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ON A POWER SERIES SOLUTION OF A SPECIAL TYPE SINGULAR CAUCHY PROBLEM

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ABSTRACT

In this paper, the singular Cauchy problem for Euler Poisson Darboux equation has been extended to a generalized Euler Poisson Darboux equation in which the real parameter k is replaced by a function as follows:

$$\Delta u = u_{tt} + (at^2 + \frac{b}{t})u_t \quad (a > 0, b > -1, t > 0)$$

$$u(x_1, x_2, \ldots, x_n; 0) = f(x_1, x_2, \ldots, x_n), u_t(x_1, x_2, \ldots, x_n; 0) = 0.$$

The solution of this singular Cauchy problem is given by an absolutely and uniformly convergent power series.

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1. INTRODUCTION

Let $x = (x_1, x_2, ..., x_n)$ be a point in R^n, k a real parameter and t the time variable. Δ is the Laplace operator in R^n and $f(x)$ is an initial function which is infinitely differentiable. Singular Cauchy problem for the Euler-Poisson-Darboux (abbreviated EPD) Equation is known

$$\Delta u = u_{tt} + \frac{k}{t}u_t \qquad (t > 0) \tag{1}$$

$$u(x_1, x_2, \ldots, x_n; 0) = f(x_1, x_2, \ldots, x_n), u_t(x_1, x_2, \ldots, x_n; 0) = 0. \tag{2}$$

Equation (1) for special values of k and n occurs in many important and classical problems since the time of Euler [8]. It is proved that the Cauchy problem (abbreviated

CP) (1),(2) does not have a unique solution when $k < 0$ and $(1-k)$th partial derivatives of the solutions with respect to t has logarithmic singularities when $k = -(2n+1)$, $n \in N$. It was also proved that these solutions are depended on the parameter k. A unified solution of (1), (2) for all real values of k was given by Weinstein [11], Diaz and Weinberger [7] and Blum [2]. For analytical initial function $f(x)$, (1),(2) singular CP was solved by Walter [9] who has given the solution in term of an absolutely and uniformly convergent power series. Special methods were utilized in solving various different cases. All of these solutions were given by quite very complicated formulas. Dernek [6] has used finite transformation method to solve the non homogenous singular EPD equation. The solution has ben obtained in a much simpler manner then by other methods mentioned above. Another initial value problem for EPD equation is the regular CP. Regular CP was solved by Davis [4]. Copson [3] gave an alternative solution of this problem in any space of even number of dimensions. A solution of the series form was given by Asral [1] for regular CP. The real parameter k is replaced by a function in the CP (1),(2) $k = \psi(t)$, where $\psi(t)$ is a regular function on the neighborhood of $t = 0$ or in all R-space. This is a generalization of CP (1),(2). Thus we obtain the following CP:

$$\Delta u = u_{tt} + \frac{\psi(t)}{t} u_t$$

$$u(x_1, x_2, \ldots, x_n; 0) = f(x_1, x_2, \ldots, x_n), u_t(x_1, x_2, \ldots, x_n; 0) = 0.$$

The main reason for doing this generalization of EPD equation is due to a work of Weinberger and Protter [9]. They have given a solution of an initial boundary value problem for equation (1) where they have chosen $n = 1, \psi(t) = 2t^2$. A solution of the series form for $\psi(t) = at^2$ is given by Dernek in [5]. In this paper we shall consider another generalization of singular CP (1),(2) as follows:

$$\Delta u = u_{tt} + (at^2 + \frac{b}{t}) u_t \quad (t > 0) \tag{3}$$

$$u(x_1, x_2, \ldots, x_n; 0, a, b) = f(x_1, x_2, \ldots, x_n), u_t(x_1, x_2, \ldots, x_n; 0, a, b) = 0 \tag{4}$$

where the initial function $f(x)$ is infinitely differentiable and the sequence $(|\Delta^n f|)$ is majorized by a suitably chosen sequence which has positive terms. We shall give a solution which is an absolutely and uniformly convergent power series for $a > 0, b > -1$.

Let us seek a special solution of the CP (3),(4) in the form

$$u(x,t,a,b) = \sum_{n=0}^{\infty} u_n(t,a,b)\Delta^n f(x) \tag{5}$$

where $u_0(t,a,b) = 1$ and $\Delta^0 f = f, \Delta^n f = \Delta(\Delta^{n-1} f)$ $(n = 1,2,\ldots)$. We can consider (5) as a power series with respect to Δf. Let us consider (5) as a formal solution of (3),(4). Substituting $u(x,t,a,b)$ and its derivatives with respect to t into (3), we obtain the following recurrence relations which are ordinary differential equations:

$$\frac{d^2}{dt^2}u_n(t,a,b) + (at^2 + \frac{b}{t})\frac{d}{dt}u_n(t,a,b) = u_{n-1}(t,a,b) \tag{6}$$

The function $u_n(t,a,b)$ satisfies the following initial conditions:

$$u_n(0,a,b) = 0, \frac{d}{dt}u_n(0,a,b) = 0 \quad (n \in N). \tag{7}$$

The Cauchy Problem (6),(7) is now $CP(n)$. We can solve the ordinary differential equations (6) with the Mathematical Induction Principle.

2. SOLUTION OF THE CP(6),(7)

Theorem 1. The $CP(n)$ has a solution of the form:

$$u_n(t,a,b) = \sum_{r=0}^{\infty} A_{n,3r} t^{3r+3n}, \quad u_0(t,a,b) \equiv 1 \tag{8}$$

The coefficients $A_{n,3r}$ are given by

$$A_{n,3r} = (-1)^r a^r \frac{\varphi_{n,3r}}{(3r+2n)(b+1)_{(3r+3n-3)}} \quad (a > 0, b > -1) \tag{9}$$

and $\varphi_{n,3r}$ are given by

$$\varphi_{n,3r} = \sum_{s_1=0}^{r} \frac{1}{3s_1 + 2n - 2} \sum_{s_2=0}^{s_1} \frac{1}{3s_2 + 2n - 4} \cdots \sum_{s_{n-1}=0}^{s_{n-2}} \frac{1}{3s_{n-2} + 2}.$$

$$\prod_{k_1=s_1}^{r} \frac{b+3n-2+3k_1}{b+2n-1+3k_1} \prod_{k_2=s_2}^{s_1} \frac{b+3n-5+3k_2}{b+2n-3+3k_2} \cdots \prod_{k_{n-1}=0}^{s_{n-2}} \frac{b+4+3k_{n-1}}{b+3+3k_{n-1}} \tag{10}$$

where $(b+1)_{(3r)} = (b+1)(b+4)\ldots(b+1+3r)$.

Proof. Let us seek a formal solution to the $CP(1)$ in the following series form:

$$u_1(t,a,b)=\sum_{r=0}^{\infty}A_{1,r}t^{r+2}.$$

When u_1 and its derivatives with respect to t are written into $CP(1)$

$$A_{1,0}=\frac{1}{2(b+1)},A_{1,1}=0,A_{1,2}=0,A_{1,3}=-a\frac{1}{5(b+1)(b+4)}\qquad(b>-1)$$

$$(r+2)(r+b+1)A_{1,r}=-a(r-1)A_{1,r-3}\qquad(r=3,6,...,3n).\tag{11}$$

is obtained.If we write the recurrence relations (11) for $r=3,6,...,3n$ and multiply them we obtain

$$A_{1,3n}=(-1)^na^n\frac{1}{(3n+2)(b+i)_{(3n)}},A_{1,0}=\frac{1}{2(b+1)}\qquad(b>-1).$$

Thus the solution of $CP(1)$ can be written as follows

$$u_1(t,a,b)=\sum_{r=0}^{\infty}\frac{(-1)^ra^rt^{3r+2}}{(3r+2)(b+1)_{(3r)}}\quad(a>0,b>-1)..$$

Let us seek a formal solution for $CP(2)$ as follows

$$u_2(t,a,b)=\sum_{r=0}^{\infty}A_{2,3r}t^{3r+4}.\tag{12}$$

We assume that $A_{2,3r}$ has the following form:

$$A_{2,3r}=(-1)^ra^r\frac{\varphi_{2,3r}}{(3r+4)(b+1)_{(3r+3)}}\quad(a>0,b>-1).\tag{13}$$

When the values of u_2 and its derivatives with respect to t are written into $CP(2)$, the following relations are obtained

$$A_{2,0}=\frac{1}{2.4(b+1)(b+3)}\qquad(b>-1)$$

$$(3r+4)(3r+3+b)A_{2,3r}+a(3r+1)A_{2,3r-3}=A_{1,3r}\qquad(r\geq 1)\tag{14}$$

then

$$\frac{b+3+3r}{b+4+3r}\varphi_{2,3r}-\varphi_{2,3r-3}=\frac{1}{3r+2}\qquad(a>0,b>-1).\tag{15}$$

From (13) we have

$$A_{2,0} = \frac{\varphi_{2,0}}{4(b+1)(b+4)}, \qquad \varphi_{2,0} = \frac{1}{2}\frac{(b+4)}{(b+3)}.$$

The solution of (15) is

$$\varphi_{2,3r} = \sum_{s=0}^{r} \frac{1}{3s+2} \prod_{k=s}^{r} \frac{b+4+3k}{b+3+3k} \qquad (r = 1, 2, \ldots; b > -1). \tag{16}$$

and the solution of $CP(2)$ can be given by (12) where $A_{2,3r}$ is given by (13) and $\varphi_{2,3r}$ is given by (16). Let

$$u_{n-1}(t,a,b) = \sum_{r=0}^{\infty} A_{n-1,3r} t^{3r+2n-2}, \quad A_{n-1,3r} = \frac{(-1)^r a^r \varphi_{n-1,3r}}{(3r+2n-2)(b+1)_{(3r+3n-6)}}$$

be a solution of the series form of $CP(n-1)$ where

$$\varphi_{n-1,3r} = \sum_{s_1=0}^{r} \frac{1}{3s_1+2n-4} \sum_{s_2=0}^{s_1} \frac{1}{3s_2+2n-6} \cdots \sum_{s_{n-2}=0}^{s_{n-3}} \frac{1}{3s_{n-2}+2}\,.$$

$$\prod_{k_1=s_1}^{r} \frac{b+3n-5+3k_1}{b+2n-3+3k_1} \prod_{k_2=s_2}^{s_1} \frac{b+3n-8+3k_2}{b+2n-5+3k_2} \cdots \prod_{k_{n-2}=0}^{s_{n-3}} \frac{b+4+3k_{n-2}}{b+3+3k_{n-2}}$$

Let us assume

$$u_n(t,a,b) = \sum_{r=0}^{\infty} A_{n,3r} t^{3r+3n} \tag{17}$$

is a formal solution of $CP(n)$, where $A_{n,3r}$ has the form (9). This is a consequence of Mathematical Induction Principle. We will find an explicit form for $\varphi_{n,3r}$. If we substitute the values of u_n and its derivatives with respect to t into $CP(n)$ we obtain the following relations:

$$A_{n,0} = \frac{1}{2^n n!(b+1)(b+3)\ldots(b+2n-1)} \qquad (b > -1) \tag{18}$$

$$(3r+2n)(3r+2n-1+b)A_{n,3r} + a(3r+2n-3)A_{n,3r-3} = A_{n-1,3r} \quad (r \geq 1).$$

Hence we have the following difference equation:

$$\frac{b+2n-1+3r}{b+3n-2+3r}\varphi_{n,3r} - \varphi_{n,3r-3} = \frac{\varphi_{n-1,3r}}{3r+2n-2}\,. \tag{19}$$

From (9) $A_{n,0} = \frac{\varphi_{n,0}}{2n(b+1)_{(3n-3)}}$ and from (18) we have

$$\varphi_{n,0} = \frac{1}{2^{n-1}(n-1)!}\frac{b+4}{b+3}\frac{b+7}{b+5}\cdots\frac{b+3n-2}{b+2n-1} \quad (a>0, b>-1). \tag{20}$$

The numbers $\varphi_{n,0}$ $(n \geq 2)$ are well defined, since if $a = 0$ is used in the the equation (3) the EPD Equation is obtained. The functions $u_n(t,a,b)$ $(n \in N)$ are continuous with respect to a. This will be prove in the next section. If we consider (20) and the solution of (19) we obtain (10). Then $CP(n)$ has a solution which is given by (17), where the coefficients $\varphi_{n,3r}$ are given by (10) and $A_{n,3r}$ by (9). The coefficients $\varphi_{n,3r}$ increase with the indices r for each $n \in N$.

3. CONVERGENCE OF THE SERIES $u_n(t,a,b)$

Lemma 1. The coefficients $\varphi_{n,3r}$ and $A_{n,3r}$ satisfy the following relations:

$$\varphi_{n,3r} = O((n!)^{2r+1}) \quad \textit{and} \quad |A_{n,3r+3}/A_{n,3r}| = O(r^{-1}) \quad (n=2,3,\ldots; r=0,1,\ldots).$$

Proof. It is clear that $|A_{1,3r+3}/A_{1,3r}| = O(r^{-1})$. Let us consider $u_2(t,a,b)$ and the numbers $\varphi_{2,3r}$ which are given by (12) and (16) respectively.It is easily seen that

$$\frac{b+4+3k}{b+3+3k} = 1 + \frac{1}{b+3+3k} < 2 \quad (b>-1, \quad k>0)$$

then

$$0 \leq \varphi_{2,3r} \leq 2^r \sum_{s=0}^{r} \frac{1}{2^s(3s+2)} \leq 2^r \frac{r+1}{2} = 2^{r-1}(r+1).$$

We have $r < 2^r$ $(r = 0,1,\ldots)$, then

$$0 \leq \frac{\varphi_{2,3r}}{2^{2r+1}} \leq \frac{2^{r-1}(r+1)}{2^{2r+1}} \leq \frac{1}{2}.$$

Hence $\varphi_{2,3r} = O(2^{2r+1})$ and we obtain $|A_{2,3r+3}/A_{2,3r}| = O(r^{-1})$. Now let us the coefficient $\varphi_{3,3r}$. These coefficients can be written as follows:

$$\varphi_{3,3r} = \sum_{s=0}^{r} \frac{1}{3s+4} \prod_{k=s}^{r} \frac{b+7+3k}{b+5+3k} \varphi_{2,3s}.$$

The coefficients $\varphi_{2,3s} \quad (s=0,1,2,\ldots,r)$ monotonously increase with s from the following inequalities,

$$\frac{r+1}{3^{r+1}} < 1, \quad \frac{b+7+3k}{b+5+3k} < 3 \quad (r=0,1,\ldots; b>-1)$$

we can write

$$0 \le \varphi_{3,3r} \le 3^r \sum_{s=0}^{r} \frac{1}{3^s(3s+4)} \varphi_{2,3s} \le \frac{3^r}{4}(r+1)\varphi_{2,3r}$$

and then $0 \le \frac{\varphi_{3,3r}}{2^{2r+1}3^{3r+1}} \le \frac{1}{2.4}$. Hence we obtain $\varphi_{3,3r} = O((3!)^{2r+1})$ and

$$|A_{3,3r+3}/A_{3,3r}| = O(r^{-1}).$$

Let us assume

$$\varphi_{n-1,3r} = O(((n-1)!)^{2r+1})$$

and

$$|A_{n-1,3r+3}/A_{n-1,3r}| = O(r^{-1}).$$

We consider the coefficients $\varphi_{n,3r}$, which are expressable as follows:

$$\varphi_{n,3r} = \sum_{s=0}^{r} \frac{1}{3s+2n-2} \prod_{k=s}^{r} \frac{b+3n-2+3k_1}{b+2n-1+3k_1} \varphi_{n-1,3s}.$$

Where the numbers $\varphi_{n-1,3r}$ increase with $s, (s=0,1,\ldots,r)$ and from the following inequalities

$$\frac{b+3n-2+3k_1}{b+2n-1+3k_1} = 1 + \frac{n-1}{b+2n-1+3k_1} < n, \quad r+1 < n^{r+1}$$

we have

$$0 \le \varphi_{n,3r} \le n^r \sum_{s=0}^{r} \frac{1}{n^s(3s+2n-2)} \varphi_{n-1,3s} \le n^r \frac{r+1}{2n-2} \varphi_{n-1,3r}.$$

and thus

$$0 \le \frac{\varphi_{n,3r}}{(n!)^{2r+1}} \le \frac{1}{2^{n-1}(n-1)!}.$$

Hence we obtain

$$\varphi_{n,3r} = O((n!)^{2r+1}) \quad and \quad |A_{n,3r+3}/A_{n,3r}| = O(r^{-1}).$$

Theorem 2. The radius of convergence of $u_n(t,a,b), n \in N$. is infinite for every $t \in R$ when $a > 0, b > -1$. The series $u_n(t,a,b)$, $(n \in N)$ is absolutely and uniformly convergent for every t.

Corollary. $u_n(t,a,b)$, $(n \in N)$ are continuous functions of the parameter a and the variable t. Then the function $u_n(t.a,b)$ can be differentiated infinitely.

4. THE UPPER BOUNDS

Lemma 2. The series (8) have the following integral transformations for $t = \xi, n = 1,2,\ldots$

$$u_n(t,a,b) = \int_0^t \int_0^\mu (\frac{\xi}{\mu})^b e^{-a(\mu^3-\xi^3)/3} u_{n-1}(\xi,a,b) d\mu d\xi \ , \quad u_0(t,a,b) \equiv 1. \tag{21}$$

Proof. We may transform the $CP(n)$ to an integral equation system. To this end we may write the equations (6) for $t = \xi$ as follows

$$\frac{d^2}{d\xi^2} u_n(\xi,a,b) + (a\xi^2 + \frac{b}{\xi}) \frac{d}{d\xi} u_n(\xi,a,b) = u_{n-1}(\xi,a,b) \ , u_0(\xi,a,b) \equiv 1. \tag{22}$$

Multiplying (22) with $\xi^b e^{a\xi^3/3}$ and integrating both sides with respect to ξ on $(0,\mu)$, and using the initial conditions (7), we obtain

$$\frac{d}{d\mu} u_n(\mu,a,b) = \mu^{-b} e^{-a\mu^3/3} \int_0^\mu \xi^b e^{a\xi^3/3} u_{n-1}(x,\xi,a,b) d\xi. \tag{23}$$

Integrating both sides of (23) which respect to μ on $(0,t)$ and using (7) we obtain the integral equations (21).Thus the integral representation for solutions of $CP(n)$ can be expressed by (21).

Lemma 3. Let us define the domain B by the points $(0,0),(t,0),(t,t)$ and the line $\xi = \mu$. Introducing the new variables

$$T: \quad \rho = (\mu^3 - \xi^3)/3, \quad r = \xi/\mu.$$

(21) is transformed to the following integrals

$$u_n(t,a,b) = \lim_{\epsilon_1,\epsilon_2 \to 0} \int_{r=0}^{1-\epsilon_1} \int_{\rho=\epsilon_2}^{s} u_{n-1}((3\rho)^{1/3} r (1-r^3)^{-1/3}, a, b) \frac{r^b dr d\rho}{(3\rho)^{1/3} e^{a\rho} (1-r^3)^{2/3}} \tag{24}$$

$u_0(t,a,b) \equiv 1.$

Proof. The functional determinant of this transformation is

$$\frac{D(\mu,\xi)}{D(\rho,r)} = 3^{-1/3}\rho^{-1/3}(1-r^3)^{-2/3} \quad (\rho \neq 0, r \neq 1).$$

The proof is trivial.

Theorem 3. The functions $u_n(t,a,b)$ have the upper bounds

$$0 \leq u_n(t,a,b) \leq \frac{d^n t^{6n}}{(2n)!} \quad (n \in N,\ d > 1) \tag{25}$$

Proof. First we proof the following inequality:

$$0 \leq u_n(t,a,b) \leq \frac{t^{6n}}{2^n n!(b+1)(b+7)\dots(b+6n-5)} \quad (n \in N) \tag{26}$$

Let us $n=1$. From (24)

$$0 \leq u_1(t,a,b) \leq \lim_{\epsilon_1,\epsilon_2\to 0}\int_{r=0}^{1-\epsilon_1}\int_{\rho=\epsilon_2}^{s} r^b 3^{-1/3}(1-r^3)^{\frac{-2}{3}}\rho e^{-a\rho}dr d\rho$$

$$= \frac{t^6}{3^{7/3}}\lim_{\epsilon_1\to 0}\int_{r=0}^{1-\epsilon_1}\frac{r^b}{(as)^2}(1-r^3)^{4/3}[1-(1+as)e^{-as}]dr$$

where $s = t^3(1-r^3)/3$. From $(1-r^3)^{4/3} < 1$ and $1-e^{-as}-ase^{-as} \leq (as)^2/2, as \geq 0$ we have

$$0 \leq u_1(t,a,b) \leq \frac{t^6}{2}\lim_{\epsilon_1\to 0}\int_{r=0}^{1-\epsilon_1} r^b dr \leq \frac{t^6}{2(b+1)}.$$

For $n=2$ and from the above inequality we obtain

$$0 \leq u_2(t,a,b) = \lim_{\epsilon_1,\epsilon_2\to 0}\int_{r=0}^{1-\epsilon_1}\int_{\rho=\epsilon_2}^{s}\frac{3^2 r^{b+6}}{3^{1/3}(1-r^3)^{\frac{8}{3}}}\rho^3 e^{-a\rho}\frac{dr d\rho}{2(b+1)}$$

$$\leq \frac{t^{12}}{3^{7/3}2(b+1)}\lim_{\epsilon_1,\epsilon_2\to 0}\int_{r=0}^{1-\epsilon_1}\int_{\rho=\epsilon_2}^{s}\frac{r^{b+6}}{s^4}(1-r^3)^{4/3}\rho^3 e^{-a\rho}dr d\rho$$

$$\leq \frac{t^{12}}{2(b+1)}\lim_{\epsilon_1\to 0}\int_{r=0}^{1-\epsilon_1}\frac{r^{b+6}}{a^4 s^4}6\{1-[1-as-\frac{3}{2}a^2s^2-\frac{1}{6}a^3s^3]e^{-as}\}dr.$$

We have

$$1-e^{-m}\sum_{p=0}^{2n-1}\frac{m^{2n-p-1}}{\Gamma(2n-\rho)} \leq \frac{m^{2n}}{(2n)!} \quad (m = as \geq 0). \tag{27}$$

Hence we can write

$$0 \le u_2(t,a,b) \le \frac{t^{12}}{2.4(b+1)} \lim_{\epsilon_1 \to 0} \int_{r=0}^{1-\epsilon_1} r^{b+6} dr \le \frac{t^{12}}{2.4(b+1)(b+7)}.$$

If we use (27) and Mathematical Induction Principle we obtain (26). On the other hand we have the following inequality

$$d^n k(k+1)\ldots(k+n-1) > n! \quad (k>0,\ d>1,\ n \in N).$$

Setting $k = b+1$ in the above inequality we have

$$(b+1)(b+7)(b+13)\ldots(b+6n-5) = k(k+6)(k+12)\ldots(k+6n-6)$$

$$> k(k+1)(k+2)\ldots(k+n-1) > \frac{n!}{d^n}.$$

Hence

$$\frac{t^{6n}}{2^n n!(b+1)(b+7)\ldots(b+6n-5)} \le \frac{d^n t^{6n}}{2^n n! n!} < \frac{d^n t^{6n}}{(2n)!} \quad (n \in N).$$

From (26)

$$0 \le u_n(t,a,b) \le \frac{d^n t^{6n}}{(2n)!} \quad (n \in N)$$

is obtained.

5. THE SOLUTION OF THE SINGULAR CP (3),(4)

Lemma 4. Let us assume that the function $f(x_1, x_2, ..., x_n)$ is infinitely differentiable.

$$v(x,t,d) = \sum_{n=0}^{\infty} \frac{d^n t^{6n}}{(2n)!} |\Delta^n f| \quad (d>1)$$

is absolutely and uniformly convergent on the whole space $R^n xR$ or on a subspace of $R^n xR$ which contains the plane $t=0$ respectively when

$$|\Delta^n f| = o((2n)!) \quad (n \in N). \tag{30}$$

$$|\Delta^n f| = O((2n)!) \quad (n \in N). \tag{31}$$

More then $v(x,t,d)$ is a majorant for the series (5).

From the above theorems and Lemmas the following theorem is easly proved.

Theorem 4. Under the conditions of Lemma 4 the power series (5) is a solution of CP (3),(4). This solution is absolutely and uniformly convergent on the whole space $R^n xR$, when f satisfies (30) or it is absolutely and uniformly convergent on a subspace of $R^n xR$ which contains the plane $t = 0$ when f satisfies (31).

References

[1] ASRAL, B., "On the regular Cauchy Problems for the EPD Equations and the method of ascent", Bull. Math. de la Soc. Sci. Math. de la R,S de Ron. Tome 25(73), nr.2 (1981)121-128.

[2] BLUM, E. K., "The EPD Equation in the exceptional cases", Proc. Amer. Math. Soc. Vol.5 (1954) 511-520.

[3] COPSON, E. T., "On a regular Cauchy problem for the EPD Equation" Proc. Royal Soc. London, A 235 (1956) 560 - 572.

[4] DAVIS, R., "On a regular Cauchy problem for the EPD Equation" Ann. Mat. Ser.IV. 42 (1956) 205-226.

[5] DERNEK, N., "Bir Hiper. Denk.için Cauchy Probleminin Çözümü Hk" Ata. Üni. Fen Bil. Der. Özel Sayı(1985) 311-314.

[6] DERNEK, N., "On the solution of the EPD Equation using finite integral trans." Turkish Jr. Math. Vol. 21 No.3 (1997) 317-325.

[7] DIAZ, J. B. and WEINBERGER, H. F., "A solution of the singular initial value problem for the EPD Equation" Proc. Amer. Math. Soc. Vol.4 (1954) 703-715.

[8] EULER, L., "Institutions Calculi Integralis III", Petropoli, 1770.

[9] PROTTER, M. H. and WEINBERGER, H. F., "Maximum Principle in differential equations" Prentice-Hall, Inc. Englewood cliffs. nd. (1967).

[10] WALTER. W, "Über die EPD Gleichung" Math. Zeit. Bd. G.7 (1957) 361-367.

[11] WEINSTEIN, A., "Sur le probleme de Cauchy pour L'equation de Poisson et L'equation des ondes" C. R. Acad. Sci, Vol 234 (1952) 2584-2585.

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