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A generalization of the connections of Schouten and Vranceanu on the f(2v+3, ε)-varieties

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Abstract. In this paper, we generalize the classic connections of Schouten and Vranceanu by adding a certain tensor of type $(1,2)$, on the differentiable varieties of a tensorial field equipped with a tensorial field f of type $(1,1)$ which satisfies the condition $f^{2v+3} + \varepsilon f = 0$, $\varepsilon = \pm 1$. We put in evidence the relation between the integrability of the f-structure and the generalized connections.

INTRODUCTION

The classic connections of Schouten and Vranceanu have been studied for the first time under invariant form in a few articles of S. Ianus and I. Popovici [2],[3]. They are about the linear connections on the varieties almost product, closely tied in structure from the point of view of integrability, of the parallelism of distributions and of geodesics.

The two types of connections have been examined and generalized for the canonical structure almost product induced on a f-variety by K.D. Singh and R.K. Vohra [5], G. Pripoae [4] and D. Demetropoulou- Psomopoulou [1].

1. THE $f(2v+3, \varepsilon)$-VARIETIES

Let M be an m-dimensional differentiable variety: $C(M)$ the affine modul of the linear connections on M; $\Im_s{}^r(M)$ the modul of the tensors of type (r,s): for $\Im_0{}^1(M)$ it is used the notation $\aleph(M)$. The geometric objects contemplated from now on will always be supposed of class C^∞.

We suppose that M is a $f(2v+3, \varepsilon)$-variety , therefore there exists $f \in \Im_1{}^1(M)$ so that

$$f^{2v+3} + \varepsilon f = 0 \quad \text{, where } \varepsilon = \pm 1.$$

By noting [1]: $l = - \varepsilon f^{2v+2}$, $m = \varepsilon f^{2v+2} + I$, where I is the identity tensor ,we have

$$l + m = I,\ lm = ml = 0,\ l^2 = l,\ m^2 = m,\ lf = fl = f,\ fm = mf = 0.$$

Namely $P = -2l+I$. It results $P^2 = I$, then P is a structure almost product on M. It defines two complementary global distributions (it uses the projectors l and m) which will be noted by D and D'.

In what follows $\nabla \in C(M)$, $B, D \in \Im_2{}^1(M)$ will be fixed.

The Shouten connection [1], [2] is the connection ∇^s defined by

$$\nabla_X{}^s = m\nabla_X mY + l\nabla_X lY$$

and the Vranceanu connection [1], [2] is the connection ∇^v definied by

$$\nabla_X{}^v Y = m\nabla_{mX} mY + l\nabla_{lX} lY + m[lX, mY] + l[mX, lY].$$

Definition. It is called generalized Schouten connection the linear connection $\overline{\nabla}$ defined by

(1.1) $\qquad \overline{\nabla}_X Y = \nabla_X{}^s Y + m\,B(X, mY) + l\,B(X, lY).$

The generalized Vranceanu connection $\widetilde{\nabla}$ is defined by

(1.2) $\qquad \widetilde{\nabla}_X Y = \nabla_X{}^v Y + m[lX, mY] + l[mX, lY] + m\,D(mX, mY) + l\,D(lX, lY)$

Remark.

Let $p: \Im_2{}^1(M) \to \Im_2{}^1(M)$, $\quad p(B) = \overline{B}$, where

$$\overline{B}\,(X,Y) = m\,B(X, mY) + l\,B(X, lY)$$

and $\overline{p}_B : C(M) \to C(M)$, $\overline{p}_B(\nabla) = \overline{\nabla} = \nabla^s + p(B)$.

At the same time, let $q : \Im_2{}^1(M) \to \Im_2{}^1(M)$ he thus $q(D) = \overline{D}$, where

$$\overline{D}\,(X,Y) = m\,D(mX, mY) + l\,D(lX, lY)$$

and $\widetilde{p}_D : C(M) \to C(M)$, $\widetilde{p}_D(\nabla) = \widetilde{\nabla} = \nabla^v + q(D)$.

It is verified then that p and q are vectorial projectors on $\aleph(M)$.and $\overline{p}_B$ and $\widetilde{p}_D$ are affine projectors:

$$p^2 = p,\ q^2 = q,\ \overline{p}^2{}_B = \overline{p}_B,\ \widetilde{p}^2{}_D = \widetilde{p}_D.$$

We have the relations:

a) $\overline{p}_B \circ \widetilde{p}_D = \widetilde{p}_D$, if $m\,B(X, mY) + l\,B(X, lY) = 0$;

b) $\overline{p}_B \circ \widetilde{p}_D = \overline{p}_B$, if $m[lX, mY] + m\,D(mX, mY) + l\,[mX, lY] + l\,D(lX, lY) = m\nabla_{lX} mY + l\nabla_{mX} lY$

c) $\widetilde{p}_D \circ \overline{p}_B = \overline{p}_B$, if $m[lX, mY] + l[mX, lY] + m\,D(mX, mY) + l\,D(lX, lY) = m\nabla_{lX} mY + l\nabla_{mX} lY + m\,B(lX, mY) + l\,B(mX, lY)$

d) $\widetilde{p}_D \circ \overline{p}_B = \widetilde{p}_D$, if $m\,B(mX, mY) + l\,B(lX, lY) = 0$, for all $X, Y \in \aleph(M)$.

2. PARALLELISM OF THE DISTRIBUTIONS *D* AND *D'*

A distribution S is called parallel in relation to ∇ if for all $X \in \aleph(M)$ and $Y \in S$, we have $\nabla_X Y \in S$.

<u>**Theorem 2.1.**</u> On a $f(2\nu + 3, \varepsilon)$-variety M, the distributions D and D'are parallel in relation to $\overline{\nabla}$ and $\widetilde{\nabla}$.

<u>**Demonstration:**</u> Since $ml = 0$, taking account of (1.1) and (1.2), we have

$$m\overline{\nabla}_X Y = 0,\ m\widetilde{\nabla}_X Y = 0,\ \text{for every } X \in \aleph(M), Y \in D.$$

In consequence, D is parallel in relation to $\overline{\nabla}$ and $\widetilde{\nabla}$.

It is the same for the distribution D'.

<u>Therem 2.2</u>. On a $f(2\nu + 3, \varepsilon)$-variety M, the tensorial fields of type $(1,1)$ l and m are constant covariants in relation to $\overline{\nabla}$ and $\widetilde{\nabla}$.

We have $(\overline{\nabla}_X \ell)(Y) = (\overline{\nabla}_X)(\ell Y) - \ell(\nabla_X Y)$. Then, from (1.1), it results:

$$(\overline{\nabla}_X l)(Y) = 0, \text{ for all } X, Y \in \aleph(M).,$$

We establish thus

$$\overline{\nabla}_X m = 0,\ \widetilde{\nabla}_X l = 0,\ \widetilde{\nabla}_X m = 0.$$

3. THE INTEGRABILITY OF THE $f(2\nu + 3, \varepsilon)$-STRUCTURE

We have from [1] that the distributions D and D'are integrable if

(3.1) $\qquad l[mX, mY] = 0$, respectively $m[lX, lY] = 0$.

<u>**Theorem 3.1.**</u> If the connection ∇ is symmetric, the tensorial field D is symmetric and the distributions D and D 'are integrable, then the generalized Vranceanu connection is symmetric.

<u>**Proof.**</u> Since the connection ∇ is symmetric, the correspondent tensor of torsion

(3.2) $\qquad T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

The tensor of torsion of the connection $\widetilde{\nabla}$ is

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y].$$

By a simple computation, taking account of (3.2) it is deduced

$$\tilde{T}\ (X,Y) = 1[mX,mY]\text{-}m[1X,1Y]\text{-}m\ D(mX,mY)\text{-}1\ D(1X,1Y) +$$
$$+\ m\ D(mY,mX) + 1\ D(1Y,1X).$$

D being a symmetric tensorial field, (3.1) proves the theorem.

Theorem 3.2. If the connection ∇ is symmetric, the tensorial field B is symmetric and one of the connections $\overline{\nabla}$ or $\tilde{\nabla}$ is symmetric, then the distributions D and D'are integrable.

Proof. If the connections $\nabla$and $\overline{\nabla}$ are symmetric, then for tensors of torsion , we have $T(X,Y) = 0 = \overline{T}\ (X,Y)$ and therefore $T(1X,1Y) = \overline{T}\ (1X,1Y) = 0,$ from where we obtain

$$\nabla_{1X}1Y - \nabla_{1Y}1X.- [1X,1Y] = 1(\nabla_{1X}1Y - \nabla_{1Y}1X.- [1X,1Y]) +$$
$$+\ m(\ B(1X,1Y) - B(1Y,1X) + m[1X,1Y] = 0,$$

that is to say $m[1X,1Y] = 0.$

Analogously, from $T(mX,mY) = \overline{T}\ (mX,mY) = 0$, it is obtained $1[mX,mY] = 0$ and thus both disributions are integrable.

We get the same result if we suppose that $\nabla$and $\tilde{\nabla}$ are symetric.

Theorem 3.3. If the connection $\overline{\nabla}$ has the torsion

$$T = \alpha\otimes 1 - 1\otimes\alpha + \beta\otimes m - m\otimes\beta,\ \text{ with } \alpha,\beta \in T_1^{\,0}(M)$$

and B is the symmetric tensorial field , then the distributions D and D'are integrable.

Proof. We have

$$[X,Y] = \overline{\nabla}_XY - \overline{\nabla}_YX.- \alpha\,(X)\ 1(Y) +\ \ \alpha\ \ (Y)\ 1(X) - \beta(X)\ m(Y) + \beta\,(Y)m(X)$$

and with (1.1), we obtain

$$[X,Y] = m\,\nabla_XmY + 1\,\nabla_X1Y + m\,B(X,mY) + 1\ B(X,1Y) - m\,\nabla_YmX - 1\,\nabla_Y1X.-$$
$$-\ m\ B(Y,mX) - 1\ B(Y,1X) -\ \ \alpha\ \ (X)\ 1Y +\ \alpha\ \ (Y)1X.- \beta\ \ (X)\ mY + \beta\ \ (Y)\ mX.$$

Then

$$[mX,mY] = m\,\nabla_{mX}mY + m\,B(mX,mY) - m\,\nabla_{mY}mX.- m\ B(mY,mX) -\ \ \beta\ \ (X)mY +$$
$$+\ \beta\ \ (Y)\ mX.$$

and finally, $1[mX,mY] = 0.$

It is proved, in a similar way, that $m[mX,mY] = 0.$

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