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AMBIVALENT GROUPS HAVING STRONGLY EMBEDDED SUBGRUOPS

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Summary : In this note we shall prove that in an ambivalent solvable group having a strongly embedded subgroup, a Sylow 2-subgroup is either a cyclic group or a generalized quaternion group.

KUVVETLİ YATIRILMIŞ ALT GRUPLARI HAİZ AMBİVALANT GRUPLAR

Özet : Bu çalışmada, kuvvetli yatırılmış bir alt grubu haiz bir ambivalant çözülebilir grupta bir 2-Sylow alt grubun ya bir devresel grup veya bir genelleştirilmiş kuaterniyon grubu olduğu ispat edilmektedir.

Definition. Let G be a finite group. A subgroup H of G is said to be strongly embedded in G if the following condition are satisfied:

(1) H is a proper subgroup of even order.

(2) For any element $x \in G-H$, the order of $H \cap H^*$ is odd (see [3], pg. 391).

Theorem 1 (Bender, see [3], pg. 391). Let G be a group having a strongly embedded subgroup H. Then, we have one of the following alternatives:

(1) Every Sylow 2-subgroup of G contains exactly one element of order 2. Thus a Sylow 2-subgroup of G is either a cyclic group or a generalized quaternion group.

(2) The group G possesses a normal series $G \rangle L \rangle M \rangle \{1\}$ such that both G/L and M are groups of odd order, and such that the factor group L/M is isomorphic to one of the simple groups PSL $(2, q)$, $Sz(q)$, or PS U $(3, q)$, where q is a power of 2.

In the first case (1), let t be any element of order two. Then, $C_G(t)$ is a proper subgroup of G, and any proper subgroup of G containing $C_G(t)$ is strongly embedded in G. In the second case (2), every strongly embedded subgroup H of G is of the form $H = N_G(S)\, O_{2'}(G)$ for some Sylow 2-subgroup S of G.

Theorem 2 (see [3], pg. 393). Let H be a strongly embedded subgroup of a group G. Let u be an element of $I(H)$, and let $C = C_G)(u)$. Then, the following poropositions hold:

(1) The set $I(G)=\{x \in G \mid x^2=1, x \neq 1\}$ is a conjugacy class of G. In other words, all involutions of G are conjugate.

(2) The set $I(H)$ is a conjugacy class of H. Furthermore, if $b = a^x$ for $a, b \in I(H)$ and $x \in G$, then we have $x \in H$.

Theorem 3 (see [1]). In the conditions of Theorem 1-part 2, a Sylow 2-subgroup of G is either homocyclic or a Suzuki 2-group (see [2]) and $G/O_{2'}(G)$ is a 2-normal group.

Definition. An ambivalent group is a finite group all whose characters are real valued.

Theorem 4. Let G be a solvable ambivalent group having a stronly embedded subgroup. Then:

i) A Sylow 2-subgroup of G is either $\mathbf{Z}_2$ or a generalized quaternion group.

ii) G is 2-nilpotent and $G = O_{2'}(G)$ where $S \in \mathrm{Syl}_2(G)$ inverts all elements of $O_{2'}(G)$.

Proof. To find the Sylow 2-subgroups of G we can suppose that $O_{2'}(G)$ is trivial.

Let $A = \langle I(G) \rangle$. Then A is a normal subgroup of G and because G is solvable, it contains an abelian minimal normal subgroup of G. Since $O_{2'}(G)$ is trivial it follows that $A = I(G) \cup \{1\}$ and A is abelian. Let H be a strongly embedded subgroup of G such that $A < H$. Let $S \in \mathrm{Syl}_2(G)$ such that $A \subseteq S \subseteq H$. Since H is a strongly embedded subgroup of G and A is a normal subgroup of G it follows that S is the single Sylow 2-group of G and S is normal in G.

Since the irreducible characters of G/S are in fact irreducible characters of G, it follows that G/S is also an ambivalent group. Since no odd order group is ambivalent it follows that $G = S$. Clearly $Z(S) \neq 1$. Since $I(S)$ must be a single conjugacy class, we have that $|I(S)|=1$, hence $G = S$ contains only one involution. Therefore S is either a cyclic group or a quaternion group. Since S is an ambivalent group it follows that S is isomorphic either to $\mathbf{Z}_2$ or to a generalized quaternion group. If $O_{2'}(G) \neq 1$, it follows that $G = O_{2'}(G)\, S$ (semidirect product) and S inverts all elements of $O_{2'}(G)$.

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