ABOUT A CONJECTURE OF DEACONESCU

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Summary : At the International Conference on Group Theory, Timişoara 1992, M. Deaconescu conjectured that the only groups all whose elements of the same order are conjugate are the symmetric groups S_1, S_2, S_3. In this paper, we shall prove that this conjecture is true if besides the group is solvable.

DEACONESCU'NUN BİR İDDİASI HAKKINDA

Özet : 1992 yılında Timişoara'da düzenlenen Uluslararası Gruplar Teorisi Konferansında M. Deaconescu, aynı mertebeden tüm elemanları birbirinin eşleniği olan grupların yalnızca S_1, S_2, S_3 simetrik grupları olduğunu iddia etmiştir. Bu çalışmada, grupların çözülebilir olması ek koşulu altında bu iddianın doğru olduğu ispat edilmektedir.

Definition. A **Q**-group is a finite group all whose characters are rational valued.

Proposition 1. A finite group G is a **Q**-group if and only if for every $x \in G$, $N_G(\langle x \rangle) / C_G(\langle x \rangle) \simeq \mathrm{Aut}(\langle x \rangle)$ (see [3]).

Theorem 2 (Thompson, see [2], pg. 511). Suppose that G is a solvable group of even order and that the Sylow 2-subgroup of G contains more than one involution. Suppose that all the involutions in G are conjugate. Then the Sylow 2-subgroup of G are either homocyclic or Suzuki 2-groups.

Theorem 3. Let G be a finite solvable group all whose elements of the same order are conjugate. Then G is isomorphic to S_1, S_2, S_3.

Proof. Clearly, such a group is a **Q**-gruop and all his involutions are conjugate. Firstly, we shall determine the Sylow 2-subgroups of G. For this purpose we can suppose that $O_{2'}(G)$ is trivial. Let $I(G) = \{x \in G \mid x^2 = 1\}$. Let A be a minimal normal subgroup of G. Since G is solvable and $O_{2'}(G)$ is trivial, then A is an elementary abelian 2-subgroup, hence $A = I(G)$.

Let S be a Sylow 2-subgroup of G. We will prove now by induction on $|G|$ that $N_G(S)=S$. If $|G|=1$ the statement is trivial. Now G/A is a solvable **Q**-group having S/A as a Sylow 2-subgroup. It follows by induction that $N_{G/A}(S/A)=S/A$, so that if $x \in N_G(S)$, then $xA \in S/A$. Set $xA=yA$ with $y \in S$. Then $y^{-1}x \in A$, therefore $x \in S$.

By Theorem 2, if G contains more than one involution, then the Sylow 2-subgroups of G are either cyclic or Suzuki 2-groups. If a Sylow 2-subgroup S is homocyclic then it is trivial that $A \subseteq Z(S)$. If S is a Suzuki 2-group, then (see [2], pg. 313) $S' = Z(S) = A = I(S)$. By Burnside fusion theorem (see [1], pg. 240) if $x, y \in Z(S)$ and $x^z = y$, with $z \in G$, then there is $t \in N_G(S)$ such that $x^t = y$. Since $N_G(S) = S$ it follows that for every $x, y \in A \subseteq S$, there is a $t \in S$ such that $x^t = y$. This contradicts $A \subseteq Z(S)$.

Therefore G contains only one involution, so that S is either a cyclic group or a generalized quaternion group.

Now, even if $O_{2'}(G)$ is not trivial, by Corollary 36, pg. 36 of [3], $S \simeq \mathbf{Z}_2$ or to the quaternion group Q_8 of order 8 and:

(a) If S is $\mathbf{Z}_2$, then $G = E_3 \mathbf{Z}_2$ where E_3 is an elementary abelian 3-group and $\mathbf{Z}_2$ inverts all elements of E_3.

(b) If $S = Q_8$, then G is one of the following groups:

(i) $E_3 Q_8$ where E_3 is a direct sum of copies of the 2-dimensional irreducible representation of Q_8 over the field F_3 of 3 elements.

(ii) the Markel group $(\mathbf{Z}_5 \times \mathbf{Z}_5) Q_8$.

It is easy to check now that among these groups only S_1, S_2, S_3 have the elements of the same order conjugate.

REFERENCES

[1] GORENSTEIN, D. : **Finite Groups**, Harper and Row, 1968.

[2] HUPPERT, B. and BLACKBURN, N. : **Finite Groups**, Vol. 2, Springer-Verlag, 1982.

[3] KLETZING, D. : **Structure and Representations of Q-Groups**, Lecture Notes in Mathematics, Springer-Verlag, 1984.