ON THE STABILITY AND THE BOUNDEDNESS PROPERTIES OF SOLUTIONS OF CERTAIN FOURTH ORDER DIFFERENTIAL EQUATIONS

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Summary: The main purpose of this paper is to study the asymptotic stability in the large of the zero solution for Eq. (1.1) with \( p \equiv 0 \) and the boundedness of solutions for Eq. (1.1) with \( p \neq 0 \).

4. MERTEBEDEN BELIRLI DİFERANSİYEL DENKLEMLERİN STABİLİTE VE SINIRLILIK ÖZELLİKLERİ HAKKINDA

Özet: Bu çalışmanın ana amacı, \( p \equiv 0 \) halinde (1.1) denklemünün sıfır çözümünün asimtotik stabilitesini ve \( p \neq 0 \) halinde (1.1) çözüm lerinin sınırlılığını incelemektir.

1. Introduction and statement of the results

We consider the equation

\[
x^{(4)} + \phi (x, \ddot{x}, \dot{x}, x) \ddot{x} + f(x, \ddot{x}) + g(x, \dot{x}) + h(x) = p(t, x, \ddot{x}, \dot{x}, x) \quad (1.1)
\]

in which the functions \( \phi, f, g, h \) and \( p \) depend at most on the arguments shown explicitly and the dots denote differentiation with respect to \( t \). Further, it will be assumed that the functions \( \phi, f, g, h \) and \( p \) are continuous for all values of their respective arguments and that the derivatives

\[
\frac{\partial}{\partial x} \phi (x, y, z, u), \frac{\partial}{\partial y} \phi (x, y, z, u), \frac{\partial}{\partial u} \phi (x, y, z, u), \frac{\partial}{\partial y} f(y, z), \frac{\partial}{\partial x} g(x, y),
\]

\[
\frac{\partial}{\partial y} g(x, y) \text{ and } h'(x) \text{ exist and are continuous for all } x, y, z \text{ and } u.
\]

All functions and solutions are supposed to be real. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed.

Key words: Nonlinear differential equations of the fourth order, \( V \)-function, Stability, Boundedness.

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It will be convenient in what follows to use the equivalent system:

\[ \begin{align*}
    \dot{x} &= y, \quad \dot{y} = z, \quad \dot{z} = u, \\
    \dot{u} &= -\phi(x, y, z, u) - f(y, z) - g(x, y) - h(x) + p(t, x, y, z, u),
\end{align*} \]

which is obtained from (1.1) by setting \( y = \dot{x}, \quad z = \ddot{x} \) and \( u = \dddot{x} \).

The boundedness and stability properties of solutions for various equations of the fourth order differential equations have been considered by many authors. Many of these results are summarized in [12].

Ezeilo [4] investigated the stability and boundedness of the solutions of the equation

\[ x^{(4)} + f(\dddot{x}) \dddot{x} + \alpha_2 \dddot{x} + g(\dot{x}) + \alpha_4 x = p(t). \]

Harrow ([6], [7], [8]) studied the problem for the simple variant of (1.1) given by

\[ x^{(4)} + ax + f(\dot{x}) + g(\dot{x}) + h(x) = p(t). \]

In [9], Lalli and Skrapek obtained a similar result for the equation

\[ x^{(4)} + f(\dot{x}) \dddot{x} + f_1(\dot{x}, \ddot{x}) + g(\dot{x}) + h(x) = p(t). \]

Abou-El-Ela [1] investigated the boundedness of the solutions of the equation

\[ x^{(4)} + f(\dot{x}, x) \dddot{x} + \alpha_2 \dddot{x} + g(\dot{x}) + \alpha_4 x = p(t). \]

Also recently, in [3], Bereketoglu dealt with the equation of the form

\[ x^{(4)} + f_1(\dot{x}, \dddot{x}, x) + f_2(\dot{x}, \dddot{x}) + g(\dot{x}) + h(x) = p(t). \]  \hspace{1cm} (1.3)

He presented sufficient conditions for the asymptotic stability in the large of the trivial solution of (1.3) with \( p(t) = 0 \) and the boundedness of solutions of (1.3) with \( p(t) \neq 0 \).

In the case \( p(t, x, y, z, u) = 0 \) we have

**Theorem 1.** Suppose the following conditions are satisfied:

(i) \( f(y, 0) = g(x, 0) = h(0) = 0. \)

(ii) There are positive constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \Delta_0 \) such that

\[ \alpha_1 \alpha_2 \alpha_3 - \alpha_3 \frac{g(x, y)}{y} - \alpha_1 \alpha_4 \varphi(x, y, z, 0) \geq \Delta_0 \text{ for all } x, z \text{ and } y \neq 0. \]

(iii) \( \varphi(x, y, z, u) \geq \alpha_1 > 0 \text{ for all } x, y, z \text{ and } u, \)

\[ \frac{f(y, z)}{z} \geq \alpha_2 \text{ for all } y, z \neq 0, \]
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\[ \frac{g(x, y)}{y} \geq a_1 \text{ for all } x, y \neq 0, \]
\[ \frac{h(x)}{x} \geq \beta \text{ for all } x \neq 0, \text{ where } \beta \text{ is a positive constant.} \]

(iv) \( \left( a_4 - \frac{a_1 \Delta_0}{4a_3} \right) < h'(x) \leq a_4 \text{ for all } x. \)

(v) \( \left( \frac{\partial}{\partial y} g(x, y) - \frac{g(x, y)}{y} \right) \leq \delta_1 \text{ for all } x, y \neq 0, \text{ where } \delta_1 \text{ is a positive constant satisfying } \delta_1 < \frac{2a_4 \Delta_0}{a_1 a_3^2}. \)

(vi) \( \left( \frac{1}{z} \right) \int_0^z \varphi(x, y, s, 0) \, ds - \varphi(x, y, z, 0) \leq \delta_2 \text{ for all } x, y \text{ and } z \neq 0, \text{ where } \delta_2 \text{ is a positive constant such that } \delta_2 < \frac{2\Delta_0}{a_1 a_3}. \)

(vii) \( \frac{\partial}{\partial y} f(y, z) \leq 0, y \frac{\partial}{\partial x} \varphi(x, y, z, 0) \leq 0, z \frac{\partial}{\partial x} \varphi(x, y, z, 0) \leq 0, \)
\[ y \frac{\partial}{\partial y} \varphi(x, y, z, 0) \leq 0 \text{ and } z \frac{\partial}{\partial y} \varphi(x, y, z, 0) \leq 0 \text{ for all } x, y \text{ and } z. \)

(viii) \( \frac{f(y, z)}{z} - \alpha_2 \leq \frac{\epsilon_0 a_3^3}{a_4^2} \text{ for all } y, z \neq 0, \text{ where } \epsilon_0 \text{ is a positive constant such that } \)
\[ \epsilon_0 < \epsilon \leq \min \left[ \frac{1}{a_1}, \frac{a_4}{a_3}, \frac{\Delta_0}{4a_1 a_3 D_0}, \frac{a_1}{4a_4 D_0} \left( \frac{2a_4 \Delta_0}{a_1 a_3^2} - \delta_1 \right), \right. \]
\[ \frac{a_1}{4D_0} \left( \frac{2\Delta_0}{a_1^2 a_3} - \delta_2 \right) \]  
(1.4)
\[ \text{with } D_0 = a_1 a_2 + \frac{a_2 a_3}{a_4}. \]

(ix) \( \left( \frac{\partial}{\partial x} g(x, y) \right)^2 \leq \frac{a_1 \Delta_0 (\epsilon - \epsilon_0)}{16} \text{ for all } x \text{ and } y, \)
\[ \text{and } \frac{1}{y} \int_0^y \frac{\partial}{\partial x} g(x, s) \, ds \leq \frac{\sigma_3 (\epsilon - \epsilon_0)}{4} \text{ for all } x, y \neq 0. \]
Then every solution of (1.1) satisfies
\[ x(t) \to 0, \dot{x}(t) \to 0, \ddot{x}(t) \to 0, \dot{\ddot{x}}(t) \to 0 \text{ as } t \to \infty. \] (1.6)

**Remark 1.** When \( \varphi(x, x, x, x) = a_3 x \), \( g(x, x) = a_2 x \) and finally \( p(t, x, x, x, x) = p(t) \), then conditions of Theorem 1 and Theorem 2 are reduced to those of Bereketoglu [3]. When \( \varphi(x, x, x, x) \) and \( g(x, x) \) depend only on \( x \), \( \dot{x} \), respectively, and \( f(x, \dot{x}) = a_2 x, \dot{\dot{x}}(x) = a_4 x \) and
\[ p(t, x, x, x, x) = p(t), \]
then conditions of Theorem 1 and Theorem 2 are reduced completely to those of Ezeilo [4]. Moreover, conditions of Theorem 1 and Theorem 2 reduce to the conditions of the relevant theorems by Lalli and Skrakep [9] and Harrow [6],
up to very small differences. These differences are due to the fact that the
Lyapunov function is not identical.

2. The Function $V(x, y, z, u)$

The main tool, in the proof of the theorems, is the function $V = V(x, y, z, u)$
defined by:

$$
2V = 2d_2 \int_0^x h(s) \, ds + [d_z \alpha_z - d_l \alpha_l] \phi^2 + 2 \int_0^x g(x, s) \, ds + 2 \int_0^x [d_1 f(y, s) - d_2 s] \, ds + \\
+ 2 \int_0^x \varphi(x, y, s, 0) \, ds + 2 d_2 z \int_0^x \varphi(x, y, s, 0) \, ds + d_1 u^2 + 2y h(x) + \\
+ 2d_4 z h(x) + 2d_4 z g(x, y) + 2d_2 y u + 2zu,
$$

where

$$
d_1 = \frac{1}{\alpha_1} + \varepsilon,
$$

$d_2$ being the constant defined by (1.5).

First discuss some important inequalities.

Let $\Phi_1$ be the function defined by

$$
\Phi_1(x, y, z, 0) = \begin{cases}
\left(\frac{1}{z}\right) \int_0^x \phi(x, y, s, 0) \, ds, z \neq 0 \\
\phi(x, y, 0, 0), z = 0.
\end{cases}
$$

Using (iii) and (vi) we obtain

$$
\Phi_1(x, y, z, 0) \geq \alpha_1 > 0 \text{ for all } x, y \text{ and } z, \quad (2.4)
$$

Further we define

$$
\Phi_3(x, y) = \begin{cases}
g(x, y), y \neq 0 \\
\frac{\partial}{\partial y} g(x, 0), y = 0.
\end{cases}
$$

We have from (iii) and (v)

$$
\Phi_3(x, y) \geq \alpha_3 \text{ for all } x \text{ and } y, \quad (2.7)
$$

$$
\frac{\partial}{\partial y} g(x, y) - \Phi_3(x, y) \leq \delta_1 \text{ for all } x \text{ and } y. \quad (2.8)
$$
From (2.2) and (1.5) we have
\[
\alpha_2 - d_1 \frac{g(x, y)}{y} - d_2 \varphi(x, y, z, 0) = \]
\[
= \left( \frac{1}{\alpha_1 \alpha_3} \right) \left[ a_1 a_2 a_3 - \alpha_3 \frac{g(x, y)}{y} - a_4 \varphi(x, y, z, 0) \right] - \varepsilon \left[ \frac{g(x, y)}{y} + \varphi(x, y, z, 0) \right].
\]
But also (ii) and (iii) imply that
\[
\frac{g(x, y)}{y} < a_1 a_2 \varphi(x, y, z, 0) < \frac{\alpha_2 \alpha_3}{\alpha_4}.
\]
Thus it follows that
\[
\alpha_2 - d_1 \frac{g(x, y)}{y} - d_2 \varphi(x, y, z, 0) > \left( \frac{\alpha_0}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) \quad \text{for all } x, z \text{ and } y \neq 0, \quad (2.9)
\]
by using (ii) and (viii).

Since \( \Phi_1(x, y, z, 0) = \varphi(x, y, \bar{z}, 0), \bar{z} = \theta z, 0 \leq \theta \leq 1, \) then
\[
\alpha_3 - d_1 \frac{g(x, y)}{y} - d_2 \Phi_1(x, y, z, 0) \geq \frac{\alpha_0}{\alpha_1 \alpha_3} - \varepsilon D_0. \quad (2.10)
\]

The following two lemmas are to prove that the function \( V(x, y, z, u) \) is a Lyapunov function of the system (1.2).

**Lemma 1.** Suppose that the conditions of Theorem 1 hold. Then there is a positive constant \( D_1 \) such that
\[
V \geq D_1 [x^2 + y^2 + z^2 + u^2] \quad (2.11)
\]
for all \( x, y, z \) and \( u \).

**Proof.** \( V(0, 0, 0, 0) = 0, \) since \( f(0, 0) = g(0, 0) = h(0) = 0. \) Rewrite the function \( 2V(x, y, z, u) \) as follows:
\[
2V(x, y, z, u) = \frac{1}{\Phi_1(x, y, z, 0)} [u + z \Phi_1(x, y, z, 0) + d_2 y \Phi_1(x, y, z, 0)]^2 + \\
+ \frac{1}{\Phi_3(x, y)} [h(x) + y \Phi_3(x, y) + d_1 z \Phi_3(x, y)]^2 + V_1 + V_2 + V_3 + V_4, \quad (2.12)
\]
where
\[
V_1 = [d_2 \alpha_2 - d_1 \alpha_4 - d_2^2 \Phi_1(x, y, z, 0)] y^2 + 2 \int_0^y g(x, s) ds - y^2 \Phi_3(x, y),
\]
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\[ V_2 = 2 d_1 \int_0^z \left[ f(y, s) - a_2 s \right] ds + [d_1 a_2 - d_2 - d_2^2 \Phi_3(x, y)] z^2 + \\
+ 2 \int_0^z s \phi(x, y, s, 0) ds - z^2 \Phi_1(x, y, z, 0), \]

\[ V_3 = 2d_2 \int_0^x h(s) ds - \frac{1}{\Phi_4(x, y)} \left[ \frac{h(x)}{x} \right]^2 x^2, \]

\[ V_4 = \left[ d_1 - \frac{1}{\Phi_4(x, y, z, 0)} \right] u^2. \]

From (1.5), (2.2), (iii) and (2.10) we obtain

\[ d_2 a_2 - d_1 a_4 - d_2^2 \Phi_1(x, y, z, 0) > \frac{a_4}{a_3} \left( \frac{\Delta_0}{a_1 a_3} - \varepsilon D_0 \right). \]

Since \( yg(x, y) = \int_0^y g(x, \eta) d\eta + \int_0^y \eta g(x, \eta) d\eta, \) then

\[ 2 \int_0^y g(x, \eta) d\eta - y^2 \Phi_3(x, y) \geq \left( - \frac{\delta_1}{2} \right) y^2, \text{ by (2.8)}. \]

Therefore we get

\[ V_1 \geq \frac{1}{2} \left[ \frac{2a_4 \Delta_0}{a_1 a_3^2} - \frac{2a_4 \Delta_0}{a_3} \varepsilon - \delta_1 \right] y^2 > \frac{1}{4} \left[ \frac{2a_4 \Delta_0}{a_1 a_3^2} - \delta_1 \right] y^2, \text{ by (1.4)}. \]

By similar estimation, using condition (iii), (1.5), (2.2) and (2.9) we get

\[ d_1 a_2 - d_2 - d_2^2 \Phi_1(x, y) = \\
d_1 [a_2 - d_1 \Phi_1(x, y) - d_2 \phi(x, y, z, 0)] + d_2 [d_4 \phi(x, y, z, 0) - 1] > \\
d_1 [a_2 - d_1 \Phi_3(x, y) - d_2 \phi(x, y, z, 0)] \left( \frac{1}{a_1} \right) \left[ \frac{\Delta_0}{a_1 a_3} - \varepsilon D_0 \right]. \] (2.13)

From the identity

\[ \int_0^z s \phi(x, y, s, 0) ds = z \int_0^z \phi(x, y, s, 0) ds - \int_0^z s \Phi_1(x, y, s, 0) ds \]

we get
\[
2 \int_0^z \varphi(x, y, s, 0) \, ds - z^3 \Phi_1(x, y, z, 0) = \left[ \int_0^z \left( \varphi(x, y, s, 0) - \Phi_1(x, y, s, 0) \right) \right] s \, ds \geq - \left( \frac{\delta_2}{2} \right) z^3, \text{ by (2.5).}
\]

Also from (iii) we obtain
\[
\int_0^z \left[ \frac{f(y, s)}{s} - a_2 \right] s \, ds \geq 0.
\]

Therefore
\[
V_2 \geq \left\{ \frac{1}{a_1} \left( \frac{\Delta_0}{a_1 a_3} - \epsilon D_0 \right) - \frac{\delta_2}{2} \right\} z^3 \geq \frac{1}{4} \left( \frac{2\Lambda_0}{a_1^2 a_3} - \delta_2 \right) z^3, \text{ by (1.4).}
\]

For the component \( V_3 \), from (i), (iii), (iv) and (1.5) it follows that
\[
V_3 \geq 2 \left(\epsilon + a_4 a_3^{-2} \right) \int_0^x h(s) \, ds - \frac{1}{a_3} \left[ \frac{h(x)}{x} \right]^2 x^2 \geq (\epsilon \beta) x^2 +
\]
\[
+ 2 \int_0^x \frac{h(s)}{s} \left[ \frac{a_4}{a_3} - \frac{1}{a_3} H'(s) \right] s \, ds \geq (\epsilon \beta) x^2.
\]

By using (2.2) and (2.4) we obtain \( V_4 \geq \epsilon u^2 \).

Combining the estimates for \( V_1, V_2, V_3 \) and \( V_4 \) with (2.12) we have
\[
2V \geq (\epsilon \beta) x^2 + \frac{1}{4} \left[ \frac{2a_4 \Delta_0}{a_1 a_2} - \delta_1 \right] y^2 + \frac{1}{4} \left( \frac{2\Lambda_0}{a_1^2 a_3} - \delta_2 \right) z^2 + \epsilon u^2,
\]
noting that all the four coefficients of the above expression are nonnegative. Then there exists a positive constant \( D_1 \) such that
\[
V \geq D_1 [x^2 + y^2 + z^2 + u^2].
\]

Thus the proof is now complete.

**Lemma 2.** Suppose that the conditions of Theorem 1 hold. Then there is a positive constant \( D_2 \) such that whenever \( (x, y, z, u) \) is any solution of (1.2) with \( p(t, x, y, z, u) \equiv 0 \), then
\[
\dot{V} = \frac{d}{dt} V(x, y, z, u) \leq - D_2 (y^2 + z^2 + u^2). \tag{2.14}
\]
Proof. A straightforward calculation using the identity
\[
\frac{d}{dt} V = \frac{\partial V}{\partial u} \dot{u} + \frac{\partial V}{\partial z} \dot{u} + \frac{\partial V}{\partial y} \dot{z} + \frac{\partial V}{\partial x} \dot{y}
\]
yields
\[
\dot{V} = -d_1 u^2 \varphi (x, y, z, u) - d_2 y f(y, z) - d_2 y g(x, y) - z f(y, z) + u^2 +
\]
\[+ d_1 z \int_0^z \frac{\partial}{\partial y} f(y, s) ds + d_2 y^2 \int_0^z \frac{\partial}{\partial x} \varphi(x, y, s, 0) ds + z \int_0^z \frac{\partial}{\partial y} \varphi(x, y, s, 0) ds +
\]
\[+ d_2 yz \int_0^z \frac{\partial}{\partial y} \varphi(x, y, s, 0) ds + d_2 z \int_0^z \varphi(x, y, s, 0) ds [d_2 \alpha_2 - d_1 \alpha_4] yz +
\]
\[+ d_1 yz \frac{\partial}{\partial x} g(x, y) + d_1 z^2 \frac{\partial}{\partial y} g(x, y) + y \int_0^z \frac{\partial}{\partial x} g(x, s) ds + y^2 h'(x) +
\]
\[+ d_1 yzh'(x) - [\varphi(x, y, z, u) - \varphi(x, y, z, 0)] z u - d_2 [\varphi(x, y, z, u) - \varphi(x, y, z, 0)] yu +
\]
\[+ y \int_0^z \frac{\partial}{\partial x} \varphi(x, y, s, 0) ds.
\]
Since
\[
z \int_0^z \frac{\partial}{\partial y} f(y, s) ds \leq 0, y \int_0^z \frac{\partial}{\partial x} \varphi(x, y, s, 0) ds \leq 0, z \int_0^z \frac{\partial}{\partial y} \varphi(x, y, s, 0) ds \leq 0,
\]
\[
z \int_0^z y \frac{\partial}{\partial y} \varphi(x, y, s, 0) ds \leq 0 \text{ and } \int_0^z \frac{\partial}{\partial x} \varphi(x, y, s, 0) ds \leq 0, \text{ by (vii),}
\]
then we obtain
\[
\dot{V} \leq - \left[ \alpha_2 - d_1 \frac{\partial}{\partial y} g(x, y) - d_2 \Phi_1 (x, y, z, 0) \right] z^2 -
\]
\[+ \left[ d_1 \varphi(x, y, z, u) - 1 \right] u^2 - V_5 - V_6 - V_7 - V_8,
\]
where

\[
(2.15)
\]
\[ V_5 = f(y, z) z + d_2 f(y, z) y - \alpha_2 x^2 - \alpha_2 d_2 y z , \]
\[ V_6 = [\varphi(x, y, z, u) - \varphi(x, y, z, 0)] z u + d_2 [\varphi(x, y, z, u) - \varphi(x, y, z, 0)] y u , \]
\[ V_7 = \left[ d_2 \frac{g(x, y)}{y} - \alpha_4 \right] y^2 - d_1 y z \frac{\partial}{\partial x} g(x, y) - y \int_0^y \frac{\partial}{\partial x} g(x, s) ds , \quad (2.16) \]
\[ V_8 = (\alpha_4 - h'(x)) y^2 + d_4 [\alpha_4 - h'(x)] y z . \]

By the same way as in (2.13), it follows that
\[ \alpha_4 - d_1 \frac{\partial}{\partial y} g(x, y) - d_2 \Phi_1(x, y, z, 0) \geq \left( \frac{\Delta_0}{\alpha_1 \alpha_3} - \varepsilon D_0 \right) \geq \frac{3\Delta_0}{4\alpha_1 \alpha_3} , \quad \text{by (1.4).} \quad (2.17) \]

By using (iii) and (2.2) we find
\[ [d_1 \varphi(x, y, z, u) - 1] \geq \varepsilon \alpha_1 . \quad (2.18) \]

The function \( V_3 \) is the same as in [3]. The estimates for \( V_3 \) as in [3] give that
\[ V_5 \geq - (\varepsilon \alpha_3) y^2 . \quad (2.19) \]

Also, from (x) we obtain for \( u \neq 0 \)
\[ V_6 = [\varepsilon \varphi_u(x, y, z, \theta u) + d_2 \varphi_u(x, y, z, \theta u)] u^2 \geq 0, \quad 0 \leq \theta \leq 1 \]
but \( V_6 = 0 \) when \( u = 0 \). Hence
\[ V_6 \geq 0 \quad \text{for all } x, y, z \text{ and } u. \quad (2.20) \]

Combining (2.16) and (2.19) we obtain
\[
V_5 + V_7 \geq - (\varepsilon \alpha_3) y^2 + \left[ d_2 \frac{g(x, y)}{y} - \alpha_4 \right] y^2 - d_1 y z \frac{\partial}{\partial x} g(x, y) - y \int_0^y \frac{\partial}{\partial x} g(x, s) ds
\]
\[
\geq (\varepsilon - \varepsilon_0) \alpha_3 y^2 - d_1 y z \frac{\partial}{\partial x} g(x, y) - y \int_0^y \frac{\partial}{\partial x} g(x, s) ds
\]
\[
\geq (\varepsilon - \varepsilon_0) \alpha_3 y^2 - d_1 y z \frac{\partial}{\partial x} g(x, y) - \left[ \frac{1}{y} \int_0^y \frac{\partial}{\partial x} g(x, s) ds \right] y^2
\]
\[
\geq \frac{3}{4} (\varepsilon - \varepsilon_0) \alpha_3 y^2 - d_1 y z \frac{\partial}{\partial x} g(x, y)
\]
\[
= \frac{1}{2} (\varepsilon - \varepsilon_0) \alpha_3 y^2 + \frac{1}{4} (\varepsilon - \varepsilon_0) \alpha_3 \left[ y^2 - \frac{4d_1}{(\varepsilon - \varepsilon_0) \alpha_3} y z \frac{\partial}{\partial x} g(x, y) \right]
\]
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\[ \frac{1}{2} (\varepsilon - \varepsilon_0) a_3 y^2 - \frac{d_j^2}{(\varepsilon - \varepsilon_0) a_3} \left[ \frac{\partial}{\partial x} g(x, y) \right]^2 z^2 \]

\[ \geq \frac{1}{2} (\varepsilon - \varepsilon_0) a_3 y^2 - \frac{\Lambda_0}{4a_1 a_3} z^2, \]  

(2.21)

by using (ii), (1.5), (ix), (2.2) and (1.4).

Now

\[ V_s = (a_4 - h'(x)) (y^2 + d_i yz) \geq - (a_4 - h'(x)) \frac{d_j^2}{4} z^2 \]

\[ > - \frac{\alpha_1 \Lambda_0}{16 a_3} \left( \frac{1}{\alpha_1} + \varepsilon \right)^2 z^2 > - \frac{\Lambda_0}{4a_1 a_3} z^2, \]  

(2.22)

by using (iv), (2.2) and (1.4).

On gathering the estimates (2.17)-(2.22) into (2.15) we deduce that

\[ \dot{V} \leq - \left( \frac{\Lambda_0}{4a_1 a_3} \right) z^2 - \frac{1}{2} (\varepsilon - \varepsilon_0) a_3 y^2 - (\varepsilon a_1) u^2 \leq - D_2 (y^2 + z^2 + u^2), \]

where \( D_2 = \min \left\{ \frac{\alpha_1 \Lambda_0}{4a_1 a_3}, \frac{1}{2} (\varepsilon - \varepsilon_0) a_3, \varepsilon a_1 \right\} \).

3. Proof of Theorem 1

By Lemma 1

\[ V(x, y, z, u) = 0, \text{ at } x^2 + y^2 + z^2 + u^2 = 0, \]

\[ V(x, y, z, u) > 0, \text{ if } x^2 + y^2 + z^2 + u^2 \neq 0 \]

\[ V(x, y, z, u) \to 0, \text{ as } x^2 + y^2 + z^2 + u^2 \to 0. \]

Also, let \( (x(t), y(t), z(t), u(t)) \) be any solution of (1.2) with \( p(t, x, y, z, u) = 0 \), such that \( x(0) = x_0, y(0) = y_0, z(0) = z_0, u(0) = u_0 \). Consider the function \( V(t) = V(x(t), y(t), z(t), u(t)) \) corresponding to this solution. By Lemma 2, we have

\[ V(t) \leq V(0) \text{ for } t \geq 0. \]

Thus, the remainder of the proof of Theorem 1 is the same as the one given by Ezeilo [4] and hence is omitted.

4. Proof of Theorem 2

The proof here is based essentially on the method devised by Antosiewicz [2]. Let \( (x(t), y(t), z(t), u(t)) \) be the solution of (1.2) satisfying the initial
conditions (1.8) and consider the function $V(t) = V(x(t), y(t), z(t), u(t))$, where $V(x, y, z, u)$ is the function $V$ used in the proof of Theorem 1. Using this function, we have that, for the system (1.2),

$$
\dot{V} \leq - D_2 (x^2 + z^2 + u^2) + (d_2 y + z + d_1 u) p(t, x, y, z, u),
$$

so that

$$
\dot{V} \leq D_3 \left( |y| + |z| + |u| \right) p(t, x, y, z, u),
$$

where $D_3 = \max \{d_2, 1, d_1\}$.

It follows from (1.7) and the obvious inequalities

$$
|y| \leq 1 + y^2, \quad |z| \leq 1 + z^2, \quad |u| \leq 1 + u^2, \quad 2 |yz| \leq y^2 + z^2, \quad 2 |yu| \leq y^2 + u^2, \quad 2 |zu| \leq z^2 + u^2,
$$

that

$$
\dot{V} \leq D_3 [3 + 4 (y^2 + z^2 + u^2)] q(t).
$$

By (2.11) we have

$$
V \geq D_4 [y^2 + z^2 + u^2].
$$

Putting $D_4 = 3D_3, D_5 = \frac{4D_3}{D_1}$ we obtain

$$
\dot{V} - D_5 q(t) V \leq D_4 q(t).
$$

Therefore we obtain the result

$$
V(t) \leq \frac{1}{x(t)} \left( V(0) + D_4 \int_0^t q(s) x(s) \, ds \right),
$$

where $x(t) = \exp \left( - D_5 \int_0^t q(s) \, ds \right)$. Since $x(t) \leq 1$ for $t \geq 0$,

$$
V(t) \leq (V(0) + D_4 A) e^{D_5 A},
$$

where $V(0) = V(x(0), y(0), z(0), u(0))$. The proof of Theorem 2 is complete.
ON THE STABILITY AND THE BOUNDEDNESS ...

REFERENCES


