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Existence and uniqueness of positive solutions for system of (p, q, r)-Laplacian fractional order boundary value problems

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Abstract

In this paper the existence of unique positive solutions for system of (p, q, r)-Lapalacian Sturm-Liouville type two-point fractional order boundary vaue problems,

$$
\begin{aligned}
{}^{\mathcal{C}}\mathfrak{D}^{\alpha}_{0+}\big(\phi_p(\mathtt{u}(t))\big) + \mathtt{f}\big(t, \mathtt{u}(t), \mathtt{v}(t), \mathtt{w}(t)\big) &= 0,\ \ 0 < t < 1,\\
{}^{\mathcal{C}}\mathfrak{D}^{\beta}_{0+}\big(\phi_q(\mathtt{v}(t))\big) + \mathtt{g}\big(t, \mathtt{v}(t), \mathtt{w}(t), \mathtt{u}(t)\big) &= 0,\ \ 0 < t < 1,\\
{}^{\mathcal{C}}\mathfrak{D}^{\gamma}_{0+}\big(\phi_r(\mathtt{w}(t))\big) + \mathtt{h}\big(t, \mathtt{w}(t), \mathtt{u}(t), \mathtt{v}(t)\big) &= 0,\ \ 0 < t < 1,\\
a_1(\phi_p\mathtt{u})(0) - b_1(\phi_p\mathtt{u})'(0) = 0, \ \ & c_1(\phi_p\mathtt{u})(1) + d_1(\phi_p\mathtt{u})'(1) = 0,\\
a_2(\phi_q\mathtt{v})(0) - b_2(\phi_q\mathtt{v})'(0) = 0, \ \ & c_2(\phi_q\mathtt{v})(1) + d_2(\phi_q\mathtt{v})'(1) = 0,\\
a_3(\phi_r\mathtt{w})(0) - b_3(\phi_r\mathtt{w})'(0) = 0, \ \ & c_3(\phi_r\mathtt{w})(1) + d_3(\phi_r\mathtt{w})'(1) = 0,
\end{aligned}
$$

where $1 < \alpha, \beta, \gamma \leq 2$, $\phi_\ell(\tau) = |\tau|^{\ell-2}\tau$, $\ell \in (1, \infty)$, ${}^{\mathcal{C}}\mathfrak{D}^{\star}_{0+}$ is a Caputo fractional derivatives of order $\star \in \{\alpha, \beta, \gamma\}$ and $a_i, b_i, c_i, d_i, i = 1, 2, 3$ are positive constants, is established by an application of n–fixed point theorem of ternary operators on partially ordered metric spaces.

Keywords: Boundary value problem, Caputo fractional derivative, n-fixed point, Positive solution, Monotone mappings, Partially ordered complete metric spaces, Contractive.
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1. Introduction

Fractional calculus is deeply related to the dynamics of complicated real-world problems. Fractional operators are non-local and describe several natural phenomena in a better and systematic manner [22]. Many mathematical models are accurately governed by fractional order differential equations. Since the classical mathematical models are special cases of the fractional order mathematical models, it implies that the results for the fractional mathematical model are more general and more accurate [10]. In recent years, there are certain papers and monographs dealing with the existence, uniqueness, multiple solutions and positive solutions of fractional order nonlinear boundary value problems, see [3, 6, 8, 7, 9, 13, 14, 15, 16, 17, 24, 25, 26] and references therein. The study of turbulent flow through porous media is important for a wide range of scientific and engineering applications such as fluidized bed combustion, compact heat exchangers, combustion in an inert porous matrix, high temperature gas-cooled reactors, chemical catalytic reactors [5] and drying of different products such as iron ore [12]. To study such type of problems, Leibenson [11] introduced the following p-Laplacian equation,

$$\big(\phi_p(\mathtt{u}'(t))\big)' = f\big(t, \mathtt{u}(t), \mathtt{u}'(t)\big),$$

where $\phi_p(\tau) = |\tau|^{p-2}\tau,\ p > 1$, is the p-Laplacian operator its inverse function is denoted by $\phi_q(\tau)$ with $\phi_q(\tau) = |\tau|^{q-2}\tau$, and p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$. It is well known fact that the p-Laplacian operator and fractional calculus arises from many applied fields such as turbulant filtration in porous media, blood flow problems, rheology, modelling of viscoplasticity, material science, it is worth studying the fractional differential equations with p-Laplacian operator. In [21] Tian et al., considered the p–Laplacian fractional order boundary value problem,

$$\mathfrak{D}^{\gamma}_{0+}\big[\phi_p\big(\mathfrak{D}^{\alpha}_{0+}\mathtt{u}(t)\big)\big] = f(t, \mathtt{u}(t)),\ 0 < t < 1,$$
$$\mathtt{u}(0) = \mathfrak{D}^{\alpha}_{0+}\mathtt{u}(0) = 0, \mathfrak{D}^{\beta}_{0+}\mathtt{u}(1) = a\mathfrak{D}^{\beta}_{0+}\mathtt{u}(\xi), \mathfrak{D}^{\alpha}_{0+}\mathtt{u}(1) = b\mathfrak{D}^{\alpha}_{0+}\mathtt{u}(\eta),$$

where $\mathfrak{D}^{\star}_{0+}$ is a RiemannâĂŞLiouville fractional derivative of order $\star \in \{\alpha, \beta, \gamma\}$ and established existence of positive solutions by applying monotone iterative method and the fixed point index theory on cones. Recently, Wang and Zhai [20] studied existence and uniqueness of solutions for a new form of fractional differential equation containing p–Laplacian operator,

$$\mathfrak{D}^{\beta}\Big(\phi_p\big(\mathfrak{D}^{\alpha}u(t) - g(t)\big)\Big) + f(t, u(t)) = 0,\ 0 < t < 1,$$

satisfying ∞–point boundary value conditions,

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,$$
$$\mathfrak{D}^{\alpha}_{0+}u(0) = 0, u^{(i)}(1) = \sum_{i=1}^{\infty}\alpha_j u(\xi_j),$$

based on a new fixed point theorem for $\phi - (h, e)$-concave operatos. Motivated by the works mentioned above, in this paper we consider the system of fractional order differential equation, for $0 < t < 1$,

$$\left.\begin{aligned} {}^{C}\mathfrak{D}^{\alpha}_{0+}\big(\phi_p(\mathtt{u}(t))\big) + \mathtt{f}\big(t, \mathtt{u}(t), \mathtt{v}(t), \mathtt{w}(t)\big) = 0 \\ {}^{C}\mathfrak{D}^{\beta}_{0+}\big(\phi_q(\mathtt{v}(t))\big) + \mathtt{g}\big(t, \mathtt{v}(t), \mathtt{w}(t), \mathtt{u}(t)\big) = 0 \\ {}^{C}\mathfrak{D}^{\gamma}_{0+}\big(\phi_r(\mathtt{w}(t))\big) + \mathtt{h}\big(t, \mathtt{w}(t), \mathtt{u}(t), \mathtt{v}(t)\big) = 0 \end{aligned}\right\} \tag{1}$$

Satisfying the boundary conditins,

$$\left.\begin{aligned} a_1(\phi_p\mathtt{u})(0) - b_1(\phi_p\mathtt{u})'(0) = 0, \quad c_1(\phi_p\mathtt{u})(1) + d_1(\phi_p\mathtt{u})'(1) = 0, \\ a_2(\phi_q\mathtt{v})(0) - b_2(\phi_q\mathtt{v})'(0) = 0, \quad c_2(\phi_q\mathtt{v})(1) + d_2(\phi_q\mathtt{v})'(1) = 0, \\ a_3(\phi_r\mathtt{w})(0) - b_3(\phi_r\mathtt{w})'(0) = 0, \quad c_3(\phi_r\mathtt{w})(1) + d_3(\phi_r\mathtt{w})'(1) = 0, \end{aligned}\right\} \tag{2}$$

where $1 < \alpha, \beta, \gamma \leq 2$, $\phi_\ell(\tau) = |\tau|^{\ell-2}\tau$, $\ell \in (1, \infty)$, ${}^{\mathcal{C}}\mathfrak{D}^{\star}_{0+}$ is Caputo fractional derivative of order $\star \in \{\alpha, \beta, \gamma\}$ and $a_i, b_i, c_i, d_i, i = 1, 2, 3$ are positive constants. The rest of the paper is organized in the following fashion. In Section 2, we provide some definitions and lemmas which will be useful in our main results. Later, we construct kernel for the homogeneous boundary value problem corresponding to (1)-(2). In Section 3, we study existence and uniqueness of n-fixed point theorems for contractive type mappings in partially ordered complete metric spaces. In Section 4, we study existence and uniqueness of solution of the boundary value problem (1)-(2) as an application n-fixed point theorem. Finally, we provide an example to check the validity of our obtained results.

2. Kernel and its bounds

In this section, we construct the kernel for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the kernel.

Definition 2.1 ([14, 9]). *Let $\alpha \in (0, +\infty)$. The operator I^{γ}_{a+} defined on $L_1[a,b]$ by*

$$I^{\gamma}_{a+} f(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} f(\tau) d\tau,$$

for $t \in [a,b]$, is called the left sided Riemann-Liouville fractional integral of order γ. Under same hypotheses, the right-sided Riemann-Liouville fractional integral operator is given by

$$_{b-}I^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (\tau - t)^{\gamma-1} f(\tau) d\tau.$$

Definition 2.2 ([14, 9]). *Suppose $\gamma > 0$ with $n = [\gamma] + 1$. Then the left and right sided Caputo fractional derivatives defined on absolutely continuous functions space $AC^n[a,b]$ are given by*

$$\big({}^{\mathcal{C}}\mathfrak{D}^{\gamma}_{a+} f\big)(t) = \big(I^{n-\gamma}_{a+} \mathfrak{D}^n f\big)(t),$$

$$\big({}^{\mathcal{C}}_{b-}\mathfrak{D}^{\gamma} f\big)(t) = (-1)^n \big({}_{b-}I^{n-\gamma} \mathfrak{D}^n f\big)(t),$$

where $\mathfrak{D}^n := \frac{d^n}{dt^n}$.

Lemma 2.3 ([14, 9]). *Let $\gamma > 0$. Then*

(i) for $f(t) \in L_1(a,b)$, we have

$$\big({}^{\mathcal{C}}\mathfrak{D}^{\gamma}_{a+} I^{\gamma}_{a+} f\big)(t) = f(t), \ \big({}^{\mathcal{C}}_{b-}\mathfrak{D}^{\gamma}\, {}_{b-}I^{\gamma} f\big)(t) = f(t).$$

(ii) for $f(t) \in AC^n[a,b]$, we have

$$\big(I^{\gamma}_{a+}\, {}^{\mathcal{C}}\mathfrak{D}^{\gamma}_{a+} f\big)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k,$$

$$\big({}_{b-}I^{\gamma}\, {}^{\mathcal{C}}_{b-}\mathfrak{D}^{\gamma} f\big)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!} (b-t)^k.$$

Lemma 2.4. *Let $h \in C(\mathbb{R})$ and $1 < \delta \leq 2$. Then the boundary value problem*

$${}^{\mathcal{C}}\mathfrak{D}^{\delta}_{0+}\big(\phi_\epsilon(z(t))\big) + h(t) = 0, \ 0 < t < 1, \tag{3}$$

$$\begin{aligned} \kappa_1(\phi_\epsilon z)(0) - \kappa_2(\phi_\epsilon z)'(0) = 0, \\ \kappa_3(\phi_\epsilon z)(1) + \kappa_4(\phi_\epsilon z)'(1) = 0, \end{aligned} \tag{4}$$

and $\frac{1}{\epsilon}+\frac{1}{\epsilon'}=1$*, where* ϵ' *is called inverse of* ϵ*, has a unique solution*

$$z(t)=\phi_{\epsilon'}\Big(\int_0^1\mathcal{N}(t,\tau)(h(\tau)d\tau\Big), \tag{5}$$

where

$$\mathcal{N}(t,\tau):=\begin{cases}\mathcal{N}_1(t,\tau), & 0<\tau\le t<1,\\ \mathcal{N}_2(t,\tau), & 0<t\le\tau<1,\end{cases} \tag{6}$$

$$\mathcal{N}_1(t,\tau)=\mathcal{N}_2(t,\tau)-\frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)},$$

$$\mathcal{N}_2(t,\tau)=\frac{\Delta}{\Gamma(\delta)}[\kappa_3(1-\tau)^{\delta-1}+\kappa_4(\delta-1)(1-\tau)^{\delta-2}](\kappa_1t+\kappa_2),$$

and $\Delta=(\kappa_2\kappa_3+\kappa_1\kappa_3+\kappa_1\kappa_4)^{-1}$.

Proof. From Lemma 2.3, the equation (3) transforms to the fractional integral equation

$$\phi_\epsilon(z)(t)=A+Bt-\int_0^t\frac{(t-\tau)^{\delta-1}}{\Gamma(\delta)}h(\tau)d\tau.$$

By the boundary conditions (4), one can determine A and B as

$$A=\frac{\Delta\kappa_2}{\Gamma(\delta)}\int_0^1[\kappa_3(1-\tau)^{\delta-1}+\kappa_4(\delta-1)(1-\tau)^{\delta-2}h(\tau)d\tau,$$

$$B=\frac{\Delta\kappa_1}{\Gamma(\delta)}\int_0^1[\kappa_3(1-\tau)^{\delta-1}+\kappa_4(\delta-1)(1-\tau)^{\delta-2}h(\tau)d\tau.$$

Thus, we have

$$\begin{aligned}(\phi_\epsilon z)(t)&=\frac{\Delta}{\Gamma(\delta)}\int_0^1[\kappa_3(1-\tau)^{\delta-1}+\kappa_4(\delta-1)(1-\tau)^{\delta-2}](\kappa_1t+\kappa_2)h(\tau)d\tau\\&\qquad-\frac{1}{\Gamma(\delta)}\int_0^t(t-\tau)^{\delta-1}h(\tau)d\tau\\&=\int_0^1\mathcal{N}(t,\tau)h(\tau)d\tau.\end{aligned}$$

Therefore,

$$u(t)=\phi_{\epsilon'}\Big(\int_0^1\mathcal{N}(t,\tau)h(\tau)d\tau\Big).$$

□

This completes the proof.

Lemma 2.5. *The kernel* $\mathcal{N}(t,\tau)$ *has the following properties:*

(i) $\mathcal{N}(t,\tau)$ *is continuous on* $(0,1)\times(0,1)$,

(ii) for $\delta>\frac{2\kappa_1+\kappa_2}{\kappa_1+\kappa_2}$*, we have* $\mathcal{N}(t,\tau)>0$ *for any* $t,\tau\in(0,1)$,

(iii) for $\delta>\frac{2\kappa_1+\kappa_2}{\kappa_1+\kappa_2}$*, we have* $\mathcal{N}(t,\tau)\le\mathcal{N}(\tau,\tau)$ *for* $t,\tau\in(0,1)$,

Proof. One can easily establish the property (i). Now, we establish (ii).
For $0 < \tau \leq t < 1$, we have

$$\frac{\partial \mathcal{N}_1(t,\tau)}{\partial t} = \frac{\kappa_1 \Delta}{\Gamma(\delta)}[\kappa_3(1-\tau)^{\delta-1} + \kappa_4(\delta-1)(1-\tau)^{\delta-2}] - \frac{(\delta-1)(t-\tau)^{\delta-2}}{\Gamma(\delta)}$$

and

$$\frac{\partial^2 \mathcal{N}_1(t,\tau)}{\partial t^2} = \frac{(\delta-1)(\delta-2)(t-\tau)^{\delta-3}}{\Gamma(\delta)} \geq 0.$$

This shows that $\frac{\partial \mathcal{N}_1(t,\tau)}{\partial t}$ is increasing on $t \in [\tau, 1)$. So by $\delta > \frac{2\kappa_1+\kappa_2}{\kappa_1+\kappa_2}$, we have

$$\begin{aligned}
\frac{\partial \mathcal{N}_1(t,\tau)}{\partial t} &\leq \frac{\partial \mathcal{N}_1(1,\tau)}{\partial t} \\
&= \frac{\kappa_1 \Delta}{\Gamma(\delta)}[\kappa_3(1-\tau)^{\delta-1} + \kappa_4(\delta-1)(1-\tau)^{\delta-2}] - \frac{(\delta-1)(1-\tau)^{\delta-2}}{\Gamma(\delta)} \\
&\leq \frac{\kappa_1\kappa_3\Delta + (\kappa_1\kappa_4\Delta - 1)(\delta-1)(1-\tau)^{\delta-2}}{\Gamma(\delta)} \leq 0.
\end{aligned}$$

Then $\mathcal{N}_1(t,\tau)$ is decreasing with respect to t on $[s, 1)$, we get

$$\mathcal{N}_1(1,\tau) \leq \mathcal{N}_1(t,\tau) \leq \mathcal{N}_1(\tau,\tau).$$

Further,

$$\begin{aligned}
\mathcal{N}_1(1,\tau) &= \frac{\Delta}{\Gamma(\delta)}[\kappa_3(1-\tau)^{\delta-1} + \kappa_4(\delta-1)(1-\tau)^{\delta-2}](\kappa_1+\kappa_2) - \frac{(1-\tau)^{\delta-1}}{\Gamma(\delta)} \\
&= \frac{(1-\tau)^{\delta-2}}{\Gamma(\delta)}[-\kappa_1\kappa_4\Delta(1-\tau) + \Delta\kappa_4(\kappa_1+\kappa_2)(\delta-1)] \\
&\geq \frac{\Delta(1-\tau)^{\delta-2}}{\Gamma(\delta)}[-\kappa_1\kappa_4(1-\tau) + \kappa_4(\kappa_1+\kappa_2)(\frac{2\kappa_1+\kappa_2}{\kappa_1+\kappa_2} - 1)] \\
&\geq \frac{\Delta(1-\tau)^{\delta-2}}{\Gamma(\delta)}\kappa_1 d\tau > 0.
\end{aligned}$$

When $0 < t \leq \tau < 1$, we have

$$\frac{\partial \mathcal{N}_2(t,\tau)}{\partial t} = \frac{\kappa_1 \Delta}{\Gamma(\delta)}[\kappa_3(1-\tau)^{\delta-1} + \kappa_4(\delta-1)(1-\tau)^{\delta-2}] \geq 0,$$

from this

$$0 < \mathcal{N}_2(0,\tau) \leq \mathcal{N}_2(t,\tau) \leq \mathcal{N}_2(\tau,\tau).$$

From the proof of (ii), we have $\mathcal{N}(t,\tau) \leq \mathcal{N}(\tau,\tau)$. □

Lemma 2.6. *The the boundary value problem for* $0 < t < 1$*,*

$$\begin{aligned}
{}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\alpha}\big(\phi_p(\mathtt{u}(t))\big) + \mathtt{f}\big(t, \mathtt{u}(t), \mathtt{v}(t), \mathtt{w}(t)\big) = 0, \\
{}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\beta}\big(\phi_q(\mathtt{v}(t))\big) + \mathtt{g}\big(t, \mathtt{v}(t), \mathtt{w}(t), \mathtt{u}(t)\big) = 0, \\
{}^{\mathcal{C}}\mathfrak{D}_{0^+}^{\gamma}\big(\phi_r(\mathtt{w}(t))\big) + \mathtt{h}\big(t, \mathtt{w}(t), \mathtt{u}(t), \mathtt{v}(t)\big) = 0,
\end{aligned}$$

$$\begin{aligned}
a_1(\phi_p\mathtt{u})(0) - b_1(\phi_p\mathtt{u})'(0) = 0, \quad c_1(\phi_p\mathtt{u})(1) + d_1(\phi_p\mathtt{u})'(1) = 0, \\
a_2(\phi_q\mathtt{v})(0) - b_2(\phi_q\mathtt{v})'(0) = 0, \quad c_2(\phi_q\mathtt{v})(1) + d_2(\phi_q\mathtt{v})'(1) = 0, \\
a_3(\phi_r\mathtt{w})(0) - b_3(\phi_r\mathtt{w})'(0) = 0, \quad c_3(\phi_r\mathtt{w})(1) + d_3(\phi_r\mathtt{w})'(1) = 0,
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1, \frac{1}{q}+\frac{1}{q'}=1$ *and* $\frac{1}{r}+\frac{1}{r'}=1$, *where* p',q',r' *are called inverses of* p,q,r *respectively, has a unique solution* $(\mathtt{u}(\mathtt{t}),\mathtt{v}(\mathtt{t}),\mathtt{w}(\mathtt{t}))$,

$$\mathtt{u}(\mathtt{t})=\phi_{p'}\Big(\int_0^1\mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{u}(\tau),\mathtt{v}(\tau),\mathtt{w}(\tau)\big)d\tau\Big),$$
$$\mathtt{v}(\mathtt{t})=\phi_{q'}\Big(\int_0^1\mathcal{N}_\beta(t,\tau)\mathtt{f}\big(\tau,\mathtt{v}(\tau),\mathtt{w}(\tau),\mathtt{u}(\tau)\big)d\tau\Big),$$
$$\mathtt{w}(\mathtt{t})=\phi_{r'}\Big(\int_0^1\mathcal{N}_\gamma(t,\tau)\mathtt{f}\big(\tau,\mathtt{w}(\tau),\mathtt{u}(\tau),\mathtt{v}(\tau)d\tau\Big),$$

where

$$\mathcal{N}_\alpha(t,\tau):=\begin{cases}\mathcal{N}_{\zeta_1}(t,\tau), & 0<\tau\le t<1,\\ \mathcal{N}_{\zeta_2}(t,\tau), & 0<t\le\tau<1,\end{cases}$$

$$\mathcal{N}_\beta(t,\tau):=\begin{cases}\mathcal{N}_{\beta_1}(t,\tau), & 0<\tau\le t<1,\\ \mathcal{N}_{\beta_2}(t,\tau), & 0<t\le\tau<1,\end{cases}$$

$$\mathcal{N}_\gamma(t,\tau):=\begin{cases}\mathcal{N}_{\gamma_1}(t,\tau), & 0<\tau\le t<1,\\ \mathcal{N}_{\gamma_2}(t,\tau), & 0<t\le\tau<1,\end{cases}$$

and

$$\mathcal{N}_{\zeta_1}(t,\tau)=\mathcal{N}_{\zeta_2}(t,\tau)-\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)},$$

$$\mathcal{N}_{\zeta_2}(t,\tau)=\frac{\Delta_1}{\Gamma(\alpha)}[c_1(1-\tau)^{\alpha-1}+d_1(\alpha-1)(1-\tau)^{\alpha-2}](a_1t+b_1),$$

$\Delta_1=(c_1b_1+a_1c_1+a_1d_1)^{-1}$,

$$\mathcal{N}_{\beta_1}(t,\tau)=\mathcal{N}_{\beta_2}(t,\tau)-\frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)},$$

$$\mathcal{N}_{\beta_2}(t,\tau)=\frac{\Delta_2}{\Gamma(\beta)}[c_2(1-\tau)^{\beta-1}+d_2(\beta-1)(1-\tau)^{\beta-2}](a_2t+b_2),$$

$\Delta_2=(c_2b_2+a_2c_2+a_2d_2)^{-1}$,

$$\mathcal{N}_{\gamma_1}(t,\tau)=\mathcal{N}_{\gamma_2}(t,\tau)-\frac{(t-\tau)^{\gamma-1}}{\Gamma(\gamma)},$$

$$\mathcal{N}_{\gamma_2}(t,\tau)=\frac{\Delta_3}{\Gamma(\gamma)}[c_3(1-\tau)^{\gamma-1}+d_3(\gamma-1)(1-\tau)^{\gamma-2}](a_3t+b_3),$$

$\Delta_3=(c_3b_3+a_3c_3+a_3d_3)^{-1}$.

3. *n*–fixed point theorems

In this section, we study the concept of n–fixed point for nonlinear and monotone mappings in partially ordered complete metric spaces and establish existence and uniqueness theorems for contractive type mappings. Our results generalize and extend the tripled fixed point theorems established by Berinde and Borcut [1, 2]. We note that the concept of n–fixed point for monotone operators is essentially different of the corresponding one for mixed monotone operators [23]. Let $(\mathcal{B},\le)$ be a partially ordered set and d be a metric on $\mathcal{B}$ such that $(\mathcal{B},d)$ is a complete metric space. Consider on the product space $\mathcal{B}^n$ the following partial order, for $(u_1,u_2,\cdots,u_n)$, $(v_1,v_2,\cdots,v_n)\in\mathcal{B}^n$,
$(u_1,u_2,\cdots,u_n)\le(v_1,v_2,\cdots,v_n) \iff$

$$\begin{matrix}u_1\ge v_1,u_2\le v_2,\cdots,u_n\le v_n, & \text{if } n \text{ is even},\\ u_1\ge v_1,u_2\le v_2,\cdots,u_n\ge v_n, & \text{if } n \text{ is odd}.\end{matrix}$$

Definition 3.1. *Let* $(\mathcal{B}, \leq)$ *be a partially ordered set and* $\mathcal{A} : \mathcal{B}^n \to \mathcal{B}$. *We say that* $\mathcal{A}$ *has mixed monotone property if* $\mathcal{A}(x_1, x_2, \cdots, x_n)$ *is monotone nondec- reasing in* $x_1, x_2, \cdots, x_n$, *i.e., for any* $(x_1, x_2, \cdots, x_n) \in \mathcal{B}^n$,

$$\begin{aligned} u_1, v_1 \in \mathcal{B},\ u_1 \leq v_1 &\implies \mathcal{A}(u_1, x_2, \cdots, x_n) \leq \mathcal{A}(v_1, x_2, \cdots, x_n), \\ u_2, v_2 \in \mathcal{B},\ u_2 \leq v_2 &\implies \mathcal{A}(x_1, u_2, \cdots, x_n) \leq \mathcal{A}(x_1, v_2, \cdots, x_n), \\ &\vdots \\ u_n, v_n \in \mathcal{B},\ u_n \leq v_n &\implies \mathcal{A}(x_1, x_2, \cdots, u_n) \leq \mathcal{A}(x_1, x_2, \cdots, v_n). \end{aligned}$$

Definition 3.2. *An element* $(u_1, u_2, \cdots, u_n) \in \mathcal{B}^n$ *is called an* n *fixed point of a mapping* $\mathcal{A} : \mathcal{B}^n \to \mathcal{B}$ *if*

$$\begin{aligned} \mathcal{A}(u_1, u_2, \cdots, u_n) &= u_1, \\ \mathcal{A}(u_2, u_3, \cdots, u_1) &= u_2, \\ &\vdots \\ \mathcal{A}(u_n, u_1, \cdots, u_{n-1}) &= u_n. \end{aligned}$$

Theorem 3.3. $\mathcal{A} : \mathcal{B}^n \to \mathcal{B}$ *be continuous and mixed monotone mapping and assume that there exist constants* $\zeta_i \in [0, 1)$, $i = 1, 2 \cdots, n$ *with* $\sum_{i=1}^{n} \zeta_i < 1$ *for which*

$$d((u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n)) \leq \sum_{i=1}^{n} \zeta_i d(u_i, v_i) \tag{7}$$

for every

$$\begin{cases} u_1 \geq v_1, u_2 \leq v_2, \cdots, u_n \leq v_n, & \text{if } n \text{ is even}, \\ u_1 \geq v_1, u_2 \leq v_2, \cdots, u_n \geq v_n, & \text{if } n \text{ is odd}. \end{cases}$$

Further, if there exist $u_{10}, u_{20}, \cdots, u_{n0} \in \mathcal{B}$ *such that*

$$\begin{aligned} u_{10} &\leq \mathcal{A}(u_{10}, x_{20}, \cdots, u_{n0}), \\ u_{20} &\leq \mathcal{A}(u_{20}, u_{30}, \cdots, u_{10}), \\ &\vdots \\ u_{n0} &\leq \mathcal{A}(u_{n0}, u_{10}, \cdots, x_{n-10}), \end{aligned}$$

then there exist $u_1, u_2, \cdots, u_n \in \mathcal{B}$ *such that*

$$\begin{aligned} u_1 &= \mathcal{A}(u_1, u_2, \cdots, u_n), \\ u_2 &= \mathcal{A}(u_2, u_3, \cdots, u_1), \\ &\vdots \\ u_n &= \mathcal{A}(u_n, u_1, \cdots, u_{n-1}). \end{aligned}$$

Proof. Without loss of generality, we may assume that n is even and the same argument work when n is odd. Since,

$$\begin{aligned} u_{10} &\leq \mathcal{A}(u_{10}, u_{20}, \cdots, u_{n0}) = u_{11}(say), \\ u_{20} &\leq \mathcal{A}(u_{20}, u_{30}, \cdots, u_{10}) = u_{21}(say), \\ &\vdots \\ u_{n0} &\leq \mathcal{A}(u_{n0}, u_{10}, \cdots, u_{10}) = u_{n1}(say). \end{aligned}$$

For $m \geq 1$, denote

$$\begin{aligned} u_{1m} =& \mathcal{A}(u_{\overline{1m-1}}, u_{\overline{2m-1}}, \cdots, u_{\overline{nm-1}}), \\ u_{2m} =& \mathcal{A}(u_{\overline{2m-1}}, u_{\overline{3k-1}}, \cdots, u_{\overline{1m-1}}), \\ &\vdots \\ u_{nm} =& \mathcal{A}(u_{\overline{nm-1}}, u_{\overline{1m-1}}, \cdots, u_{\overline{n-1m-1}}). \end{aligned}$$

Since $\mathcal{A}$ has mixed monotone property, it follows that

$$\begin{aligned} u_{12} =& \mathcal{A}(u_{11}, u_{21}, \cdots, u_{n1}) \geq \mathcal{A}(u_{10}, u_{20}, \cdots, u_{n0}) = u_{11}, \\ u_{22} =& \mathcal{A}(u_{21,31}, \cdots, u_{11}) \geq \mathcal{A}(u_{20}, u_{30}, \cdots, u_{10}) = u_{21}, \\ &\vdots \\ u_{n2} =& \mathcal{A}(u_{n1}, u_{11}, \cdots, u_{\overline{n-11}}) \geq \mathcal{A}(u_{n0}, u_{10}, \cdots, u_{\overline{n-10}}) = u_{n1}. \end{aligned}$$

Thus, we obtain n sequences satisfying the following conditions

$$\begin{aligned} &u_{10} \leq u_{11} \leq u_{12} \leq \cdot\tau \leq u_{1n} \leq \cdots, \\ &u_{20} \leq u_{21} \leq u_{22} \leq \cdot\tau \leq u_{2n} \leq \cdots, \\ &\vdots \\ &u_{n0} \leq u_{n1} \leq u_{n2} \leq \cdot\tau \leq u_{nn} \leq \cdots. \end{aligned}$$

For simplicity, we denote

$$D_n^{u_i} = d(u_{\overline{in-1}}, u_{in}), \ 1 \leq i \leq n.$$

Then by (7), we have

$$\begin{aligned} D_2^{u_1} = d(u_{11}, u_{12}) &= d\big(\mathcal{A}(u_{10}, u_{20}, \cdots, u_{n0}), \mathcal{A}(u_{11}, u_{21}, \cdots, u_{n1})\big) \\ &\leq \sum_{i=1}^{n} \zeta_i d(u_{i0}, u_{i1}) \\ &\leq \sum_{i=1}^{n} \zeta_i D_1^{u_i}. \end{aligned}$$

and

$$\begin{aligned} D_2^{u_2} = d(u_{21}, u_{22}) &= d\big(\mathcal{A}(u_{20}, u_{30}, \cdots, u_{10}), \mathcal{A}(u_{21}, u_{31}, \cdots, u_{11})\big) \\ &\leq \zeta_n D_1^{u_1} + \sum_{i=1}^{n-1} \zeta_i D_1^{u_{i+1}}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} D_2^{u_n} = d(u_{n1}, u_{n2}) &= d\big(\mathcal{A}(u_{n0}, u_{10}, \cdots, u_{\overline{n-10}}), \mathcal{A}(u_{n1}, u_{11}, \cdots, u_{\overline{n-11}})\big) \\ &\leq \zeta_1 d(u_{n0}, u_{11}) + \sum_{i=1}^{n-1} \zeta_{i+1} d(u_{i0}, u_{i1}) \\ &\leq \zeta_1 D_1^{u_n} + \sum_{i=1}^{n-1} \zeta_{i+1} D_1^{u_i}. \end{aligned}$$

and

$$
\begin{aligned}
D_3^{x_1} \leq {} & \left[\zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i}\right] D_1^{u_1} + \left[2\zeta_1\zeta_2 + \sum_{i=3}^{n} \zeta_i\zeta_{n+3-i}\right] D_1^{u_2} + \cdots \\
& + \left[2\zeta_1\zeta_n + \sum_{i=1}^{n-1} \zeta_{i+1}\zeta_{n-i}\right] D_1^{u_n} \\
D_3^{u_2} \leq {} & \left[2\zeta_1\zeta_n + \sum_{i=1}^{n-1} \zeta_{i+1}\zeta_{n-i}\right] D_1^{u_1} + \left[\zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i}\right] D_1^{u_2} + \cdots \\
& + \left[\zeta_n^2 + \sum_{i=1}^{n-1} \zeta_i\zeta_{n-i}\right] D_1^{u_n} \\
& \vdots \\
D_3^{u_n} \leq {} & \left[2\zeta_1\zeta_2 + \sum_{i=3}^{n} \zeta_i\zeta_{n+3-i}\right] D_1^{u_1} + \left[2\zeta_1\zeta_3 + \zeta_2^2 + \sum_{i=0}^{n-4} \zeta_{i+4}\zeta_{n-i}\right] D_1^{u_2} + \cdots \\
& + \left[\zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i}\right] D_1^{u_n}.
\end{aligned}
$$

To simplify writing, we consider the matrix

$$
M = \begin{bmatrix}
\zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{n-2} & \zeta_{n-1} & \zeta_n \\
\zeta_n & \zeta_1 & \zeta_2 & \cdots & \zeta_{n-3} & \zeta_{n-2} & \zeta_{n-1} \\
\zeta_{n-1} & \zeta_n & \zeta_1 & \cdots & \zeta_{n-4} & \zeta_{n-3} & \zeta_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\zeta_4 & \zeta_5 & \zeta_6 & \cdots & \zeta_1 & \zeta_2 & \zeta_3 \\
\zeta_3 & \zeta_4 & \zeta_5 & \cdots & \zeta_n & \zeta_1 & \zeta_2 \\
\zeta_2 & \zeta_3 & \zeta_4 & \cdots & \zeta_{n-1} & \zeta_n & \zeta_1
\end{bmatrix}
$$

denoted by

$$
\begin{bmatrix}
k^1_{11} & k^1_{12} & k^1_{13} & \cdots & k^1_{1\overline{n-2}} & k^1_{1\overline{n-1}} & k^1_{1n} \\
k^1_{21} & k^1_{22} & k^1_{23} & \cdots & k^1_{2\overline{n-2}} & k^1_{2\overline{n-1}} & k^1_{2n} \\
k^1_{31} & k^1_{32} & k^1_{33} & \cdots & k^1_{3\overline{n-2}} & k^1_{3\overline{n-1}} & k^1_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
k^1_{\overline{n-2}1} & k^1_{\overline{n-2}2} & k^1_{\overline{n-2}3} & \cdots & k^1_{\overline{n-2}\,\overline{n-2}} & k^1_{\overline{n-2}\overline{n-1}} & k^1_{\overline{n-2}n} \\
k^1_{\overline{n-1}1} & k^1_{\overline{n-1}2} & k^1_{\overline{n-1}3} & \cdots & k^1_{\overline{n-1}\overline{n-2}} & k^1_{\overline{n-1}\,\overline{n-1}} & k^1_{\overline{n-1}n} \\
k^1_{n1} & k^1_{n2} & k^1_{n3} & \cdots & k^1_{n\overline{n-2}} & k^1_{n\overline{n-1}} & k^1_{nn}
\end{bmatrix}.
$$

Also denote $M^2 =$

$$
\begin{bmatrix}
\zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i} & 2\zeta_1\zeta_2 + \sum_{i=3}^{n} \zeta_i\zeta_{n+3-i} & \cdots & 2\zeta_1\zeta_n + \sum_{i=1}^{n-1} \zeta_{i+1}\zeta_{n-i} \\
2\zeta_1\zeta_n + \sum_{i=1}^{n-1} \zeta_{i+1}\zeta_{n-i} & \zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i} & \cdots & \zeta_n^2 + \sum_{i=1}^{n-1} \zeta_i\zeta_{n-i} \\
\vdots & \vdots & \ddots & \vdots \\
2\zeta_1\zeta_2 + \sum_{i=3}^{n} \zeta_i\zeta_{n+3-i} & 2\zeta_1\zeta_3 + \zeta_2^2 + \sum_{i=0}^{n-4} \zeta_{i+4}\zeta_{n-i} & \cdots & \zeta_1^2 + \sum_{i=0}^{n-2} \zeta_{i+2}\zeta_{n-i}
\end{bmatrix}
$$

by

$$\begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} & \dots & k^2_{1\overline{n-2}} & k^2_{1\overline{n-1}} & k^2_{1n} \\ k^2_{21} & k^2_{22} & k^2_{23} & \dots & k^2_{2\overline{n-2}} & k^2_{2\overline{n-1}} & k^2_{2n} \\ k^2_{31} & k^2_{32} & k^2_{33} & \dots & k^2_{3\overline{n-2}} & k^2_{3\overline{n-1}} & k^2_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ k^2_{\overline{n-2}1} & k^2_{\overline{n-2}2} & k^2_{\overline{n-2}3} & \dots & k^2_{\overline{n-2}\,\overline{n-2}} & k^2_{\overline{n-2}\overline{n-1}} & k^2_{\overline{n-2}n} \\ k^2_{\overline{n-1}1} & k^2_{\overline{n-1}2} & k^2_{\overline{n-1}3} & \dots & k^2_{\overline{n-1}\overline{n-2}} & k^2_{\overline{n-1}\,\overline{n-1}} & k^2_{\overline{n-1}n} \\ k^2_{n1} & k^2_{n2} & k^2_{n3} & \dots & k^2_{n\overline{n-2}} & k^2_{n\overline{n-1}} & k^2_{nn} \end{bmatrix}.$$

where

$$\begin{aligned} \sum_{i=1}^{n} k^2_{1i} = \sum_{i=1}^{n} k^2_{2i} = \dots = \sum_{i=1}^{n} k^2_{ni} \\ = \left[\sum_{i=1}^{n} \zeta_i\right]^2 < \sum_{i=1}^{n} \zeta_i < 1. \end{aligned}$$

Now we prove by induction on m that

$$M^m = \begin{bmatrix} k^m_{11} & k^m_{12} & k^m_{13} & \dots & k^m_{1\overline{n-2}} & k^m_{1\overline{n-1}} & k^m_{1n} \\ k^m_{21} & k^m_{22} & k^m_{23} & \dots & k^m_{2\overline{n-2}} & k^m_{2\overline{n-1}} & k^m_{2n} \\ k^m_{31} & k^m_{32} & k^m_{33} & \dots & k^m_{3\overline{n-2}} & k^m_{3\overline{n-1}} & k^m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ k^m_{\overline{n-2}1} & k^m_{\overline{n-2}2} & k^m_{\overline{n-2}3} & \dots & k^m_{\overline{n-2}\,\overline{n-2}} & k^m_{\overline{n-2}\overline{n-1}} & k^m_{\overline{n-2}n} \\ k^m_{\overline{n-1}1} & k^m_{\overline{n-1}2} & k^m_{\overline{n-1}3} & \dots & k^m_{\overline{n-1}\overline{n-2}} & k^m_{\overline{n-1}\,\overline{n-1}} & k^m_{\overline{n-1}n} \\ k^m_{n1} & k^m_{n2} & k^m_{n3} & \dots & k^m_{n\overline{n-2}} & k^m_{n\overline{n-1}} & k^m_{nn} \end{bmatrix},$$

where

$$\begin{aligned} \sum_{i=1}^{n} k^m_{1i} = \sum_{i=1}^{n} k^m_{2i} = \dots = \sum_{i=1}^{n} k^m_{ni} \\ = \left[\sum_{i=1}^{n} \zeta_i\right]^m < \sum_{i=1}^{n} \zeta_i < 1. \end{aligned} \tag{8}$$

Suppose (8) is true for m. Then

$$\begin{aligned} M^{m+1} &= M^m \cdot M \\ &= \begin{bmatrix} k^m_{11} & k^m_{12} & k^m_{13} & \dots & k^m_{1\overline{n-2}} & k^m_{1\overline{n-1}} & k^m_{1n} \\ k^m_{21} & k^m_{22} & k^m_{23} & \dots & k^m_{2\overline{n-2}} & k^m_{2\overline{n-1}} & k^m_{2n} \\ k^m_{31} & k^m_{32} & k^m_{33} & \dots & k^m_{3\overline{n-2}} & k^m_{3\overline{n-1}} & k^m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ k^m_{\overline{n-2}1} & k^m_{\overline{n-2}2} & k^m_{\overline{n-2}3} & \dots & k^m_{\overline{n-2}\,\overline{n-2}} & k^m_{\overline{n-2}\overline{n-1}} & k^m_{\overline{n-2}n} \\ k^m_{\overline{n-1}1} & k^m_{\overline{n-1}2} & k^m_{\overline{n-1}3} & \dots & k^m_{\overline{n-1}\overline{n-2}} & k^m_{\overline{n-1}\,\overline{n-1}} & k^m_{\overline{n-1}n} \\ k^m_{n1} & k^m_{n2} & k^m_{n3} & \dots & k^m_{n\overline{n-2}} & k^m_{n\overline{n-1}} & k^m_{nn} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \dots & \zeta_{n-2} & \zeta_{n-1} & \zeta_n \\ \zeta_n & \zeta_1 & \zeta_2 & \dots & \zeta_{n-3} & \zeta_{n-2} & \zeta_{n-1} \\ \zeta_{n-1} & \zeta_n & \zeta_1 & \dots & \zeta_{n-4} & \zeta_{n-3} & \zeta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \zeta_4 & \zeta_5 & \zeta_6 & \dots & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_3 & \zeta_4 & \zeta_5 & \dots & \zeta_n & \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 & \zeta_4 & \dots & \zeta_{n-1} & \zeta_n & \zeta_1 \end{bmatrix}. \end{aligned}$$

After multiplying above two matrices, we obtain

$$M^{m+1} = \begin{bmatrix} k_{11}^m\zeta_1 + \sum_{i=0}^{n-2} k_{1(i+2)}^m \zeta_{n-i} & k_{11}^m\zeta_2 + k_{12}^m\zeta_1 + \sum_{i=0}^{n-3} k_{1(i+3)}^m \zeta_{n-i} \\ \cdots & \sum_{i=0}^{n-1} k_{1(i+1)}^m \zeta_{n-i} \\ k_{21}^m\zeta_1 + \sum_{i=0}^{n-2} k_{2(i+2)}^m \zeta_{n-i} & k_{21}^m\zeta_2 + k_{22}^m\zeta_1 + \sum_{i=0}^{n-3} k_{2(i+3)}^m \zeta_{n-i} \\ \cdots & \sum_{i=0}^{n-1} k_{2(i+1)}^m \zeta_{n-i} \\ \vdots & \vdots \\ k_{n1}^m\zeta_1 + \sum_{i=0}^{n-2} k_{n(i+2)}^m \zeta_{n-i} & k_{n1}^m\zeta_2 + k_{n2}^m\zeta_1 + \sum_{i=0}^{n-3} k_{n(i+3)}^m \zeta_{n-i} \\ \cdots & \sum_{i=0}^{n-1} k_{n(i+1)}^m \zeta_{n-i} \end{bmatrix}$$

we have, from (8) that

$$\begin{aligned} \sum_{i=1}^{n} k_{1i}^{m+1} &= k_{11}^m\zeta_1 + \sum_{i=0}^{n-2} k_{1(i+2)}^m \zeta_{n-i} + k_{11}^m\zeta_2 + k_{12}^m\zeta_1 + \sum_{i=0}^{n-3} k_{1(i+3)}^m \zeta_{n-i} + \cdots \\ &\quad + \sum_{i=0}^{n-1} k_{1(i+1)}^m \zeta_{n-i} \\ &= k_{11}^m \left[\sum_{i=1}^{n} \zeta_i\right] + k_{12}^m \left[\sum_{i=1}^{n} \zeta_i\right] + k_{13}^m \left[\sum_{i=1}^{n} \zeta_i\right] + \cdots + k_{1n}^m \left[\sum_{i=1}^{n} \zeta_i\right] \\ &= \left[\sum_{i=1}^{n} \zeta_i\right]^n \left[\sum_{i=1}^{n} \xi_i\right] \\ &= \left[\sum_{i=1}^{n} \zeta_i\right]^{n+1} < \sum_{i=1}^{n} \zeta_i < 1. \end{aligned}$$

Similarly, we can prove

$$k_{21}^{k+1} + a_{22}^{k+1} + \sum_{i=3}^{n} a_{1i}^{k+1} = \cdots = a_{n1}^{k+1} + \sum_{i=2}^{n-1} a_{1i}^{k+1} + a_{nn}^{k+1} = \left[\sum_{i=1}^{n} \xi_i\right]^{n+1} < 1.$$

Therefore,

$$\begin{bmatrix} D_{m+1}^{u_1} \\ D_{m+1}^{u_2} \\ \vdots \\ D_{m+1}^{u_n} \end{bmatrix} \leq \begin{bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{n-2} & \zeta_{n-1} & \zeta_n \\ \zeta_n & \zeta_1 & \zeta_2 & \cdots & \zeta_{n-3} & \zeta_{n-2} & \zeta_{n-1} \\ \zeta_{n-1} & \zeta_n & \zeta_1 & \cdots & \zeta_{n-4} & \zeta_{n-3} & \zeta_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \zeta_4 & \zeta_5 & \zeta_6 & \cdots & \zeta_1 & \zeta_2 & \zeta_3 \\ \zeta_3 & \zeta_4 & \zeta_5 & \cdots & \zeta_n & \zeta_1 & \zeta_2 \\ \zeta_2 & \zeta_3 & \zeta_4 & \cdots & \zeta_{n-1} & \zeta_n & \zeta_1 \end{bmatrix}^m \cdot \begin{bmatrix} D_1^{u_1} \\ D_1^{u_2} \\ \vdots \\ D_1^{u_n} \end{bmatrix}.$$

That is

$$\left.\begin{aligned} D^{u_1}_{m+1} &\le \sum_{i=1}^{n} k^m_{1i} D^{u_i}_1, \\ D^{u_2}_{m+1} &\le \sum_{i=1}^{n} k^m_{2i} D^{u_i}_1 \\ D^{u_3}_{m+1} &\le \sum_{i=1}^{n} k^m_{3i} D^{u_i}_1 \\ &\vdots \\ D^{u_n}_{m+1} &\le \sum_{i=1}^{n} k^m_{ni} D^{u_i}_1 \end{aligned}\right\} \tag{9}$$

Next, we show that the sequences $\{u_{1m}\}, \{u_{2m}\}, \cdots, \{u_{nm}\}$ are Cauchy sequences. For $l > m$, we have from (9) that

$$\begin{aligned} d(u_{1l}, u_{1m}) &\le d(u_{1l}, u_{1\overline{l-1}}) + \cdots + d(u_{1\overline{m+1}}, u_{1m}) \\ &= D^{u_1}_l + D^{u_1}_{l-1} + \cdots + D^{u_1}_{m+1} \\ &\le \sum_{i=1}^{n} k^{l-1}_{1i} D^{u_i}_1 + \sum_{i=1}^{n} k^{l-2}_{1i} D^{u_i}_1 + \cdots + \sum_{i=1}^{n} k^m_{1i} D^{u_i}_1 \\ &\le (k^m_{11} + k^{m+1}_{11} + \cdots + k^{l-1}_{11}) D^{u_1}_1 + (k^m_{12} + k^{m+1}_{12} + \cdots + k^{l-1}_{12}) D^{u_2}_1 \\ &\quad \cdots + (k^m_{1n} + k^{m+1}_{1n} + \cdots + k^{l-1}_{1n}) D^{u_n}_1 \\ &\le (\vartheta^m + \vartheta^{m+1} + \cdots + \vartheta^{l-1}) D^{u_1}_1 + (\vartheta^m + \vartheta^{m+1} + \cdots + \vartheta^{l-1}) D^{u_2}_1 \\ &\quad \cdots + (\vartheta^m + \vartheta^{m+1} + \cdots + \vartheta^{l-1}) D^{u_n}_1 \\ &\le (\vartheta^m + \vartheta^{m+1} + \cdots + \vartheta^{l-1})(D^{u_1}_1 + D^{u_2}_1 + \cdots + D^{u_n}_1) \\ &\le \vartheta^k \frac{1 - \vartheta^{l-k}}{1 - \vartheta} (D^{x_1}_1 + D^{x_2}_1 + \cdots + D^{x_n}_1), \end{aligned}$$

where $\vartheta = \sum_{i=1}^{n} \zeta_i < 1$. Which shows that $\{u_{1m}\}$ is a Cauchy sequence. Similarly, we can show that $\{u_{2m}\}, \{u_{3m}\}, \cdots, \{u_{nm}\}$ are also Cauchy sequences. Since $\mathcal{B}$ is complete, there exist $u_1, u_2, \cdots, u_n \in \mathcal{B}$ such that, for $1 \le i \le n$,

$$\lim_{m\to\infty} u_{im} = u_i. \tag{10}$$

Finally, we show that

$$u_1 = \mathcal{A}(u_1, u_2, \cdots, u_n),\ u_2 = \mathcal{A}(u_2, u_3, \cdots, u_1), \cdots, u_n = \mathcal{A}(u_n, u_1, \cdots, u_{n-1}).$$

Let $\varepsilon > 0$. Since $\mathcal{A}$ is continuous at $(u_1, u_2, \cdots, u_n)$, there exists a $\delta > 0$ such that

$$\begin{aligned} d(u_1, v_1) + d(u_2, v_2) + \cdots + d(u_n, v_n) < \delta \implies& \\ d(\mathcal{A}(u_1, u_2, \cdots, u_n), \mathcal{A}(v_1, v_2, \cdots, v_n)) &< \frac{\varepsilon}{n}. \end{aligned}$$

Then by (10), it follows that, for $\delta^* = \min\{\frac{\delta}{n}, \frac{\varepsilon}{n}\}$, there exists $\eta_1, \eta_2, \cdots, \eta_n$ such that, for $\eta > \max\{\eta_1, \eta_2, \cdots, \eta_n\}$, we have $d(u_{im}, u_i) < \delta^*$ for $1 \le i \le n$. Now, take $m_0 = \max\{\eta_1, \eta_2, \cdots, \eta_n\}$. Then, for any integer $n \ge m_0$, we have

$$\begin{aligned} d(\mathcal{A}(u_1, u_2, \cdots, u_n), u_1) &\leq d(\mathcal{A}(u_1, u_2, \cdots, u_n), u_{1\overline{k+1}}) + d(u_{1\overline{k+1}}, u_1) \\ &= d(\mathcal{A}(u_1, u_2, \cdots, u_n), \mathcal{A}(u_{1k}, u_{2k}, \cdots, u_{nk})) \\ &\qquad + d(u_{1\overline{k+1}}, u_1) \\ &< \frac{\varepsilon}{n} + \delta^* \leq \varepsilon. \end{aligned}$$

Hence, $u_1 = \mathcal{A}(u_1, u_2, \cdots, u_n)$. In the same way we can show that

$$u_2 = \mathcal{A}(u_2, u_3, \cdots, u_1), \cdots, u_n = \mathcal{A}(u_n, u_1, \cdots, u_{n-1}).$$

□

In the next theorem, without taking $\mathcal{A}$ is continuous(Instead, we take additional property on $\mathcal{B}$), we prove above theorem.

Theorem 3.4. *Let $(\mathcal{B}, \leq)$ be a partially ordered set and suppose that there exists a metric d in $\mathcal{B}$ such that $(\mathcal{B}, d)$ is a complete metric space. Let $\mathcal{A} : \mathcal{B}^n \to \mathcal{B}$ be a mixed monotone mapping and assume that there exist the constants $\zeta_i \in [0, 1)$ with $\sum_{i=1}^{n} \zeta_i < 1$ such that*

$$d((u_1, u_2, \cdots, u_n), (v_1, v_2, \cdots, v_n)) \leq \sum_{i=1}^{n} \zeta_i d(u_i, v_i)$$

for every

$$\begin{cases} u_1 \geq v_1, u_2 \leq v_2, \cdots, u_n \leq v_n, & \text{if } n \text{ is even}, \\ u_1 \geq v_1, u_2 \leq v_2, \cdots, u_n \geq v_n, & \text{if } n \text{ is odd}. \end{cases}$$

Further, assume that $\mathcal{B}$ has the following properties:

(i) If a nondecreasing sequence $\{u_n\} \to u$, then $u_n \leq u$ for all n.

(ii) If there exist $u_{10}, u_{20}, \cdots, u_{n0} \in \mathcal{B}$ such that

$$\begin{aligned} u_{10} &\leq \mathcal{A}(u_{10}, u_{20}, \cdots, u_{n0}), \\ u_{20} &\leq \mathcal{A}(u_{20}, u_{30}, \cdots, u_{10}), \\ &\vdots \\ u_{n0} &\leq \mathcal{A}(u_{n0}, u_{10}, \cdots, u_{(n-1)0}), \end{aligned}$$

then there exist $u_1, u_2, \cdots, u_n \in \mathcal{B}$ such that

$$\begin{aligned} u_1 &= \mathcal{A}(u_1, u_2, \cdots, u_n), \\ u_2 &= \mathcal{A}(u_2, u_3, \cdots, u_1), \\ &\vdots \\ u_n &= \mathcal{A}(u_n, u_1, \cdots, u_{n-1}). \end{aligned}$$

Proof. Following the proof of Theorem 3.3, we only have to show that

$$u_1 = \mathcal{A}(u_1, u_2, \cdots, u_n),\ u_2 = \mathcal{A}(u_2, u_3, \cdots, u_1), \cdots, u_n = \mathcal{A}(u_n, u_1, \cdots, u_{n-1}).$$

Let $\varepsilon > 0$. Then by (10), we have

$$\begin{aligned}
&\lim_{m\to\infty} \mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}) = u_1,\\
&\lim_{m\to\infty} \mathcal{A}^m(u_{20}, u_{30}, \cdots, u_{10}) = u_2,\\
&\qquad\vdots\\
&\lim_{m\to\infty} \mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0}) = u_n.
\end{aligned}$$

So, there exists $m_1, m_2, \cdots, m_n \in \mathbb{N}$ such that for $m > \max\{m_1, m_2, \cdots, m_n\}$, we have

$$\begin{aligned}
&d(\mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}), u_1) < \frac{\varepsilon}{n+1}\\
&d(\mathcal{A}^m(u_{20}, u_{30}, \cdots, u_{n0}), u_2) < \frac{\varepsilon}{n+1}\\
&\qquad\vdots\\
&d(\mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0}), u_n) < \frac{\varepsilon}{n+1}
\end{aligned}$$

Using

$$\begin{aligned}
&\mathcal{A}^m(u_{10}, u_{20}, \cdots, xun0) \le u_1,\\
&\mathcal{A}^m(u_{20}, u_{30}, \cdots, u_{10}) \le u_2,\\
&\qquad\vdots\\
&\mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0}) \le u_n,
\end{aligned}$$

we have

$$\begin{aligned}
d(\mathcal{A}(u_1, u_2, \cdots, u_n), u_1) &\le d(\mathcal{A}(u_1, u_2, \cdots, u_n), \mathcal{A}^{m+1}(u_{10}, u_{20}, \cdots, u_{n0}))\\
&\qquad + d(\mathcal{A}^{m+1}(u_{10}, u_{20}, \cdots, u_{n0}), u_1)\\
&\le d(\mathcal{A}(u_1, u_2, \cdots, u_n), \mathcal{A}(\mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}),\\
&\quad \mathcal{A}^m(u_{20}, u_{30}, \cdots, u_{10}), \cdots, \mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0})))\\
&\qquad + d(\mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}), u_1)\\
&\le \zeta_1 d(u_1, \mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}))\\
&\quad + \zeta_2 d(u_2, \mathcal{A}^m(u_{20}, u_{30}, \cdots, u_{10}))\\
&\qquad\vdots\\
&\quad + \zeta_n d(u_n, \mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0}))\\
&\quad + d(\mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}), u_1)\\
&< (\zeta_1 + \zeta_2 + \cdots + \zeta_n)\frac{\varepsilon}{n+1} + \frac{\varepsilon}{n+1} < \varepsilon.
\end{aligned}$$

Which implies that $\mathcal{A}(u_1, u_2, \cdots, u_n) = u_1$. Similarly, we can show that $u_2 = \mathcal{A}(u_2, u_3, \cdots, u_1), \cdots, u_n = \mathcal{A}(u_n, u_1, \cdots, u_{n-1})$. □

Indeed, the n fixed point in Theorems 3.3 and 3.4 is in fact unique, if the product space $\mathcal{B}^n$ endowed with the partial order mentioned earlier have the following property:

- (B) For every $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n) \in \mathcal{B}^n$, there exits a $(u_1, u_2, \cdots, u_n) \in \mathcal{B}^n$ that is comparable to $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$.

Theorem 3.5. *In addition to the hypothesis of Theorem 3.3, assume that (B) holds. Then $\mathcal{A}$ has a unique n fixed point.*

Proof. If $(u_1^*, u_2^*, \cdots, u_n^*) \in \mathcal{B}$ is another n fixed point of $\mathcal{A}$, then we show that

$$d((u_1, u_2, \cdots, u_n), (u_1^*, u_2^*, \cdots, u_n^*)) = 0,$$

where

$$\begin{aligned}\lim_{m\to\infty} \mathcal{A}^m(u_{10}, u_{20}, \cdots, u_{n0}) &= u_1,\\ \lim_{m\to\infty} \mathcal{A}^k(u_{20}, u_{30}, \cdots, u_{10}) &= u_2,\\ &\vdots\\ \lim_{m\to\infty} \mathcal{A}^m(u_{n0}, u_{10}, \cdots, u_{(n-1)0}) &= u_n.\end{aligned}$$

Now we consider two cases:

Case 1: If $(u_1, u_2, \cdots, u_n)$ is comparable to $(u_1^*, u_2^*, \cdots, u_n^*)$ with respect to the ordering in $\mathcal{B}$, then, for every $n = 0, 1, 2, \cdots$,

$$\begin{aligned}(\mathcal{A}^m(u_1, u_2, \cdots, u_n), \mathcal{A}^m(u_2, u_3, \cdots, u_1), \cdots, \mathcal{A}^m(u_n, u_1, \cdots, u_{n-1}))\\ = (u_1, u_2, \cdots, u_n)\end{aligned}$$

is comparable to

$$\begin{aligned}(\mathcal{A}^m(u_1^*, u_2^*, \cdots, u_n^*), \mathcal{A}^m(u_2^*, u_3^*, \cdots, u_n^*), \cdots, \mathcal{A}^m(u_n^*, u_1^*, \cdots, u_{n-1}^*))\\ = (u_1^*, u_2^*, \cdots, u_n^*).\end{aligned}$$

and

$$\begin{aligned}d((u_1, u_2, \cdots, u_n), &(u_1^*, u_2^*, \cdots, u_n^*))\\ &= d(u_1, u_1^*) + d(u_2, u_2^*) + \cdots + d(u_n, u_n^*)\\ &= d(\mathcal{A}^m(u_1, u_2, \cdots, u_n), \mathcal{A}^m(u_1^*, u_2^*, \cdots, u_n^*))\\ &\quad + d(\mathcal{A}^m(u_2, u_3, \cdots, u_1), \mathcal{A}^m(u_2^*, u_3^*, \cdots, u_1^*))\\ &\qquad\vdots\\ &\quad + d(\mathcal{A}^m(u_n, u_1, \cdots, u_{n-1}), \mathcal{A}^m(u_n^*, u_1^*, \cdots, u_{n-1}^*))\\ &\le \vartheta^m[d(u_1, u_1^*) + d(u_2, u_2^*) + \cdots + d(u_n, u_n^*)]\\ &= \vartheta^m d((u_1, u_2, \cdots, u_n), (u_1^*, u_2^*, \cdots, u_n^*)),\end{aligned}$$

where $\vartheta = \sum_{i=1}^n \zeta_i < 1$. This shows that

$$d((u_1, u_2, \cdots, u_n), (u_1^*, u_2^*, \cdots, u_n^*)) = 0.$$

Case 2: If $(u_1, u_2, \cdots, u_n)$ is not comparable to $(u_1^*, u_2^*, \cdots, u_n^*)$ then there exists an upper or lower bound $(v_1, v_2, \cdots, v_n) \in \mathcal{B}^n$ of $(u_1, u_2, \cdots, u_n)$ and $(u_1^*, u_2^*, \cdots, u_n^*)$. Then, for all $m = 0, 1, \cdots$,

$$(\mathcal{A}^m(v_1, v_2, \cdots, v_n), \mathcal{A}^m(v_2, v_3, \cdots, v_1), \cdots, \mathcal{A}^m(v_n, v_1, \cdots, v_{n-1})$$

is comparable to

$$\begin{aligned}(\mathcal{A}^m(u_1, u_2, \cdots, u_n), \mathcal{A}^m(u_2, u_3, \cdots, u_1), \cdots, \mathcal{A}^m(u_n, u_1, \cdots, u_{n-1}))\\ = (u_1, u_2, \cdots, u_n)\end{aligned}$$

and to

$$\begin{aligned}(\mathcal{A}^m(u_1^*, u_2^*, \cdots, u_n^*), \mathcal{A}^m(u_2^*, u_3^*, \cdots, u_1^*), \cdots, \mathcal{A}^m(u_n^*, u_1^*, \cdots, u_{n-1}^*))\\ = (u_1^*, u_2^*, \cdots, u_n^*).\end{aligned}$$

We have,

$$\begin{aligned}
d((u_1,u_2,\cdots,u_n),&(u_1^*,u_2^*,\cdots,u_n^*))\\
&= d(u_1,u_1^*)+d(u_2,u_2^*)+\cdots+d(u_n,u_n^*)\\
&= d(\mathcal{A}^m(u_1,u_2,\cdots,u_n),\mathcal{A}^m(u_1^*,u_2^*,\cdots,u_n^*))\\
&\quad + d(\mathcal{A}^m(u_2,u_3,\cdots,u_1),\mathcal{A}^m(u_2^*,u_3^*,\cdots,u_1^*))\\
&\qquad\qquad\vdots\\
&\quad + d(\mathcal{A}^m(u_n,u_1,\cdots,u_{n-1}),\mathcal{A}^m(u_n^*,u_1^*,\cdots,u_{n-1}^*))\\
&\le d(\mathcal{A}^m(u_1,u_2,\cdots,u_n),\mathcal{A}^m(v_1,v_2,\cdots,v_n))\\
&\quad + d(\mathcal{A}^m(u_2,u_3,\cdots,u_1),\mathcal{A}^m(v_2,v_3,\cdots,v_1))\\
&\qquad\qquad\vdots\\
&\quad + d(\mathcal{A}^m(u_n,u_1,\cdots,u_{n-1}),\mathcal{A}^m(v_n,v_1,\cdots,v_{n-1}))\\
&\quad + d(\mathcal{A}^m(v_1,v_2,\cdots,v_n),\mathcal{A}^k(u_1^*,u_2^*,\cdots,u_n^*))\\
&\quad + d(\mathcal{A}^m(v_2,v_3,\cdots,v_1),\mathcal{A}^m(u_2^*,u_3^*,\cdots,u_1^*))\\
&\qquad\qquad\vdots\\
&\quad + d(\mathcal{A}^m(v_n,v_1,\cdots,v_{n-1}),\mathcal{A}^m(u_n^*,u_1^*,\cdots,u_{n-1}^*))\\
&\le \vartheta^k[d(u_1,v_1)+d(u_2,v_2)+\cdots+d(u_n,v_n)\\
&\quad + d(v_1,u_1^*)+d(v_2,u_2^*)+\cdots+d(v_n,u_n^*)]\\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

So that $d((u_1,u_2,\cdots,u_n),(u_1^*,u_2^*,\cdots,u_n^*))=0$. □

Following theorems are easy to prove. So, we omit details here.

Theorem 3.6. *In addition to the hypothesis of Theorem 3.3, suppose that every group of elements which contains n elements of $\mathcal{B}$ has an upper bound or a lower bound in $\mathcal{B}$. Then the components of the n fixed point are equal, i.e., $x_1 = x_2 = \cdots = x_n$.*

Theorem 3.7. *In addition to the hypothesis of Theorem 3.3 (or) Theorem 3.4), suppose that x_{10} is comparable with $x_{20},\cdots,x_{n0}$ in $\mathcal{B}$. Then the components of the n fixed point are equal, i.e., $u_1 = u_2 = \cdots = u_n$.*

4. Main results

In this section, we derive sufficient conditions for the existence of solution to the 3 dimensional $(p,q,r)-$Laplacian fractional order boundary value problem (1)-(2) as an application of Theorem (3.3). We note that $(\mathtt{u},\mathtt{v},\mathtt{w})\in\mathcal{B}^3$ is a solution of (1)-(2) if, and only if,

$$\begin{aligned}
\mathtt{u}(\mathtt{t}) &= \phi_{p'}\Big(\int_0^1 \mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{u}(\tau),\mathtt{v}(\tau),\mathtt{w}(\tau)\big)d\tau\Big),\\
\mathtt{v}(\mathtt{t}) &= \phi_{q'}\Big(\int_0^1 \mathcal{N}_\beta(t,\tau)\mathtt{f}\big(\tau,\mathtt{v}(\tau),\mathtt{w}(\tau),\mathtt{u}(\tau)\big)d\tau\Big),\\
\mathtt{w}(\mathtt{t}) &= \phi_{r'}\Big(\int_0^1 \mathcal{N}_\gamma(t,\tau)\mathtt{f}\big(\tau,\mathtt{w}(\tau),\mathtt{u}(\tau),\mathtt{v}(\tau)d\tau\Big),
\end{aligned}$$

where $0<t<1$.
Suppose the following holds:

($\mathcal{A}_1$) Denote $\|G\| = \max\{\|G\|_p, \|G\|_q, \|G\|_r\}$, where

$$\|G\|_\vartheta = \Big[\int_0^1 |\mathcal{N}(\tau,\tau)|d\tau\Big]^{\vartheta'},$$

and $\frac{1}{\vartheta} + \frac{1}{\vartheta'} = 1$.

($\mathcal{A}_2$) There exist some positive constants $l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2$ and n_3 such that

$$l_1 + l_2 + l_3 + m_1 + m_2 + m_3 + n_1 + n_2 + n_3 \leq \frac{1}{4\|G\|}$$

and for$(u, v, w), (x, y, z) \in X^3$ with $u \geq x, v \leq y, w \geq z$ implies

$$|f(\tau, u, v, w) - f(\tau, x, y, z)| \leq \big[l_1 d(u, x) + l_2 d(v, y) + l_3 d(w, z)\big]^{\frac{1}{p'}},$$
$$|g(\tau, v, w, u) - g(\tau, y, z, x)| \leq \big[m_1 d(v, y) + m_2 d(w, z) + m_3 d(u, x)\big]^{\frac{1}{q'}},$$
$$|h(\tau, w, u, v) - h(\tau, z, x, y)| \leq \big[n_1 d(w, z) + n_2 d(u, x) + n_3 d(v, y)\big]^{\frac{1}{r'}},$$

where

$$l_1 d(u, x) + l_2 d(v, y) + l_3 d(w, z) \leq 1,$$
$$m_1 d(v, y) + m_2 d(w, z) + m_3 d(u, x) \leq 1,$$

and

$$n_1 d(w, z) + n_2 d(u, x) + n_3 d(v, y) \leq 1.$$

($\mathcal{A}_3$) There exist $\mathtt{u}_0, \mathtt{v}_0, \mathtt{w}_0 \in C([0, 1])$ such that

$$\mathtt{u}_0(\mathtt{t}) \leq \phi_{p'}\Big(\int_0^1 \mathcal{N}_\alpha(t, \tau)\mathtt{f}\big(\tau, \mathtt{u}_0(\tau), \mathtt{v}_0(\tau), \mathtt{w}_0(\tau)\big)d\tau\Big),$$
$$\mathtt{v}_0(\mathtt{t}) \leq \phi_{q'}\Big(\int_0^1 \mathcal{N}_\beta(t, \tau)\mathtt{f}\big(\tau, \mathtt{v}_0(\tau), \mathtt{w}_0(\tau), \mathtt{u}_0(\tau)\big)d\tau\Big),$$
$$\mathtt{w}_0(\mathtt{t}) \leq \phi_{r'}\Big(\int_0^1 \mathcal{N}_\gamma(t, \tau)\mathtt{f}\big(\tau, \mathtt{w}_0(\tau), \mathtt{u}_0(\tau), \mathtt{v}_0(\tau)\big)d\tau\Big).$$

Theorem 4.1. *Suppose* ($\mathcal{A}_1$), ($\mathcal{A}_2$) *and* ($\mathcal{A}_3$) *are hold. Then the boundary value problem* (1)-(2) *has a solution.*

Proof. Let $\mathcal{B} = C[0, 1]$ be a partially ordered set such that for $u, v \in \mathcal{B}$,

$$u \leq v \iff u(t) \leq v(t) \quad \text{for all} \quad t \in [0, 1].$$

If $\mathcal{B}$ is endowed with the sup metric:

$$d(u, v) = \sup_{t\in[0,1]} |u(t) - v(t)|,$$

for all $u, v \in \mathcal{B}$. Then $(\mathcal{B}, d)$ is a complete metric space. We also introduce a generalization of the Bielecki metric [4, 19] by the relation

$$d_{\mathfrak{B}}(u, v) = \sup_{t\in[0,1]} \frac{|u(t) - v(t)|}{\sigma(t)},$$

where $\sigma : [0, 1] \to (0, \infty)$ is a nondecreasing continuous function. Define an operator $T : \mathcal{B}^3 \to \mathcal{B}$ by

$$Tu(t) = (T_\alpha \mathtt{u}(\mathtt{t}), T_\beta \mathtt{v}(\mathtt{t}), T_\gamma \mathtt{w}(\mathtt{t}))$$

where

$$T_\alpha \mathtt{u(t)} = \phi_{p'}\Big(\int_0^1 \mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{u}(\tau),\mathtt{v}(\tau),\mathtt{w}(\tau)\big)d\tau\Big),$$
$$T_\beta \mathtt{v(t)} = \phi_{q'}\Big(\int_0^1 \mathcal{N}_\beta(t,\tau)\mathtt{g}\big(\tau,\mathtt{v}(\tau),\mathtt{w}(\tau),\mathtt{u}(\tau)\big)d\tau\Big),$$
$$T_\gamma \mathtt{w(t)} = \phi_{r'}\Big(\int_0^1 \mathcal{N}_\gamma(t,\tau)\mathtt{h}\big(\tau,\mathtt{w}(\tau),\mathtt{u}(\tau),\mathtt{v}(\tau)\big)d\tau\Big),$$

For any $u = (\mathtt{u},\mathtt{v},\mathtt{w}), x = (\mathtt{x},\mathtt{y},\mathtt{z}) \in X^3$ with $\mathtt{u} \geq \mathtt{x}, \mathtt{v} \leq \mathtt{y}$, $\mathtt{w} \geq \mathtt{z}$ and $t \in [0,1)$, we have

$$\begin{aligned}
|T_\alpha\mathtt{u(t)} - T_\alpha\mathtt{x(t)}| &\leq \Big|\phi_{p'}\Big(\int_0^1 \mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{u}(\tau),\mathtt{v}(\tau),\mathtt{w}(\tau)\big)d\tau\Big) \\
&\qquad - \phi_{p'}\Big(\int_0^1 \mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{x}(\tau),\mathtt{y}(\tau),\mathtt{z}(\tau)\big)d\tau\Big)\Big| \\
&\leq \Big[\int_0^1 |\mathcal{N}(\tau,\tau)|\big|\mathtt{f}(\tau,\mathtt{u}(\tau),\mathtt{v}(\tau),\mathtt{w}(\tau)) - \mathtt{f}(\tau,\mathtt{x}(\tau),\mathtt{y}(\tau),\mathtt{z}(\tau))\big|d\tau\Big]^{p'} \\
&\leq \Big[\int_0^1 |\mathcal{N}(\tau,\tau)|\big[l_1 d(\mathtt{u},\mathtt{x}) + l_2 d(\mathtt{v},\mathtt{y}) + l_3 d(\mathtt{w},\mathtt{z})\big]^{\frac{1}{p'}} d\tau\Big]^{p'} \\
&\leq \|G\|_p[l_1 d(\mathtt{u},\mathtt{x}) + l_2 d(\mathtt{v},\mathtt{y}) + l_3 d(\mathtt{w},\mathtt{z})].
\end{aligned}$$

Similarly,

$$|T_\beta\mathtt{v(t)} - T_\beta\mathtt{y(t)}| \leq \|G\|_q[m_1 d(\mathtt{v},\mathtt{y}) + m_2 d(\mathtt{w},\mathtt{z}) + m_3 d(\mathtt{u},\mathtt{x})],$$
$$|T_\gamma\mathtt{w(t)} - T_\gamma\mathtt{z(t)}| \leq \|G\|_r[n_1 d(\mathtt{w},\mathtt{z}) + n_2 d(\mathtt{u},\mathtt{x}) + n_3 d(\mathtt{v},\mathtt{y})].$$

so,

$$\begin{aligned}
|Tu(t) - Tx(t)| &= |T_\alpha\mathtt{u(t)} - T_\alpha\mathtt{x(t)}| + |T_\beta\mathtt{v(t)} - T_\beta\mathtt{y(t)}| + |T_\gamma\mathtt{w(t)} - T_\gamma\mathtt{z(t)}| \\
&\leq \|G\|\big[(l_1 + m_3 + n_2)d(\mathtt{u},\mathtt{x}) + (l_2 + m_1 + n_3)d(\mathtt{v},\mathtt{y}) \\
&\qquad + (l_3 + m_2 + n_1)d(\mathtt{w},\mathtt{z})\big] \\
&\leq \|G\|[l_1 + l_2 + l_3 + m_1 + m_2 + m_3 + n_1 + n_2 + n_3] \\
&\qquad \times [d(\mathtt{u},\mathtt{x}) + d(\mathtt{v},\mathtt{y}) + d(\mathtt{w},\mathtt{z})] \\
&\leq \frac{1}{4}\sigma(t)\big[d_{\mathfrak{B}}(\mathtt{u},\mathtt{x}) + d_{\mathfrak{B}}(\mathtt{v},\mathtt{y}) + d_{\mathfrak{B}}(\mathtt{w},\mathtt{z})\big]
\end{aligned}$$

Thus,

$$\frac{|Tu(t) - Tx(t)|}{\sigma(t)} \leq \frac{1}{4}d_{\mathfrak{B}}(\mathtt{u},\mathtt{x}) + \frac{1}{4}d_{\mathfrak{B}}(\mathtt{v},\mathtt{y}) + \frac{1}{4}d_{\mathfrak{B}}(\mathtt{w},\mathtt{z}).$$

That is,

$$d_{\mathfrak{B}}(u,x) \leq \frac{1}{4}d_{\mathfrak{B}}(\mathtt{u},\mathtt{x}) + \frac{1}{4}d_{\mathfrak{B}}(\mathtt{v},\mathtt{y}) + \frac{1}{4}d_{\mathfrak{B}}(\mathtt{w},\mathtt{z}).$$

Hence, by Theorem 4.1 , the boundary value problem has a solution

$$u^* = (u^*(t), v^*(t), w^*(t)) \in (C[0,1])^3.$$

□

Example 4.1. Consider the following system of fractional order differential equation, for $0 < t < 1$,

$$\left.\begin{aligned}
{}^{C}\mathfrak{D}^{\alpha}_{0^+}\big(\phi_p(\mathtt{u}(t))\big) + \mathtt{f}\big(t,\mathtt{u}(t),\mathtt{v}(t),\mathtt{w}(t)\big) = 0 \\
{}^{C}\mathfrak{D}^{\beta}_{0^+}\big(\phi_q(\mathtt{v}(t))\big) + \mathtt{g}\big(t,\mathtt{v}(t),\mathtt{w}(t),\mathtt{u}(t)\big) = 0 \\
{}^{C}\mathfrak{D}^{\gamma}_{0^+}\big(\phi_r(\mathtt{w}(t))\big) + \mathtt{h}\big(t,\mathtt{w}(t),\mathtt{u}(t),\mathtt{v}(t)\big) = 0
\end{aligned}\right\} \tag{11}$$

satisfying the boundary conditions,

$$\left.\begin{aligned} a_1(\phi_p\mathtt{u})(0)-b_1(\phi_p\mathtt{u})'(0)=0,\quad c_1(\phi_p\mathtt{u})(1)+d_1(\phi_p\mathtt{u})'(1)=0,\\ a_2(\phi_q\mathtt{v})(0)-b_2(\phi_q\mathtt{v})'(0)=0,\quad c_2(\phi_q\mathtt{v})(1)+d_2(\phi_q\mathtt{v})'(1)=0,\\ a_3(\phi_r\mathtt{w})(0)-b_3(\phi_r\mathtt{w})'(0)=0,\quad c_3(\phi_r\mathtt{w})(1)+d_3(\phi_r\mathtt{w})'(1)=0.\end{aligned}\right\} \tag{12}$$

Where $\alpha=2,\ \beta=2,\ \gamma=2,\quad a_i=0,\ b_i=c_i=d_i=1\quad$ for $i=1,2,3$,

$$f(t,\mathtt{u},\mathtt{v},\mathtt{w})=\frac{1}{450}(1-t)^{\frac{1}{3}}\left[\frac{1}{1+\mathtt{u}}+\mathtt{v}+\mathtt{w}\right],$$

$$g(t,\mathtt{v},\mathtt{w},\mathtt{u})=\frac{1}{25}(1-t)\left[\frac{1}{1+\mathtt{v}}+\mathtt{w}+\mathtt{u}\right],$$

and

$$h(t,\mathtt{w},\mathtt{u},\mathtt{v})=\frac{1}{450}t(1-t)^{\frac{1}{3}}\left[\frac{1}{1+\mathtt{w}}+\mathtt{u}+\mathtt{v}\right].$$

Now, let $(\mathtt{u},\mathtt{v},\mathtt{w}),(\mathtt{x},\mathtt{y},\mathtt{z})\in\mathcal{B}^3$ with $\mathtt{u}\geq\mathtt{x},\mathtt{v}\leq\mathtt{y},\mathtt{w}\geq\mathtt{z}$ implies

$$\begin{aligned}|f(t,\mathtt{u},\mathtt{v},\mathtt{w})-f(t,\mathtt{x},\mathtt{y},\mathtt{z})| &\leq \frac{1}{450}|(1-t)^{1/3}|\left|\frac{1}{1+\mathtt{u}}-\frac{1}{1+\mathtt{x}}\right|+\frac{1}{450}|(1-t)^{1/3}||\mathtt{v}-\mathtt{y}|\\ &\quad+\frac{1}{450}|(1-t)^{1/3}||\mathtt{w}-\mathtt{z}|\\ &\leq l_1d(\mathtt{u},\mathtt{x})+l_2d(\mathtt{v},\mathtt{y})+l_3d(\mathtt{w},\mathtt{z})\\ &\leq\big[l_1d(\mathtt{u},\mathtt{x})+l_2d(\mathtt{v},\mathtt{y})+l_3d(\mathtt{w},\mathtt{z})\big]^{p'}.\end{aligned}$$

Similarly,

$$\begin{aligned}|g(t,\mathtt{v},\mathtt{w},\mathtt{u})-g(t,\mathtt{y},\mathtt{z},\mathtt{x})| &\leq\big[m_1d(\mathtt{v},\mathtt{y})+m_2d(\mathtt{w},\mathtt{z})+m_3d(\mathtt{u},\mathtt{x})\big]^{q'}\\ |h(t,\mathtt{w},\mathtt{u},\mathtt{v})-h(t,\mathtt{z},\mathtt{x},\mathtt{y})| &\leq\big[n_1d(\mathtt{w},\mathtt{z})+m_2d(\mathtt{u},\mathtt{x})+m_3d(\mathtt{v},\mathtt{y})\big]^{r'}.\end{aligned}$$

Let we set $p=r=4,\ q=5$. Then $p'=r'=\frac{4}{3}$,$q'=\frac{5}{4}$. After certain calculations, we get $\|G\|=1.7170$ and

$$l_1+l_2+l_3+m_1+m_2+m_3+n_1+n_2+n_3=0.1333\leq\frac{1}{4\|G\|}=0.4156.$$

Set $u_0=0,v_0=\frac{1}{40},w_0=0$ Then

$$\begin{aligned}\mathtt{u}_0(\mathtt{t})&=\phi_{p'}\Big(\int_0^1\mathcal{N}_\alpha(t,\tau)\mathtt{f}\big(\tau,\mathtt{u}_0(\tau),\mathtt{v}_0(\tau),\mathtt{w}_0(\tau)\big)d\tau\Big)\\ &\leq\phi_{p'}\Big(\int_0^1\mathcal{N}(\tau,\tau)\mathtt{f}\big(\tau,\mathtt{u}_0(\tau),\mathtt{v}_0(\tau),\mathtt{w}_0(\tau)\big)d\tau\Big)\\ &\leq 0.00268.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathtt{v}_0(\mathtt{t})&=\phi_{q'}\Big(\int_0^1\mathcal{N}_\alpha(t,\tau)\mathtt{g}\big(\tau,\mathtt{v}_0(\tau),\mathtt{u}_0(\tau),\mathtt{w}_0(\tau)\big)d\tau\Big)\\ &\leq\phi_{q'}\Big(\int_0^1\mathcal{N}(\tau,\tau)\mathtt{g}\big(\tau,\mathtt{v}_0(\tau),\mathtt{w}_0(\tau),\mathtt{u}_0(\tau)\big)d\tau\Big)\\ &\leq 0.0325.\\ \mathtt{w}_0(\mathtt{t})&=\phi_{r'}\Big(\int_0^1\mathcal{N}_\gamma(t,\tau)\mathtt{h}\big(\tau,\mathtt{w}_0(\tau),\mathtt{v}_0(\tau),\mathtt{u}_0(\tau)\big)d\tau\Big)\\ &\leq\phi_{r'}\Big(\int_0^1\mathcal{N}(\tau,\tau)\mathtt{h}\big(\tau,\mathtt{w}_0(\tau),\mathtt{u}_0(\tau),\mathtt{v}_0(\tau)\big)d\tau\Big)\\ &\leq 0.001023.\end{aligned}$$

All the conditions of Theorem 4.1 are satisfied. Hence, from Theorem 4.1, there exist a solution for (11)-(12).

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