} = \alpha_l g_{jk} - \alpha_k g_{jl} \tag{8}$$

where

$$\alpha_l = \frac{\partial \alpha}{\partial x^l} .$$

Also (5) takes the form

$$\nabla_r B_j^{\beta} = \varepsilon \alpha g_{rj} N^{\beta} \tag{9}$$

$$\nabla_r N^{\alpha} = - \alpha B_r^{\alpha} . \tag{10}$$

UMBILICAL HYPERSURFACE OF A GK_n

Applying the operator ∇_r to (7) and using (9), we obtain

$$\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l + \varepsilon\,\alpha\, g_{ri} \overline{R}_{\alpha\beta\gamma\delta} N^\alpha B^\beta_j B^\gamma_k B^\delta_l -$$
$$- \varepsilon\,\alpha\, g_{rk} \overline{R}_{\gamma\delta\alpha\beta} N^\gamma B^\delta_l B^\alpha_i B^\beta_j - \varepsilon\,\alpha\, g_{rl} \overline{R}_{\delta\gamma\alpha\beta} N^\delta B^\gamma_k B^\alpha_i B^\beta_j =$$
$$= \nabla_r R_{ijkl} - 2\varepsilon\,\alpha\,\alpha_r (g_{il} g_{jk} - g_{ik} g_{jl}).$$

Substituting (8) into this equation, we find

$$\nabla_r R_{ijkl} = \nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l +$$
$$+ 2\varepsilon\,\alpha\,\alpha_r (g_{il} g_{jk} - g_{ik} g_{jl}) + \varepsilon\,\alpha\, g_{ri} (\alpha_l g_{jk} - \alpha_k g_{jl}) - \tag{11}$$
$$- \varepsilon\,\alpha\, g_{rk} (\alpha_j g_{li} - \alpha_i g_{lj}) - \varepsilon\,\alpha\, g_{rl} (\alpha_j g_{ik} - \alpha_i g_{jk}).$$

Now, let us suppose that the $(\overline{M}, \overline{g})$ is a generalized recurrent space, i.e.,

$$\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} = \lambda_\rho \overline{R}_{\alpha\beta\gamma\delta} + \mu_\rho (\overline{g}_\alpha \overline{g}_{\beta\gamma} - \overline{g}_{\alpha\gamma} \overline{g}_{\beta\delta}). \tag{12}$$

Taking into account (7), the relation (12) becomes

$$\nabla_r R_{ijkl} = \lambda_r R_{ijkl} + \mu_r (g_{il} g_{jk} - g_{ik} g_{jl}) +$$
$$+ 2\varepsilon\,\alpha\,\alpha_r (g_{il} g_{jk} - g_{ik} g_{jl}) + \varepsilon\,\alpha\, g_{ri} (\alpha_l g_{jk} - \alpha_k g_{il}) - \tag{13}$$
$$- \varepsilon\,\alpha\, g_{rk} (\alpha_j g_{li} - \alpha_i g_{lj}) - \varepsilon\,\alpha\, g_{rl} (\alpha_j g_{ik} - \alpha_i g_{jk})$$

where

$$\lambda_r = \lambda_\rho B^\rho_r$$

and

$$\mu_r = \lambda_\rho B^\rho_r .$$

From (13), we have

$$\nabla_r R_{jk} = \lambda_r R_{jk} + (n-1) \mu_r g_{jk} + 2\varepsilon\, n\, \alpha\,\alpha_r g_{jk} - \varepsilon\,\alpha\,\alpha_k g_{jr} - \varepsilon\, n\,\alpha\,\alpha_j g_{rk} \tag{14}$$

and

$$\nabla_r R = \lambda_r R + n(n-1) \mu_r + 2n(n-1)\, \varepsilon\,\alpha\,\alpha_r . \tag{15}$$

Now, let us consider the covariant derivative of the conformai curvature tensor C_{ijkl} of the hypersurface (M, g):

$$\nabla_r C_{ijkl} = \nabla_r R_{ijkl} - \frac{1}{(n-2)} [g_{jk} \nabla_r R_{il} - g_{jl} \nabla_r R_{ik} +$$
$$+ g_{il} \nabla_r R_{jk} - g_{ik} \nabla_r R_{jl}] + \frac{\nabla_r R}{(n-1)(n-2)} [g_{jk} g_{jl} - g_{jl} g_{ik}].$$

Substituting (13), (14) and (15) into this relations we obtain, after some calculation

$$\nabla_r C_{ijkl} = \lambda_r C_{ijkl} + 2\varepsilon\alpha\alpha_r (g_{il} g_{jk} - g_{ik} g_{jl}) + \varepsilon\alpha g_{ri} (\alpha_l g_{jk} - \alpha_k g_{jl}) -$$
$$- \varepsilon\alpha g_{rk} (\alpha_j g_{li} - \alpha_i g_{lj}) - \varepsilon\alpha g_{rl} (\alpha_j g_{ik} - \alpha_i g_{jk}) -$$
$$- \frac{1}{(n-2)} [g_{jk} (2n\varepsilon\alpha\alpha_r g_{il} - \varepsilon\alpha\alpha_l g_{ir} - \varepsilon n\alpha\alpha_i g_{rl}) - \tag{16}$$
$$- g_{jl} (2n\varepsilon\alpha\alpha_r g_{ik} - \varepsilon\alpha\alpha_k g_{ir} - \varepsilon n\alpha\alpha_i g_{rk}) +$$
$$+ g_{il} (2n\varepsilon\alpha\alpha_r g_{jk} - \varepsilon\alpha\alpha_k g_{jr} - \varepsilon n\alpha\alpha_j g_{rk}) -$$
$$- g_{ik} (2n\varepsilon\alpha\alpha_r g_{jl} - \varepsilon\alpha\alpha_l g_{jr} - \varepsilon\alpha\alpha_j g_{rl})] +$$
$$+ \frac{2n\varepsilon\alpha\alpha_r}{(n-2)} (g_{jk} g_{il} - g_{jl} g_{ik}).$$

If $\alpha = 0$, (16) reduces to

$$\nabla_r C_{ijkl} = \lambda_r C_{ijkl} \tag{17}$$

i.e., the hypersurface is conformally recurrent or (in the case $\lambda_r = 0$) conformally symmetric. If $\alpha = \text{constant} \neq 0$, (16) reduces to

$$\nabla_r C_{ijkl} = \lambda_r C_{ijkl} \tag{18}$$

i.e., the hypersurface is conformally recurrent. Conversely, if (18) holds, then from (16) we get

$$2\varepsilon\alpha\alpha_r (g_{il} g_{jk} - g_{ik} g_{jl}) + \varepsilon\alpha g_{ri} (\alpha_l g_{jk} - \alpha_k g_{jl}) -$$
$$- \varepsilon\alpha g_{rk} (\alpha_j g_{li} - \alpha_i g_{lj}) - \varepsilon\alpha g_{rl} (\alpha_j g_{ik} - \alpha_i g_{jk}) -$$
$$- \frac{1}{(n-2)} [g_{jk} (2n\varepsilon\alpha\alpha_r g_{il} - \varepsilon\alpha\alpha_l g_{ir} - \varepsilon n\alpha\alpha_i g_{rl}) -$$
$$- g_{jl} (2n\varepsilon\alpha\alpha_r g_{ik} - \varepsilon\alpha\alpha_k g_{ir} - \varepsilon n\alpha\alpha_i g_{rk}) + \tag{19}$$
$$+ g_{il} (2n\varepsilon\alpha\alpha_r g_{jk} - \varepsilon\alpha\alpha_k g_{jr} - \varepsilon n\alpha\alpha_j g_{rk}) -$$
$$- g_{ik} (2n\varepsilon\alpha\alpha_r g_{jl} - \varepsilon\alpha\alpha_l g_{jr} - \varepsilon\alpha\alpha_j g_{rl})] +$$
$$+ \frac{2n\varepsilon\alpha\alpha_r}{(n-2)} (g_{jk} g_{il} - g_{jl} g_{ik}).$$

Transvecting (19) with g^{il} and g^{jk} respectively we get

$$\alpha_r = 0$$

i.e., $\alpha = \text{constant}$.

Thus we have the following theorem:

Theorem. Let $(\overline{M}, \overline{g})$ be a generalized recurrent Riemannian space with λ_ρ and μ_ρ as its associated vector field. Let (M, g) be its umbilical hypersurface. Then

(i) if $\alpha = 0$, (M, g) is a conformally recurrent space with $\lambda_r = \lambda_\rho B_r^\rho$ as a recurrence vector field.

(ii) (M, g) is a conformally recurrent space if and only if α = constant $\neq 0$.

If $\mu_p = 0$, the space reduces to a recurrent space.

Thus we have the following corollary of the above theorem:

Corollary. Umbilical hypersurface of a recurrent Riemannian space is conformally recurrent if and only if α = constant $\neq 0$. The above corollary has been proved by M. Prvanovic [7] in another way.

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