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DETERMINISTIC AND STOCHASTIC MODELS FOR SPREADING TWO-SPECIES POPULATION

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Abstract :

The deterministic and the stochastic equations that describe the system of two-species spreading population with interaction agmongst themselves, have been discussed. The stochastic partial differential equations for such population can be derived from a general model proposed earlier by De (1987) and more recently from random ecological niche factors-consideration by De (1995). The stationary solutions for the Fokker-Planck equation corresponding to these stochastic partial differential equations for this spreading population system might be given as the solitary wave or as the periodic wave solutions. The solitary wave solutions of the deterministic equations for the same population have also been presented.

Discussion has also been made on the utility of the deterministic equations as the tool for the approximate evaluation of the transition probabilites for the statistical case.

1. INTRODUCTION

The study of the growth of the interacting populations that spread in the region of space is generally done with the help of either the deterministic equations of reaction-diffusion type or by the stochastic partial differential equations. These stochastic partial differential equations are constructed from the corresponding deterministic evolution equations of reaction diffusion type describing such a population system, by the perturbation of all or some of the parameters of the equations with the white noises. Thus, the deterministic equations are primarily important in the study of an eco-system since they are the basic ingredients for the stochastic model that may be more relevant to the actual situations for the system; apart from their own merit of describing the eco-system in a simpler way. Here, we shall study the equations of both the types for a two-species spreading population with a special kind of interaction among them. But, in the present case we shall begin with the stochastic equations generated from a more general model proposed earlier by the present author (De, 1987; 1991; 1995) and then discuss the corresponding deterministic equations that arise, as the 'byproduct' in the approximate evaluation of the transition probabilities in the stochastic cases. Of course, these equations are themselves relevant for the two-species population because they can describe the deterministic nature of the population growth and pattern.

In section 2, we introduce the stochastic equations for the two-species population, that can be derived from the general model. In section 3, the solitary wave as well as the periodic solutions for the stationary cases of the Fokker-Planck equations corresponding to these stochastic equations have been obtained. In section 4, the corresponding deterministic equations have been introduced as the tool for approximate evaluation of the transition probabilities. In the subsequent section, the soliton type solutions or steady waves are shown to be the possible solutions if the parameters of the model satisfy some relations among themselves.

2. THE STOCHASTIC EQUATIONS FOR TWO-SPECIES POPULATION

It has been shown earilier (De, 1987; 1991; 1995) that the population density of the i^{th} species, $X_i\ (x, t)$ $(i=1, 2, \ldots\ldots, n)$ at time t and at the space point x of R^n (n=2 or 3), satisfy the following basic equations

$$\frac{\partial X_i(x,t)}{\partial t} = -\frac{\delta S}{\delta X_i} + \eta_i\ (x,\ t) \qquad (1)$$

$$(i = 1,\ 2,\ \ldots\ldots,\ n)$$

Where S is functional of X_i 's and $\frac{\delta}{\delta X_i}$ stands for the functional derivative w.r. to X_i. The variables η_i (x, t)are characterized by the following statistical properties :

$$<\eta_i (x, t) > = 0$$

$$<\eta_i (x, t)\, \eta_j (x', t') > = 2\, \delta_I\, \delta_j\, (x - x')\, \delta\, (t-t') \qquad (2)$$

The brackets mean the averaging over η_i 's with the Gaussian probability distribution. It has been shown there that the population densities are connected with the'random causes' for the system and these random variables are supposed to describe the stochastic processes to Itô (or diffusional) type.

The different choices for the functional S can produce the governing stochastic partial differential equations for different systems of populations with or without interaction among the species. Presently, we want to study a two-species spreading population with a special kind of interaction among them for which the choice of the functional S will be the folliwing :

$$S = \frac{1}{2}\int_v [d_1(\nabla X_1(x, t))^2 + d_2(\nabla X_2(x, t))^2 - \alpha_1(X_1(x, t))^2$$

$$- \alpha_2(X_2(x, t))^2 + \frac{2\alpha_1}{3K_1}(X_1(x, t))^3 + \frac{2\alpha_2}{3K_2}(X_2(x, t))^3$$

$$+ \varepsilon_{12}(X_1(x, t)\, X_2(x, t))^2]\, dx \qquad (3)$$

where d_1, d_2 are the diffusion coefficients, α_1, α_2 are the intrinsic growth rates and K_1, K_2 are carrying capacity parameters for the corresponding species. ε_{12} is the interaction parameters. This functional can result the following stochastic equations for the population under discussion.

$$\frac{\partial X_i\ (x,\ t)}{\partial t} = d_i\ \nabla^2\ X_i\ (x,\ t) + \alpha_1\ \ 1 - \frac{X_i\ (x,\ t)}{K_i}\ X_i(x,t)$$

$$- \varepsilon_{12}(X_i(x,\ t)\ X_j(x,\ t))^2 + \eta_i(x,\ t) \qquad (4)$$

$$(i = 1,2;\ j \neq i,\ j = 1,\ 2)$$

The existence results for a similar class of the stochastic differential equations have been considered by Da Prato (1983) and Da Prato, et al.(1979; 1982).

3. THE STATIONARY SOLUTIONS OF THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation corresponding to the stochastic differential equations (1) can be derived (De, 1987) in the following form :

$$\frac{\partial P}{\partial t} = \sum \int dx\ \frac{\delta}{\delta x_i}\ [(\frac{\delta}{\delta X_i} + \frac{\delta S}{\delta X_i})P] \qquad (5)$$

where P is the transition probability,

$$P \equiv P\ (X',\ t'\ |\ X'',\ t''] \qquad (6)$$

The equilibrium or stationary solutions of the Fokker-Planck equation (5) can be shown (De, 1987) to be of the form :

$$P_{eq} = C_o \exp\{-S[X]\}$$
$$X = (X_1, X_2, \ldots\ldots, X_n) \tag{7}$$

Where C_o is the normalization constant. Thus, the stationary solution of the Fokker-Planck equation corresponding to the stochastic equations (4) describing the two-species interacting and spreading population, is given by (using the functional S for such system, as given in (3)).

$$P_{eq} = c_o \exp\ -\frac{1}{2}\int_v [d_1 (\nabla x_1(x, t))^2 + d_2(\nabla x_2(x, t))^2$$
$$- \alpha_1 (X_1(x, t))^2 - \alpha_2 (X_2(x, t))^2 + \frac{2\alpha_1}{3K_1} (X_1(x, t))^3$$
$$+ \frac{2\alpha_2}{3K_2} (X_1(x, t))^3 + \varepsilon_{12}(X_1(x, t))^2 X_2(x, t))^2] dx \tag{8}$$

For such situation, we must have

$$d_1(\nabla X_1)^2 + d_2(\nabla X_2)^2 - \alpha_1 X_1^2 - \alpha_2 X_2^2 + \frac{2\alpha_1}{3K_1} X_1^3$$
$$+ \frac{2\alpha_2}{3K_2} X_2^3 + \varepsilon_{12} X_1^2 X_2^2$$

be independent of time in the region concerned, i.e., V of R^n (n = 2 or 3). Let us consider the two-dimensional case. The above expression will be equal to its initial value obtainable from the initial values of X_1, X_2, ∇X_1 and ∇X_2. Let this initial value of the above expression be $\phi_o(X)$, that is, we write

$$d_1(\nabla X_1)^2 + d_2(\nabla X_2)^2 - \alpha_1 X_1^2 - \alpha_2 X_2^2 + \frac{2\alpha_1}{3K_1} X_1^3$$

$$+ \frac{2\alpha_2}{3K_2} X_2^3 + \varepsilon_{12} X_1^2 X_2^2 = \phi_o(x) \quad (9)$$

If we seek the solutions of this equation with the parameters satisfying $d_1, d_2 \geq 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $K_1, K_2 > 0$, to be of the form :

$$X_i \equiv X_i(u),\ i = 1, 2$$

$$\text{Where } u = ax + by - ct \quad (10)$$

And take the function $\phi_o(X)=0$, we can achieve to the solition or steady wave solutions. With (10), the equation (9) becomes

$$d_1 \left(\frac{dX_1}{du}\right)^2 + d_2 + \left(\frac{dx_2}{du}\right)^2 (a^2 + b^2) - \alpha_1 X_1^2 - \alpha_2 X_2^2$$

$$+ \frac{2\alpha_1}{3K_1} X_1^3 + \frac{2\alpha_1}{3K_2} X_2^3\ \varepsilon_{12}^2\ X_1^3 = 0 \quad (11)$$

and we have the solution

$$\begin{matrix} X_1 \\ X_2 \end{matrix} = \begin{matrix} \delta_1 \\ \delta_2 \end{matrix} \text{Sech}(\lambda u) \quad (12)$$

Where δ_1, δ_2 and λ satisfy the following equations

$$\frac{\alpha_1}{K_1} \delta_1^3 + \frac{\alpha_2}{K_2} \delta_2^3 = 0$$

$$\lambda^2 (a^2 + b^2) (d_1 \delta_1^2 + d_2 \delta_2^2) = \alpha_1 \delta_1^2 + \alpha_2 \delta_2^2 \quad (13)$$

and $\varepsilon_{12} \delta_1^2 \delta_2^2 = \alpha_1 \delta_1^2 + \alpha_2 \delta_2^2$

A set of solutions of equations (13) for δ_1, δ_2 and λ is given by

$$\lambda = \left[\frac{\alpha_1 + \alpha_2\, p^{2/3}}{(a^2 + b^2)\ (d_1 + p^{2/3} d_2)} \right]^{1/2}$$

$$\delta_1 = \left[\frac{\alpha_1 + p^{-2/3} + \alpha_2}{\varepsilon_{12}} \right]^{1/2} \tag{14}$$

$$\delta_2 = \left[\frac{\alpha_1 + \alpha_2\, p^{2/3}}{\varepsilon_{12}} \right]^{1/2}$$

where $P = - \dfrac{\alpha_1 K_2}{\alpha_2 K_1} > 0$

For real values of δ_1 and δ_2 we must have either $K_1^2\,\alpha_2 + K_2^2\alpha_2 > 0$ and $\varepsilon_{12} > 0$ or $K_1^2\alpha_1 + K_2^2\alpha_2 < 0$ and $\varepsilon_{12} < 0$. In the latter case, we have imaginary λ and hence we have the periodic (wave) solutions for X_1 and X_2 instead of solitary wave solutions.

4. THE APPROXIMATE TRANSITION PROBABILITIES AND THE DETERMINISTIC EQUATIONS.

Now, we describe how the deterministic equations for the eco-systems become relevant in the approximate evaluation of the transition probabilities of the system which is described by the stochastic equation. This method was originally applied by Inagaki (1982) in the problem of random mutations in stochastic Eigen model and later by De

(1984) in a non-linear model of population system. The essential point of this method is to find a set of locally stable points of the exponent in the following path integral solution of (5) for the transition probability :

$$P[X', t' | X'', t''] = \hat{c} \int \exp\{- \frac{1}{2} \int_{t''}^{t'} dt \wedge (X, \dot{X}, t)\} D [X(t)] \quad (15)$$

with the boundary conditions

$$\begin{aligned} X_i (x, t') &= X_i' (x) \\ X_i (x, t'') &= X_i'' (x) \end{aligned} \quad (16)$$

and $\wedge (X, \dot{X}, t) = \frac{1}{2} \int dx \sum_i \{\dot{X}_i(x, t) + \frac{\delta S}{\delta X_i(x, t)}\}^2$ (17)

with $X_i = \frac{\partial X_i}{\partial t}$ and $\hat{C}$ is a normalization constant.

In fact, this set of locally stable points will determine the special path from all the possible paths indicated in (15) and thus, they form a class of solutions which satisfy the condition (De, 1987).

$$\delta H = 0 \quad (18)$$

where $H = \int_{t''}^{t'} dt \int \tilde{H} dx$ (19)

with $\tilde{H} = \frac{1}{2} \sum_i \dot{X}_i(x, t) + \frac{\delta S}{\delta x_i(x,t)}^2$ (20)

It has been proved there that the variational equation (18) corresponds to the following Euler-Lagrange type equations.

$$\frac{\partial \tilde{H}}{\partial X_i} - \frac{\partial}{\partial t}\left(\frac{\partial \tilde{H}}{\partial X_i}\right) + \nabla^2\left(\frac{\partial \tilde{H}}{\partial(\nabla^2 X_i}\right) = 0 \qquad (21)$$

$$(i = 1, 2 \ldots\ldots\ldots, n)$$

For the single-species population, we have

$$\tilde{H} = \frac{1}{2}\{\dot{X}(x, t) - d\nabla^2 X(x, t) - \alpha x(x, t) + \frac{\alpha}{K}X^2(x, t)\}^2$$

and it can be shown that for this case the equation (21) is equivalent to the following coupled equations :

$$Y = X - d\nabla^2 X - \alpha X + \frac{\alpha}{K}X^2$$

$$\frac{\partial Y}{\partial t} + d\nabla^2 Y + \alpha Y = \frac{2\alpha}{K}XY \qquad (22)$$

We can take as a class of solutions by the trivial solution $Y = 0$ of the second of the quations (22). This gives the required solution $X^c(x, t)$ which satisfies the equation

$$\frac{\partial x}{\partial t} = d\nabla^2 X + \alpha X\left(1 - \frac{X}{K}\right) \qquad (23)$$

Expanding $X(x, t)$ around $X^c(x, t)$, that is,

$$X(x, t) = X^c(x, t), \xi(x, t)$$

where $\xi(x, t)$ is small and vanishes on the boundary ∂V and at $t=t''$ and t', we can also expand $\tilde{H}$. Retaining only the first order terms in $\xi(x, t)$ the path integral for transition probability can be approximated to give

$$P[X', t' \mid X'', t''] = \hat{c} \exp \{- \frac{1}{2} H [X^c, \nabla^2 X^c, X^c\}$$

$$\times \int D(\xi) \exp (- \frac{1}{2} \int 0(\xi^2) dt\ dx) \qquad (24)$$

Thus, the approximate evaluation depends on finding certain locally stable solutions $X^C(x, t)$ which satisfy (23) and this is a deterministic equation that describes a spreading population regulated by the logistic growth. We can apply the same procedure to the two-species spreading and interacting population described by the functional S, given in (3). It can be shown that the locally stable solutions which are necessary for the approximation of the transition probabilities are the solutions of the following deterministic equations which describe the deterministic growth and spread of that interacting population :

$$\frac{\partial X_i (x, t)}{\partial t} = d_i \nabla^2 X_i (x, t) + \alpha_i X_i (x, t) (1 - \frac{X_i (x, t)}{K_i}$$

$$- \varepsilon_{12} \{X_i (x, t) (X_j (x, t)\}^2 \qquad (25)$$

$$(i, j = 1, 2; i \neq j)$$

Thus, we find another use for the deterministic equations for the eco-systems. In fact, they are helpful to the statistical ecology as well, apart from their conventional use as the primary model to study an eco-system.

5. SOLITARY WAVE SOLUTIONS OF THE DETERMINISTIC EQUATIONS

Let us seek the solutions of the deterministic equations (25) for the two-species interacting and spreading population in the form:

$$X_i \equiv U_i(X) \quad i = 1, 2$$

Where (26)

$$X = ax + by - ct$$

Then we have

$$- c \frac{DU_1}{DX} = d_1(a^2 + b^2) \frac{d^2U_1}{d^2X^2} + \alpha_1 U_1 \ (1 - \frac{U_1}{K_1}) - \varepsilon_{12} \ U_1 U_2^2$$

$$- c \frac{DU_2}{DX} = d_2(a^2 + b^2) \frac{d^2U_2}{d^2X^2} + \alpha_2 U_2 \ (1 - \frac{U_2}{K_2}) - \varepsilon_{12} \ U_1^2 U_2 \qquad (27)$$

Let us set

$$\frac{du_1}{dX} = V_1 \text{ and } \frac{du_2}{dX} = V_2$$

Then we have the following set of simultaneous first-order differential equations :

$$\frac{dU_2}{dX} = V_1$$

$$\frac{dU_2}{dX} = V_2$$

$$\frac{dV_1}{dX} = - a_1 U_1 (1 - \frac{U_1}{K_1}) - C_1 V_1 + \frac{\beta_{12}}{d_2} U_1 U_2^2 \qquad (28)$$

$$\frac{dV_2}{dX} = - a_2 U_2 (1 - \frac{U_2}{K_2}) - C_2 V_2 + \frac{\beta_{12}}{d_2} U_1^2 U_2$$

where

$$a_1 = \frac{\alpha_1}{d_1(a^2+b^2)} \qquad a_2 = \frac{\alpha_2}{d_2(a^2+b^2)}$$

$$C_1 = \frac{c}{d_1(a^2+b^2)}, \qquad C_2 = \frac{c}{d_2(a^2+b^2)}, \quad \beta_{12} = \frac{\varepsilon_{12}}{(a^2+b^2)} \tag{29}$$

As the boundary conditions we can take

$$U_1(-\infty) = 0 = U_2(-\infty)$$

$$\text{and} \quad U_1(+\infty) = D_1, \; U_2(+\infty) = D_2 \tag{30}$$

The critical or rest points of (28) are given by

$$V_1 = 0 = V_2$$

$$a_1U_1\left(1 - \frac{U_1}{K_1}\right) - \frac{\beta_{12}}{d_1} U_1U_2^2 = 0$$

$$a_2U_2\left(1 - \frac{U_2}{K_2}\right) - \frac{\beta_{12}}{d_2} U_1^2 U_2 = 0 \tag{31}$$

From these, it follows that $(U_1, U_2, V_1, V_2) = (0, 0, 0, 0)$ is one of the rest points, the other points are given by

$$V_1 = 0 = V_2$$

$$a_1\left(1 - \frac{U_1}{K_1}\right) - \frac{\beta_{12}}{d_1} U_2^2 = 0 \tag{32}$$

$$a_2\left(1 - \frac{U_2}{K_2}\right) - \frac{\beta_{12}}{d_2} U_1^2 = 0$$

Now, linearizing about the rest point (0, 0, 0, 0) we obtain as the coefficient matrix

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & 0 & -C_1 & 0 \\ 0 & -a_2 & 0 & -C_2 \end{matrix} \tag{33}$$

whose eigenvalues are given by the roots of the equation

$$\{\lambda\ (C_1 + \lambda) + a_1\}\ \{\lambda\ (C_2 + \lambda) + a_2\} = 0$$

They are

$$\lambda_i = \frac{- C_i \pm (C_i^2 - 4a_i)^{1/2}}{2} \qquad (i = 1, 2) \qquad (34)$$

In order to satisfy the boundary condition (30) at $X \rightarrow -\infty$, we must C_1, $C_2 < 0$. Also, if $|C_1| < 2a_1^{1/2}$, $|C_2| < 2a_2^{1/2}$, this rest point is a spiral point and therefore any solution which approaches to it, must eventually be negative and thus unacceptable. Therefore we should have $|C_1| < 2a_1^{1/2}$, $|C_2| < 2a_2^{1/2}$ and the solution (wave) is represented by the trajectory joining the points (0, 0, 0, 0) and (D_1, D_2, 0, 0). The other rest points can be considered in this way and we can proceed towards the existence of the trajectories representing the solutions (waves). Much discussions have been made by Kennedy, et. al. (1980) for a similar case and therefore we shall not proceed in that direction. Rather, we like to find the form of the solution of the equations (28). In fact, we have a set of solutions of the form :

$$U_i = \frac{\alpha \exp\ (\lambda X)}{d_i^{1/2}\ \{1 + \exp(\lambda X)\}^2} \qquad (i = 1, 2)$$

$$V_i = \frac{\lambda\alpha \exp\ (\lambda X)}{d_i^{1/2}\ \{1 + \exp(\lambda X)\}^2} \qquad (i = 1, 2) \qquad (35)$$

if the following relations are satisfied :

- $\lambda^2 = a_1 + C_1\lambda = a_2 + C_2\lambda$

$$2\lambda^2 = \frac{\beta_{12}\alpha^2}{d_1 d_2} = a_1 \left(1 - \frac{\alpha}{K_1 d_1^{172}} \right. \qquad (36)$$

$$= a_2 \left(1 - \frac{\alpha}{K_2 d_2^{172}}\right)$$

From these relations we find, after using (29), the following relations in terms of the parameters of the model.

$$\alpha = \frac{(\alpha_1 d_2 - \alpha_2 d_1)\ K_1 K_2\ d_1^{1/2}\ d_2^{1/2}}{\alpha_1 K_2 d_2^{3/2} - \alpha_2 K_1 d_1^{3/2}}$$

$$\lambda = (\alpha_1 d_2 - \alpha_2 d_1)/c(d_1 - d_2)\ (\alpha_1 d_2 - \alpha_2 d_1)^2$$

$$= \frac{c^2}{a^2 + b^2}\ (d_2 - d_1)\ (\alpha_1 - \alpha_2) \qquad (37)$$

$$= \frac{\alpha}{\varepsilon_{12} K_1^2 K_2^2}\ (K_2 d_2^{1/2} - K_2 d_1^{1/2})\ (\alpha_1 K_2 d_2^{3/2} - \alpha_2 K_2 d_1^{3/2})$$

$$\frac{c^2}{a^2 + b^2} = \frac{2(\alpha_1 K_2 d_2^{3/2} - \alpha_2 K_1 d_1^{3/2})^2}{\varepsilon_{12} K_1^2 K_2^2 (d_1 - d_2)^2}$$

Eliminating $c^2/(a^2 + b^2)$ we have two relations among the parameters α_1, α_2, d_1, d_2 and K_1, K_2. They are

$$(\alpha_1 d_2 - \alpha_2 d_1)^2 = \frac{2(\alpha_1 - \alpha_2)\ (\alpha_1\ K_2 d_2^{3/2} - \alpha_2\ K_1 d_1^{3/2})^2}{\varepsilon_{12}\ K_1^2 K_2^2 (d_2 - d_1)}$$

$$2(\alpha_1 - \alpha_2)\ (\alpha_1\ K_2 d_2^{3/2} - \alpha_2\ K_1 d_1^{3/2}) \qquad (38)$$

$$= \alpha_1\alpha_2\ (d_2 - d_1)\ (K_2 d_2^{1/2} - K_1 d_1^{1/2})$$

Thus, among the six parameters only four of them are independent for all existence of the solutions of the form

(35), which represent the solitons or steady progressive waves. We can solve, from the relations (38), for K_1, K_2 in terms of others and they are given by

$$K_1^2 = \frac{2\alpha_1^2 d_2 \ (\alpha_1 - \alpha_2)(d_2 - d_1)}{\varepsilon_{12} \ \{(2d_1(\alpha_2 - \alpha_1) + \alpha_1(d_2 - d_1)\}^2}$$

$$K_2^2 = \frac{2\alpha_2^2 d_1 \ (\alpha_1 - \alpha_2)(d_2 - d_1)}{\varepsilon_{12} \ \{(2d_2(\alpha_2 - \alpha_1) + \alpha_2(d_2 - d_1)\}^2} \tag{39}$$

When such values of the parameters (the carrying capacity parameters) are attained by them for an environment, the above types of soliton solutions will manifest for the system of two-species interacting population. In deriving relations (37) and also the subsequent relations we have assumed

$$\alpha_1 d_2 - \alpha_2 d_1 \neq 0$$

and $\alpha_1 K_2 d_2^{3/2} - \alpha_2 K_1 d_1^{3/2} \neq 0$

But when

$$\alpha_1 d_2 - \alpha_2 d_1 = 0 = \alpha_1 K_2 d_2^{2/3} - \alpha_2 K_1 d_1^{3/2} \tag{40}$$

which leads to

$$\frac{\alpha_1}{\alpha_2} = \frac{d_1}{d_2} = \frac{K_2^2}{K_1^2},$$

we have from (36), $C_1 = C_2$ and consequently $\alpha_1 = \alpha_2$, $K_1 = K_2$ and $d_1 = d_2$ and then we have only three relations left. They are

$$-\lambda^2 = a_1 + C_1\lambda$$

$$2\lambda^2 = \beta_{12}\alpha^2/d_1d_2 = a_1 \left(1 - \frac{\alpha}{K_1 d_1^{1/2}}\right) \qquad (41)$$

These relations determine α, λc and $c^2/(a^2 + b^2)$ for the solutions (35). Here the values of λ coincide with the eigenvalues of the coefficient matrix of the linearized problem, given by (34).

Lastly, we like to point out that if the relations (38) are satisfied among the parameters then it can be shown that the following set is also the solutions of the deterministic equations (25) :

$$U_i = \frac{\alpha}{d_i^{1/2}} \frac{e^{\lambda x}}{(A + e^{\lambda x})} + \frac{\varepsilon}{d_i^{1/2}} \operatorname{sech}^2 \frac{\lambda x}{2} \quad (i = 1, 2) \qquad (42)$$

where ε is small and A is any arbitary constant.

Thus we have obtained the soliton solutions of the type (35) or (42) for the case of two competing species of identical growth and spread parameters (that is, for similar type of species) in an environment which is equally capable of carrying them, as evident from equations (41). Also, such types of solutions are possible for this case of two-species population when the parameters of the model satisfy certain relations [equation (38) or (39)] among themselves. It is to be noticed that these solutions satisfy similar boundary conditions as stated in (30).

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