2'-R GROUPS WITH EVEN CHARACTERS

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Summary : In this note we shall study the structure of the groups all whose irreducible characters are rational valued on the 2-regular elements and all whose nonlinear irreducible characters have even degrees.

ÇİFT KARAKTERLİ 2'-R GRUPLAR

Özet : Bu çalışmada, 2-regüler elemanlar üzerinde bütün indirgenemez karakterleri rasyonel değerli olan ve lineer olmayan bütün indirgenemez karakterleri çift dereceli olan grupların yapısı incelenmektedir.

The notations and terminology are standard (see [2, 3]). All groups will be finite.

Definition. A 2'-r group is a finite group all whose characters are rational valued on the 2-regular elements.

Proposition 1. A group G is a 2'-r group if and only if $N_G(\langle x\rangle)/C_G(x) \simeq \simeq \mathrm{Aut}(\langle x\rangle)$ for every 2-regular $x \in G$ (see [1]).

Theorem 2. Let G be a 2'-r gruop such that $2 \mid \chi(1)$ for every nonlinear $\chi \in Irr(G)$. Let $S \in Syl_2(G)$. Then G is a 2-nilpotent group, S is fusion free in G and $N_G(S) = S$.

Proof. Let $\chi \in Irr(G)$. Define $det\ \chi : G \longrightarrow \mathbf{C}$ as flollows: Choose a representasion X affording χ and set $(det\ \chi)(x) = det(X(x))$. Then $det\ \chi$ is a linear character of G uniquely determined by χ. The order of $det\ \chi$ as an element of the group of linear characters of G, denoted by $o(\chi) = |G : ker(det)|$ is the determinantal order of χ (see [2]).

Let $O^2(G)$ be the unique minimal subgroup of G such that $G/O^2(G)$ is a 2-group. The group $G/O^2(G)$ is the 2-residual of G. Let $O^{2'}(G)$ be the unique minimal subgroup of G such that $|G/O^{2'}(G)|$ is odd.

Let $Irr_{2'}(G) = \{\chi \in Irr(G) \mid \chi(1)$ and $o(\chi)$ are odd$\}$ and $s(G) = \Sigma_{\chi \in Irr_{2'}(G)} \chi(1)^2$. Let $K = O^2(G)$. We shall prove that K has odd order.

Clearly $O^2(K) = K$, hence $Irr(K) = Irr_{2'}(K) \cup \{\tau \in Irr(K) \mid 2/\tau(1)\}$. Then $|K| = \Sigma_{\chi \in Irr(K)} \chi(1)^2 \equiv s(K) \mod 2$. By Proposition 1, $O^{2'}(G) = G$. Thus all $\chi \in Irr_{2'}(G)$ are linear and $\ker \chi \supseteq O^{2'}(G) = G$. Therefore $|Irr_{2'}(G)| = 1 = s(G)$.

G acts on $Irr_{2'}(K)$ by $\chi^g(x) = \chi(g x g^{-1})$. Since $K = O^2(G)$ acts trivially on $Irr_{2'}(K)$ and G/K is a 2-group, the resulting orbits are of 2-power size. Let I_0 be the G-invariant characters of $Irr_{2'}(K)$. Then, $s(K) = \Sigma_{\chi \in I_0} \chi(1)^2 \mod 2$. Let $\chi \in I_0$. Since $(|G : K, o(\chi) \times \chi(1)) = 1$, χ is extendible to G, hence $|Irr_{2'}(G)| = |I_0| = 1$. It follows that $s(K) \equiv s(G) \mod 2 \equiv |K|$ and hence $K = O^2(G)$ has odd order. Thus $O^2(G)$ is a normal 2-complement.

By Feit-Thompson Theorem (see [4]) $O^2(G)$ is solvable, hence G is solvable and by Gow's Theorem [1] $|G| = 2^a 3^b 5^c$.

Clearly $G/O^2(G) \simeq S$, thus S is also a Q-group. By Wielandt fusion theorem (see [4], p. 258) S is embedded without fusion in G.

We shall prove now by induction on the order of G that every odd order element of $N_G(S)$ is nonreal. Thus $N_G(S) = S$.

When $|G| = 1$ the statement is trivial. Let T be a minimal normal subgroup of G. Since G is solvable, T is an elementary abelian p-group, for a prime p. Let h be an odd order element of $N_G(S)$. If T does not contain h, the image of h in G/T is nonreal by induction, so h is nonreal. If T contains h, since T is elementary abelian p-group, then $[h, S] \subseteq T \cap S = 1$. It follows that $C_G(h)$ contains a Sylow 2-subgroup of G, hence the order of $N_G(\langle h \rangle)/C_G(h)$ is odd and h is nonreal.

REFERENCES

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