İstanbul Üniv. Fen Fak. Mat. Dergisi 55 - 56 (1996 - 1997), 43-49

SEMI PSEUDO SYMMETRIC MANIFOLD

M. Tarafdar and A. Mayra

ABSTRACT : In the present paper, the question whether a semi Pseudo Symmetric Manifold may be a P-Sasakian or nearly Sasakian manifold has been answered in the negative.

INTRODUCTION

In a recent paper [4] M. Tarafdar and Musa A. A. Jawarneh introduced semi Pseudo Symmetric Manifold $(SPS)_n$ i.e. a non-flat n-dimensional Riemannian manifold M^n $(n > 3)$ whose curvature tensor R satisfies the condition

1) $(\nabla_X R)(Y, Z)W = 2\pi(X)R(Y, Z)W + \pi(Y)R(X, Z)W + \pi(Z)R(Y, X)W + \pi(W)R(Y, Z)X$

where π is a non-zero 1-form.

2) $g(X, P) = \pi(X)$

for every vector field X and ∇ denotes the operator of covariant diffrentiation with respect to the metric g. Such a manifold shall be called a semi Pseudo Symmetric Manifold and the 1-form π shall be called its associated 1-form. An n-dimensional semi Pseudo Symmetric Manifold shall be denoted by $(SPS)_n$.

In the present paper the question whether a semi Pseudo Symmetric Manifold may be a P-Sasakian or nearly Sasakian manifold has been answered in the negative.

1. PRELIMINARIES

In this section we first consider some formulas which hold in a $(SPS)_n$. Let r denote the scalar curvature and L denote the symmetric endomorphism of the tangent space at each point of a Riemannian manifold (M^n, g) corresponding to the Ricci tensor S i.e.

1.1) $g(LX, Y) = S(X, Y)$

for any vector fields X, Y. From 1) we get

1.2) $(\nabla_X S)(Y, Z) = 2\pi(X)S(Y, Z) + \pi(Y)S(X, Z) + \pi(Z)S(Y, X) + \pi(R(X, Y) Z)$

Contracting 1.2) we get

1.3) $dr(X) = 2\pi(X)r + 3\pi(LX)$ where r denotes the scalar curvature of M_n and L has the meaning already defined by 1.1).

2. SEMI PSEUDO SYMMETRIC P-SASAKIAN MANIFOLD

In this section we suppose that an n-dimensional $(SPS)_n$ $(n > 3)$ is a P-Sasakian manifold.

Let (M, g) be an n-dimensional Riemannian manifold admitting a 1-form η, a vector field ξ and an (1-1) tensor field ϕ which satisfy the following conditions

2.1) $(\nabla_X\eta)Y - (\nabla_Y\eta)(X) = 0$

2.2) $(\nabla_X\nabla_Y \eta)(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)$

2.3) $g(X, \xi) = \eta(X)$ for all vector fields X

2.4) $\eta(\xi) = 1$

2.5) $\nabla_X \xi = \phi X$

Such a manifold is called a Para-Sasakian manifold or briefly a P-Sasakian manifold [3]. It is known that in a P-Sasakian manifold besides 2.1) - 2.5) the following relations hold

2.6) $\phi\xi = 0$

2.7) $R(\xi, X)Y = -g(X, Y)\,\xi + \eta(Y)X$

2.8) $S(X, \xi) = -(n-1)\eta(X)$

2.9) $g(\phi X, Y) = g(X, \phi Y)$

2.10) $S(\phi X, Y) = S(X, \phi Y)$

2.11) $(\nabla_X\eta)Y = g(\phi X, Y)$

Now

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi)$$

Using 2.5), 2.8) and 2.11 the above equation reduces to

2.11) $(\nabla_X S)(Y, \xi) = -(n-1)g(\phi X, Y) - S(Y, \phi X)$

Taking $Z = \xi$ in 1.2) and using 2.8) we get

2.12) $(\nabla_X S)(Y, \xi) = -2(n-1)\,\pi(X)\eta(Y) - (n-1)\pi(Y)\eta(X) + \pi(\xi)S(Y, X) + \pi(R(X, Y)\xi)$

Again

$$\pi(R(X, Y)\xi) = g(R(X, Y)\xi, P) = g(R(\xi, Y)\,X - R(\xi, X)\,Y, P)$$

Using 2.7) we find

2.13) $\pi(R(X, Y)\xi) = \eta(X)\pi(Y) - \eta(Y)\pi(X)$

Thus 2.12) reduces to on using 2.13)

2.14) $(\nabla_X S)(Y, \xi) = (-2n + 1)\pi(X)\eta(Y) - (n-2)\pi(Y)\eta(X) + \eta(P)S(X, Y)$

From 2.11) and 2.14) we get

$(-2n + 1)\pi(X)\eta(Y) - (n-2)\pi(Y)\eta(X) + \eta(P)S(X, Y) = -(n-1)g(\phi X, Y) - S(Y, \phi X)$

Taking $X = \xi$ and using 2.4) and 2.6) we find

2.15) $(-3n + 2)\eta(P)\eta(Y) - (n-2)\pi(Y) = 0.$

Finally taking $Y = \xi$ in above we get

$\eta(P) = 0$ as $n > 3$

Hence from 2.15) we find

$\pi(Y) = 0$

which is inadmissible by the definition of $(SPS)_n$. Thus we state

THEOREM 1 : A $(SPS)_n$ $(n > 3)$ cannot be a **P**-Sasakian Manifold.

3. SEMI PSEUDO SYMMETRIC NEARLY SASAKIAN MANIFOLD

In this section we suppose that an n-dimensional $(SPS)_n$ $(n > 3)$ is a nearly Sasakian manifold. Let (M, g) be an n-dimensional differential manifold ($n = 2m + 1$, $m > 1$) with almost contact metric structure (ϕ, ξ, η, g). If in such a manifold the following relation hold

3.1) $(\nabla_X\phi) + (\nabla_Y\phi) = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X$

then the manifold is said to be nearly Sasakian [1]. It is known that in a nearly Sasakian manifold the following relations hold [1] :

3.2) $\phi\xi = 0$

3.3) $\eta(\xi) = 1$

3.4) $\phi^2 X = -X + \eta(X)\xi$

3.5) $\nabla_X \xi = -\phi X$

3.6) $S(X, \xi) = (n-1)\eta(X)$

3.7) $(\nabla_X \eta)Y = g(\phi X, Y)$

3.8) $R(X, \xi)\xi = X - \eta(X)\xi$

3.9) $R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X$

Using 3.5), 3.6) and 3.7) we find that

3.10) $(\nabla_X S)(Y, \xi) = (n-1)g(\phi X, Y) + S(Y, \phi X)$

Taking $Z = \xi$ in 1.2) and using 3.6) we get

3.11) $(\nabla_X S)(Y, \xi) = 2(n-1)\pi(X)\eta(Y) + (n-1)\pi(Y)\eta(X) + \eta(P)S(X, Y) + \pi(R(X, Y)\xi)$

Again, on using 3.9) we find

3.12) $\pi(R(X, Y)\xi) = \eta(Y)\pi(X) - \eta(X)\pi(Y)$

Thus 3.11) reduces to

3.13) $(\nabla_X S)(Y, \xi) = (2n - 1)\pi(X)\eta(Y) + (n - 2)\eta(X)\pi(Y) + \eta(P)S(X, Y)$

From 3.10) and 3.13) we get

3.14) $(n - 1)g(\phi X, Y)\, S(Y, \phi X) = (2n - 1)\pi(X)\eta(Y) + (n - 2)\eta(X)\pi(Y) + \eta(P)S(X, Y)$

Taking $X = \xi$ in 3.14) and using 3.2) and 3.3) we find

3.15) $(3n - 2)\eta(P)\eta(Y) + (n - 2)\pi(Y) = 0$

Finally taking $Y = \xi$ in 3.15) we get

$$\eta(P) = 0 \text{ as } n > 3$$

Hence from 3.15) we find

$$\pi(Y) = 0$$

which is inadmissible by the definition of $(SPS)_n$. Thus we state

THEOREM 2 : A $(SPS)_n$ $(n > 3)$ cannot be a nearly Sasakian Manifold.

REFERENCES

[1] Blair, D.E.; Showers, D.K.; Yano, K. : Nearly Sasakian Structure, Kodai Math. Sem. Rep. 27 (1976), 175-180.

[2] Chaki, M.C. : On Pseudo Symmetrie Manifolds, Ann. St. Univ. "AI. I. CUZA." Iasi, XXXIII (1987), 54-58.

[3] Sato, I. and Matsumoto, K. : On P-Sasakian Manifold Satisfying Certain Conditions, Tensor N.S., Vol. 33, 1979, 173-178.

[4] Tarafdar, M. and Jawarneh, A.A. Musa : Semi Pseudo Symmetric Manifold, Ann. St Univ. "AI. I. Cuza.", lasi Tomul XLI (1995), 145-152.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35, BALLYGUNGE CIRCULAR ROAD
CALCUTTA 700 019
INDIA

E-MAIL : manjusha@cubmb.ernet.in