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PARAMETER ANALYSIS IN QUALITY CONTROL *

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Abstract

In this study, first, the problem of optimization of the function of net income was investigated considering quality as a dynamic factor. Afterwards, the net income at time t was reformulated by taking into account the cost arising from the inspection of a product and the cost caused by a faulty product which was not the subject of the final quality control and gave birth to a dissatisfaction and the problem is transformed to a linear queuing problem. The expected value of net income is expressed in terms of the lose and gain rate and the optimum value of the upper limits of the amount of the faulty products in the sample which was the subject of quality control and rate of defective was determined.

1 Introduction

Quality is a measurable characteristic of a process, a product or a service. It must be measurable in order to be controlable. In other words various levels of these characteristics must be measured in terms of numerical values or must be transformable to numerical values [1].

Quality control is a collection of works containing qualitative and quantitative factors and aiming to realize the established standards.

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The production process takes place in an interval of time and controlled at certain intervals and at certain steps of the process. Product control is realized according to a certain predetermined principles. The techniques concerning production and control have been changing very rapidly in time. Mathematical models of the systems changing in time is known as stochastics process [2].

Then quality control must be considered as a stochastics process and be evaluated in that fashion. On the other hand, quality control problems can be investigated by means of the quantitative techniques such as decision theory and queuing theory.

The role played by statistics in the investigation of quality standards and in the determination of the path followed by production process is so important that one can say: quality control has gained its contemporary meaning with the help of statistical methods.

The developments observed in the field of competition, the aid of quality factor to competition and the affects of quality control on quality have played an important role in the importance of quality control activities. Naturally, the quality control activities have caused the arisal of the quality cost problem. The measure of quality is determined by controlling the costs of product quality. And this approach necessitates the determination of quality costs and its classification.

The quality control costs caused by the quality of goods and service are generally divided into three different categories:

1. Cost of Prevention,
2. Cost of Appraisal,
3. Cost of Internal and External failures.

The quality costs concerning the quality of the product or the service, in other words the total of prevention, appraisal and internal-external failure costs form the total quality cost. Total quality cost as a function of the quality level may be formulated in the following way [3]:

q : quality level, $0 \leq q \leq 1$

$C_1(q)$: internal and external failure costs per unit of the product at quality level q , $0 \leq q \leq 1$

$C_2(q)$: prevention and appraisal costs per unit of the product at quality level q , $0 \leq q \leq 1$

$C_{12}(q)$: the total quality cost per unit of the product at quality level q , $0 \leq q \leq 1$ and

$$C_{12}(q) = C_1(q) + C_2(q).$$

In this work developing some new approaches to quality control problem by making use of the quality control costs is aimed.

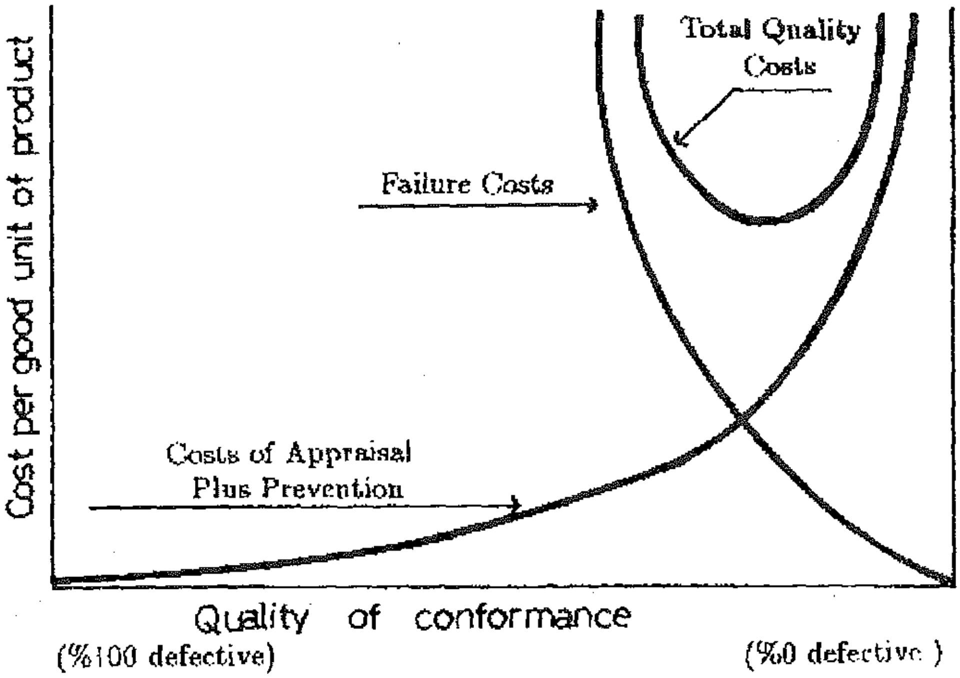


Figure 1. Juran's Model of Optimum Quality Costs [3].

2 The Optimum Quality Path Model

2.1 Optimum Quality Cost and The Concept of Zero Defect

Optimum quality cost is the minimum value of the total quality cost. The quality level corresponding to this minimum value is the optimum quality level.

The problem of determining the minimum of total quality cost is an optimization problem. Optimization is a process aiming to choose the way which minimizes or maximizes a decision problem among other alternative ways [4].

The total quality cost curve seen in Figure 1 is a convex curve and there is a q_0 which minimizes $C_{12}(q)$. q_0 is the value [5] which satisfies the inequality $C''_{12}(q_o) > 0$ and is the root of the equation $C'_{12}(q) = 0$. The total quality cost which was formulated above may be reformulated in the following way by taking into account the changes in the quality level.

$q(t)$: quality level at time t , $0 \leq q(t) \leq 1$
$C_1[q(t)]$: internal and external failure costs,
$C_2[q(t)]$: prevention and appraisal costs,
$C_{12}[q(t)]$: total quality cost,

$$C_{12}[q(t)] = C_1[q(t)] + C_2[q(t)], \quad q(t) \in [0,1]. \tag{1}$$

The minimum value of the total quality cost $q_0(t_0)$, which is a function of quality level at time t , here $t > 0$ and $q(t) \in [0,1]$, is

$$C'_{12}[q(t)]\frac{dq}{dt} = 0$$

such that for this root

$$C''_{12}[q(t)]\left(\frac{dq}{dt}\right)^2 + C'_{12}[q(t)]\frac{d^2q}{dt^2} > 0$$

If there are more than one root which satisfies these two conditions, the one which minimizes the $C_{12}[q(t)]$ is chosen and is called the absolute minimum.

In general the goal of the firms is "zero defect". In fact "zero defect" means producing nothing faulty. In other words, it means determining the conditions which do not suit the standards immediately, reorganising them up to the standards and "having the correct and up-to-standards products at the first time" and in this way reducing faulty product cost. The decrease in the cost of internal and external faulty products means the increase in the cost of appraisal and prevention while some approaches toward the target "zero defect" on the other hand, cause a rapid increase of these costs toward infinity [6]. We see in Figure 1 that total quality reaches its minimum value not at the zero defect point but at some point to the left of the zero defect point and for some amount of faulty product. Up to this point the efforts to improve the product pulls down the total quality cost. The effort to decrease the rate of faulty product from that point on brings extra costs although it develops the production process.

2.2 The Analysis of Zero Defect by Means of Quality Cost

In works which aims to analyze the zero defect the concept total quality cost has formed the starting point and this approach is called as traditional approach [5]. Fine asserts that the knowledge obtained through high quality production is more useful in the decrease of production cost than the knowledge obtained through low quality level production. By making use of this assertion a dynamic model was formulated using the knowledge which depends on the quality of production. The work of Fine is the first one saying something about the dynamic nature of the quality of production. But Fine had not considered [5] the effect of quality on the sale and the technological developments which cause a decrease in the cost of appraisal and prevantion in time. Hsiang and Lee have developed the model by taking into account the effect of these factors on the cost of appraisal and prevention and the effect of quality on the sale.

In the following model it was assumed that the accumulation of knowledge concerning quality and technological developments produce a decrease in the cost of external and internal products and so made possible a more comprehensive analysis of the problem possible. In the model the effect of quality level changing by time on the price of a unit product and the demand or the sale was used to maximize the net income of the firm(which was designeted by $\pi(t)$) and to determine the best path which will be followed by the quality level $q(t)$ which changes on the interval $[0,1]$ by taking into account the accumulation of knowledge concerning quality and the effect of technological improvements on the cost of internal and external faulty products.

Let us show by $D[q(t)]$ the demand of a product of quality level $q(t)$ at time t. $0 \leq q(t) \leq 1$ and $q'(t) > 0$. On the other hand assuming that $q(t)$ will not increase indefinitely one can say that $q''(t) < 0$, in other words $q(t)$ is a concave function of t.

$P[q(t)]$ is the price of a unit product at quality level $q(t)$. While the quality level increases the price will increase but not infinitly so we may assume that $P'[q(t)] > 0$ and $P''[q(t)] < 0$, in other words $P[q(t)]$ is a concave increasing function of $q(t)$.

Let $C_1[q(t)]$ and $C_2[q(t)]$ be respectively the cost of internal and external faulty products and the cost of appraisal and prevention. $C_1[q(t)]$ is a decreasing convex function of $q(t)$ and $C_2[q(t)]$ is an increasing convex function of $q(t)$. In other words $C_1'[q(t)] < 0$, $C_1''[q(t)] > 0$ and $C_2'[q(t)] > 0$, $C_2''[q(t)] > 0$.

Let $C_3(t)$ is the cost of production of a unit product when the cost of quality is not concerned. Assuming that the cost of production will decrease due to improvements in the production technology but this decrease will not be infinit so one can say that $C_3'(t) < 0$ and $C_3''(t) > 0$.

The total quality cost is $C_{12}[q(t)] = C_1[q(t)] + C_2[q(t)]$ and is a convex function of $q(t)$. In other words for a q_o which satisfies the equation $C_{12}'[q(t)] = 0$, $C_{12}''[q_o] > 0$ and so it is possible to find a minimum value of $C_{12}[q(t)]$.

Let the experience gained up to time t and depending upon the level of the quality of the product be $z(t)$ [5]. One can write

$$z(t) = z(0) + \int_0^t h(\tau)q(\tau)d\tau \qquad (2)$$

Here,

$z(0)$: the experience owned at time $t = 0$

$q(\tau)$: the quality level of the product produced at time τ

$h(\tau)$: an increasing function of τ and serves as a discount factor. It is a factor expressing that the experience gained from the last experiments is more important than the ones gained from the earlier experiments.

Let $a_1(t)$ be decreasing convex function representing the decrease in the cost of the internal and external faulty products due to technological improvement and let $b_1[z(t)]$ be decreasing convex function representing the decrease which arises due to the accumulation of knowledge. Let the correspondences of $a_1(t)$ and $b_1[z(t)]$ for the cost of appraisal and prevention be $a_2(t)$ and $b_2[z(t)]$. Also, $a_2(t)$ and $b_2[z(t)]$ are decreasing convex functions. These functions are connected in the following fashion:

$$\begin{aligned} C_1(q,t) &= a_1(t)b_1[z(t)]C_1[q(t)] \\ C_2(q,t) &= a_2(t)b_2[z(t)]C_2[q(t)] \end{aligned}$$

and the net income of the firm at time t is

$$\begin{aligned} \pi(t) &= \{P[q(t)] - a_1(t)b_1[z(t)]C_1[q(t)] \\ &- a_2(t)b_2[z(t)]C_2[q(t)] - C_3(t)\}D[q(t)]. \end{aligned} \qquad (3)$$

And our goal is to find the best quality path to maximize $\pi(t)$.

2.3 The Maximization of Net Income

For a planning period T the maximization of $\pi(t)$ is equal to the maximization of the following functional [5],

$$\varphi(T) = \int_0^T e^{-rt}\pi(t)dt \tag{4}$$

The value of the $\varphi(T)$ is the current value of $\pi(t)$ for a period T. r is the discount factor used to determine the current value of an income accumulated in an interval $[0, T]$. In the determination of the numerical value of r inflation plays an important role. While the rate of inflation increases the value of r increases, otherwise decreases.

The optimization of this functional is a problem of the calculus of variations. In order to maximize $\varphi(T)$ we will make use of the following theorem.

THEOREM 1. Let $J[z]$ be a functional of the form

$$\int_a^b F(t, z, z')dt,$$

defined on the set of functions $z(t)$ which have continuous first derivatives in $[a, b]$ and satisfy the boundary conditions $z(a) = A$, $z(b) = B$. Then a necessary condition for $J[z]$ to have an extremum for a given function $z(t)$ is that $z(t)$ satisfy Euler's equation [7]

$$F_z - \frac{d}{dt}F_{z'} = 0.$$

According to this theorem $F(t, z, z') = e^{-rt}\pi(t)$ and the necessary condition for the maximization of $\varphi(T)$ is the following relation which is known as Euler condition

$$\frac{\partial}{\partial z}[e^{-rt}\pi(t)] = \frac{d}{dt}\left\{\frac{\partial}{\partial z'}[e^{-rt}\pi(t)]\right\}$$

Here, z' is the derivative of $z(t)$ with respect to t and $z' = hq$ and $q = z'/h$. The following lemma will be used to find the best quality path in order to maximize the functional (4).

LEMMA. The optimal quality path $q^*(t)$ which maximizies the functional given by (4) satisfies $\partial\pi(t)/\partial q \leq 0$.

Proof: If the net income $\pi(t)$ satisfies the inequality $\partial\pi(t)/\partial q > 0$ then it is possible to increase $\pi(t)$ by increasing $q(t)$.

An increase in $q(t)$ will cause on increase in $z(t)$. And this will cause a decrease in the future costs of appraisal and prevention and internal and external faulty products. Since the profit can not be infinite we deduce that any value of $q(t)$ for which $\partial\pi(t)/\partial q$ is positive can not be optimum. So, $\partial\pi(t)/\partial q$ must be non-positive. In other words on the optimal quality path the following inequality must be satisfied

$$\begin{aligned}\{P'[q(t)] - a_1(t)b_1[z(t)]C_1'[q(t)] - a_2(t)b_2[z(t)]C_2'[q(t)]\}D[q(t)] \\ +\{P[q(t)] - a_1(t)b_1[z(t)]C_1[q(t)] \\ -a_2(t)b_2[z(t)]C_2[q(t)] - C_3(t)\}D'[q(t)] \leq 0.\end{aligned}$$

THEOREM 2. If the net income of the firm at time t is a function of $q(t)$ and is expressed in the form of (3) then the optimal strategy is to increase the quality q of the product as long as $(d/dt)(C_1D) \geq 0$.

Proof: Let $F(t,z,z') = e^{-rt}\pi(t)$. Then

$$\begin{aligned}F(t,z,z') &= e^{-rt}\{P[q(t)] - a_1(t)b_1[z(t)]C_1[q(t)] \\ &\quad -a_2(t)b_2[z(t)]C_2[q(t)] - C_3(t)\}D[q(t)].\end{aligned}$$

Let us write $q = z'/h$. We get

$$\begin{aligned}F(t,z,z') &= e^{-rt}\{P\Big(\frac{z'}{h}\Big) - a_1(t)b_1[z(t)]C_1\Big(\frac{z'}{h}\Big) \\ &\quad -a_2(t)b_2[z(t)]C_2\Big(\frac{z'}{h}\Big) - C_3(t)\}D\Big(\frac{z'}{h}\Big)\end{aligned}$$

If we use the Euler condition

$$\frac{\partial F}{\partial z} = \frac{d}{dt}\Big\{\frac{\partial F}{\partial z'}\Big\} \text{ or } \frac{\partial F}{\partial z} - \frac{d}{dt}\Big\{\frac{\partial F}{\partial z'}\Big\} = 0$$

we get

$$\begin{aligned}&\frac{q'}{h}[-(P'' - a_1b_1(z)C_1'' - a_2b_2(z)C_2'')D - 2(P' - a_1b_1(z)C_1' \\ &-a_2b_2(z)C_2')D' - (P - a_1b_1(z)C_1 - a_2b_2(z)C_2 - C_3)D''] = \\ &[a_1b_1'(z)C_1 + a_2b_2'(z)C_2]D - \Big(\frac{r}{h} + \frac{h'}{h^2}\Big)\{[P' - a_1b_1(z)C_1' \\ &-a_2b_2(z)C_2']D + [P - a_1b_1(z)C_1 - a_2b_2(z)C_2 - C_3]D'\} \\ &-\Big(\frac{a_1'b_1(z)}{h} + qa_1b_1'(z)\Big)[C_1D' + C_1'D]\end{aligned}$$

$$-\Big(\frac{a_2'b_2(z)}{h}+qa_2b_2'(z)\Big)[C_2D'+C_2'D]-\frac{C_3'D'}{h} \qquad (5)$$

In the left hand side of the equality (5), since $P[q(t)]$ is an increasing and concave function of $q(t)$ we say $P'>0,\ P''<0$. $C_1[q(t)]$ is decreasing and convex, $C_2[q(t)]$ is increasing and convex and so $C_1'(q)<0,\ C_1''(q)>0$ and $C_2'(q)>0, C_2''(q)>0$. On the other hand $D'>0, D''<0$ and

$$P''-a_1b_1(z)C_1''-a_2b_2(z)C_2''<0$$

and from the Lemma

$$P'-a_1b_1(z)C_1'-a_2b_2(z)C_2'<0$$

and since the third term on the left hand side satisfies the following inequality

$$P-a_1b_1(z)C_1-a_2b_2(z)C_2-C_3>0,\ D''<0$$

the left side of the equation (5) is non-negative.

From Lemma and the relation $h'>0$ one can easily see that the second term on the right hand side of the equation (5) is non-negative. On the other hand, since $C_3'<0$, so $-(C_3'D'/h)\geq 0$. If we rewrite the remaining terms of the right hand side and reorder them we get

$$[a_1b_1'(z)C_1+a_2b_2'(z)C_2]D-\Big[qa_1b_1'(z)+\frac{1}{h}a_1'b_1(z)\Big](C_1D'+C_1'D)$$

$$-\Big[qa_2b_2'(z)+\frac{1}{h}a_2'b_2(z)\Big](C_2D'+C_2'D)=a_1b_1'(z)D[C_1-qC_1']$$

$$+a_2b_2'(z)D[C_2-qC_2']-\frac{a_1'b_1(z)}{h}[C_1D'+C_1'D]$$

$$-qa_1b_1'(z)C_1D'-\Big[\frac{a_2'b_2(z)}{h}+qa_2b_2'(z)\Big]C_2D'-\frac{a_2'b_2(z)}{h}C_2'D. \qquad (6)$$

C_1 and C_2 are convex functions and so $C_1\leq qC_1'$ and $C_2\leq qC_2'$. On the other hand, $b_1(z)$ and $b_2(z)$ are decreasing functions and $b_1'(z)<0$ and $b_2'(z)<0$. So the first and second term of (6) are non-negative. If

$$\frac{d}{dt}(C_1D)\geq 0$$

then the third term is also non-negative so the right hand side of the equality (5) is non-negative.

As a result, in the optimal solution (5) which was obtained according to Euler condition q' must be non-negative as long as $(d/dt)(C_1 D) \geq 0$. This means that the optimal quality path $q^*(t)$ which was expressed in Lemma must satisfy the relation $[q^*(t)]' \geq 0$. This relation suggest that the level of quality must be increased in a constant fashion. So the optimal strategy of the producer must be to increase the level of the quality in a constant fashion and to try to reach the target of zero defect.

In the analysis which we tried to realize the cost of the internal and external faulty products and appraisal and prevention components of the quality control costs were taken into account and the optimum quality path was determined from the point of view of the producer.

2.4 The Case of Constant Demand

Up to now we supposed that demand is a function of the quality changing in time and tried to determine the optimum quality strategy of the producer. At this point we will suppose that the demand is constant, that is $D' = D'' = 0$. In this case, (5) transforms to

$$\begin{aligned}\frac{q'}{h}[-(P'' - a_1 b_1(z)C_1'' - a_2 b_2(z)C_2'')D] = \\ [a_1 b_1'(z)C_1 + a_2 b_2'(z)C_2]D \\ -\Big(\frac{r}{h} + \frac{h'}{h^2}\Big)[P' - a_1 b_1(z)C_1' - a_2 b_2(z)C_2']D \\ -\Big(\frac{a_1' b_1(z)}{h} + q a_1 b_1'(z)\Big)C_1' D - \Big(\frac{a_2' b_2(z)}{h} + q a_2 b_2'(z)\Big)C_2' D. \qquad (7)\end{aligned}$$

If the investigation made for Theorem 2 is repeated here, one can see that the term in paranthesis on the left hand side is non-negative. On the right hand side of the equation (7) $h' > 0$ and the second term is non-negative according to Lemma. Let us rewrite the remaining terms and reorder them and obtain

$$\begin{aligned}[a_1 b_1'(z)C_1 + a_2 b_2'(z)C_2]D - \left(\frac{a_1' b_1(z)}{h} + q a_1 b_1'(z)\right) C_1' D \\ - \left(\frac{a_2' b_2(z)}{h} + q a_2 b_2'(z)\right) C_2' D = a_1 b_1'(z) D (C_1 - q C_1')\end{aligned}$$

$$+a_2 b_2'(z) D(C_2 - qC_2') - \frac{1}{h} a_1' b_1(z) C_1' D - \frac{1}{h} a_2' b_2(z) C_2' D.$$

This expression is non-negative if

$$u_1 + u_2 < v_1 + v_2 \tag{8}$$

here $u_i = a_i' b_i(z) C_i'$ and $v_i = h a_i b_i'(z)(C_i - qC_i')$, $(i = 1, 2)$. So the right hand side of (7) becomes non-negative and $q' \geq 0$. This means that an increase in the level of quality will produce an increase in the income of the firm.

3 The Role of Quality Control in The Optimization of The Profit by means of Queuing Theory

3.1 The Factors Effecting The Demand

Up to now we assumed that demand is a function of the level of quality. Here we will assume that the most important factor in the determination of the demand is the level of quality and the demand is equal to the amount of the production $D[q(t)] = X(t)$. Here by $X(t)$ we mean the amount of production realized during a period of time t at quality level $q(t)$. Let $X(t) = x$. As mentioned in the paper of Lee and Tapiero, the factors other than quality level affecting the amount of demand, consequently the amount of production are

-consumers who have used faulty products and decide not to repeat purchase,

-consumers who have switched to other products,

-consumers who have been affected by advertising, seller reputation, warranties, servicing reputation and switched from competitors to this firm,

-consumers who have been affected by those customers who have used products free from defects,

-consumers who have decided not to repeat purchase due to negative word-of-mouth effect.

The effects of these factors to the problem are reflected by the symbols $\alpha, \delta, \theta, \sigma$, and β respectively[8]. Here, we will formulate the net income of the firm from sales by taking into account the cost arising from the inspection of a product and the cost caused by a faulty product which was not the subject of the final quality control and gave birth to a dissatisfaction and will try to find the expected value of it. Later on, we will consider the problem as a queuing problem and will find the optimum value of the faulty products in the sample with the goal of maximizing the expected profit.

3.2 The Way of Forming Quality Control Samples and Formulating The Expected Value of The Income

Let us suppose that the proportion of substandard quality products in a batch of product is p , where p is a random variable and its the probability density function is $f(p)$. A is the rate of product which has been the subject of the final control in the total product and $0 < A \leq 1$. In other words, if the amount of product is $X(t) = x$,we assume that the amount of product which has been the subject of the final control is Ax . Let us denote the cost of control of one unit of product by $C_i[q(t)]$, the loss caused by one unit of product which has not been the subject of the final control by $C_s[q(t)]$. The loss in question may occur due to any one of factors such as extra service, repair, replacement, the lose of consumers or any combination of these factors.

If the number of defective items in Ax items which were inspected is less than a predetermined number r , it is assumed that the level of a desirable quality is attained and the whole lot except those found defective in the sample is sold. If the number of defective items exceeds r , the whole lot is inspected, the defective ones are discarded and the noudefective ones are sold. Let us denote the cost of the total inspection by $C(x,p)$ where the amount of product is x and the rate of defective in product is p . In other words, the cost which arises when $y \leq r$,so Ax of x units was the subject of the final control or $y > r$ and all of the units were the subject of the final control. y is the amount of the faulty product in the sample and the total inspection cost can be formulated as

$$C(x,p) = \begin{cases} AxC_i + (x - Ax)pC_s, & y \leq r \\ xC_i & , \quad y > r \end{cases} \tag{9}$$

$\pi_1(t)$, the net profit which is obtained from the sales per unit of the product at time t . is

$$\pi_1(t) = P[q(t)] - C_k[q(t)] - C_e[q(t)] - C_3(t),$$

where $C_k[q(t)]$ is the internal failure cost per unit of the product arising due to the quality level. $C_1[q(t)]$ is the internal and external failure cost per unit of the product at quality level $q(t)$, and is given by

$$C_1[q(t)] = C_k[q(t)] + C_s[q(t)].$$

$C_e[q(t)]$ is the other prevention and appraisal cost per unit of the product except final control cost, i.e.

$$C_e[q(t)] = C_2[q(t)] - C_i[q(t)].$$

If $C_3(t)$ is the production cost per unit of the product except the costs concerning quality. the net profit of the firm at time t will be formulated as

$$\pi(t) = \begin{cases} \pi_1(x - Axp) - [AxC_i + (x - Ax)pC_s], & \text{if} \quad y \leq r \\ \pi_1 x(1-p) - xC_i & , \text{if} \quad y > r \end{cases}$$

If the number of defective items in a sample of size Ax is less than r , then Ax the defective ones are discarded and the rest which is equal to $x - Axp$ is sold. Otherwise the whole of the lot is controlled, defective ones are discarded and the rest is sold. In this case the expected value of the income is

$$\begin{aligned} E[\pi(t)|x(t).p] &= \{\pi_1(x - Axp) - [AxC_i + (x - Ax)pC_s]\} * \\ &* \quad P(y \leq r) + [\pi_1 x(1-p) - xC_i]P(y > r). \end{aligned}$$

and if $h(x)$, is the probability density function of x, the expected value of the net profit will be

$$\begin{aligned} E[\pi(t)|p] &= \sum_x [\{\pi_1(x - Axp) - [AxC_i + (x - Ax)pC_s]\}P(y \leq r) + \\ &+ \quad [\pi_1 x(1-p) - xC_i]P(y > r)]h(x) \\ &= \quad E[x(t)][\{\pi_1(1 - Ap) - [AC_i + (1 - A)pC_s]P(y \leq r) + \\ &+ \quad [\pi_1(1-p) - C_i]P(y > r)]. \end{aligned} \tag{10}$$

If quality control samples are drawn randomly, we may assume that the probability of obtaining m faulty products among Ax units is binomial with parameters Ax and p. Then

$$P(y \leq r) = \sum_{m=0}^{r} \binom{Ax}{m} p^m (1-p)^{Ax-m}, \tag{11}$$

and $P(y > r) = 1 - P(y \leq r)$, where Ax is an integer.

3.3 The Rates of Loss and Gain

The customers using the products of the firm in question may switch to the product of the rival firms and vice versa. Let us call the number of customers which are lost by the firm and the number of customers which are gained in unit time respectively as the rate of loss and the rate of gain. Let us suppose that these quantities are proportional to the amount x produced by the firm.

The losses may occur due to faulty products and due to non-faulty products. Let α, and δ respectively be the rate of losses occuring due to these two different situations. Then the rate of loss for one product is

$$\begin{aligned} \mu &= \alpha(1-A)pP(y \leq r) + \delta P(y \leq r)[A(1-p) + (1-A)(1-p)] \\ &+ P(y > r)(1-p)\} \\ &= \alpha(1-A)pP(y \leq r) + \delta(1-p). \end{aligned} \tag{12}$$

and for x product is

$$x\mu = x[\alpha(1-A)pP(y \leq r) + \delta(1-p)] \ .$$

The gains may occur due to (i) consumers who have been affected by advertising, seller reputation, warranties, servicing reputation and switched from competitors to this firm, (ii) consumers who have been affected by those customers who have used products free from defects, (iii) consumers who have decided not to repeat purchase due to negative word-of-mouth effect. Let θ, σ, and β respectively be parameters to these actions. The gain which is a combination of the actions symbolized by the parameters θ, σ, and β will form the base for formulating the rate of gain for one product as

$$\begin{aligned} \lambda &= \theta + \sigma[(A - Ap)(1) + (1-A)(1-p)]P(y \leq r) + \\ &+ \sigma[(1-p)P(y > r)(1)] - \beta(1-A)pP(y \leq r) \\ &= \theta + \sigma(1-p) - \beta(1-A)pP(y \leq r). \end{aligned} \tag{13}$$

and for x product is

$$x\lambda = x[\theta + \sigma(1-p) - \beta(1-A)pP(y \leq r)] \ .$$

This formulation gives us the possibility of interpreting the problem as a queuing problem with linear rate of arrivals and linear rate of service if we consider the rate of gain as rate of arrival and the rate of loss as rate of service[9].

3.4 The Expected Value of The Elements in The System for Queues of Linear Arrival and Service Rates

Let the number of the elements exists in the system describe the situation of the system. Let us assume that there are x elements in the system. Let us call the number of arrivals in unit time as arrival rate and the number of those who have completed their service and left the system as service rate. Let arrival rate and service rate be variables. When there are x elements in the system, let us describe this situation by E_x and corresponding arrival and service rates respectively λ_x and μ_x. During a time interval of length h let the passage from a situation to neighboring situation be possible. By choosing h sufficiently small, the other passages can be made impossible. We will formulate the differential- difference equation system of this model in terms of $P_x(t)$ which is the probability of having x elements in the system at time t . $P_x(t+h)$ is equal to the sum of following these three mutually exclusive events.

(i) There is x elements at time t and no arrival occurs during h and no service is completed during h .

(ii) There is $x-1$ elements at time t and one arrival occurs during h and no service is completed during h .

(iii) There is $x+1$ elements at time t and no arrival occurs during h and one service is completed during h .

If we calculate the probabilities $P_x(t+h)$ and $P_o(t+h)$ and take limit for $h \to 0$, we obtain the following differential- difference equation system.

$$\begin{aligned} P'_x(t) &= -(\lambda_x+\mu_x)P_x(t)+\lambda_{x-1}P_{x-1}(t)+\mu_{x+1}P_{x+1}(t) \\ P'_o(t) &= -\lambda_o P_o(t)+\mu_1 P_1(t). \end{aligned}$$

If let us substitute $\lambda_x = \lambda x$, $\mu_x = \mu x$, we get

$$P'_x(t) = -(\lambda+\mu)xP_x(t)+\lambda(x-1)P_{x-1}(t)+\mu(x+1)P_{x+1}(t), \quad (14)$$

$$P'_o(t) = \mu P_1(t). \quad (15)$$

Let $E[x(t)]$ be the expected value of the elements in the system at time t.

$$E[x(t)] = \sum_{x=0}^{\infty} xP_x(t). \quad (16)$$

For our model $E[x(t)]$ means the expected value of the amount of production for time t. If we multiply the both sides of (14) by x and sum on $x = 1, 2, \cdots$ we obtain

$$\begin{aligned} E_t[x(t)] &= \sum_{x=1}^{\infty} x P'_x(t) \\ &= (\lambda - \mu) E[x(t)], \end{aligned}$$

and from this equation

$$E[x(t)] = c e^{(\lambda-\mu)t}.$$

If $E[x(0)] = i$, then it will be [10]

$$E[x(t)] = i e^{(\lambda-\mu)t}$$

Substituting the values of loss and gain rates obtained for one product, respectively (12) and (13), in this equation we get

$$\begin{aligned} E[x(t)] = i.exp\{[\theta + (\sigma - \delta)(1-p) - \\ (\alpha+\beta)(1-A)pP(y \leq r)]t\} \end{aligned} \tag{17}$$

3.5 The Determination of The Optimum Value of $P(y \leq r)$, r and p for Maximizing The Expected Value of Income

Let the rate of faulty products p have a degenerate probability density function. That is

$$f(p) = \begin{cases} 1 \ , & p = p_o \\ 0 \ , & p \neq p_o \end{cases}$$

For $p = p_o$,

$$E[\pi(t)] = \sum_{p=p_o} E[\pi(t)|p] f(p) = E[\pi(t)|p_o]$$

If we substitute $p_o = p$ and $P(y \leq r) = v$ for the sake of brevity and simplicity, we get from (10)

$$E[\pi(t)] = E[x(t)](1-A)(\pi_1 p + C_i - pC_s)v + \pi_1(1-p) - C_i.$$

If we substitute the value of $E[x(t)]$ given by (17) we get

$$\begin{aligned} E[\pi(t)] &= i[(1-A)(\pi_1 p + C_i - pC_s)v + \pi_1(1-p) - C_i]* \\ &* \ exp\{[\theta + (\sigma - \delta)(1-p) - (\alpha+\beta)(1-A)pv]t\}. \quad (18) \end{aligned}$$

In (18). let us substitute

$$\begin{aligned} a &= (1-A)(\pi_1 p + C_i - pC_s) \\ b &= \pi_1(1-p) - C_i \\ c &= (1-A)(\alpha+\beta)p \\ d &= \theta + (\sigma-\delta)(1-p). \end{aligned}$$

we get

$$E[\pi(t)] = i(av+b)e^{(-cv+d)t}. \quad (19)$$

In this section, the optimum value of r is r^* which is the upper boundary for the faulty products in the sample and the optimum value of p is p^* which is the defective rate will determine in order to maximize the expected value of the profit.

In the equation (19), if the first derivative with respect to v is equal to zero, it will be

$$v^* = \frac{a - bct}{act}. \quad (20)$$

If we substitute the value of v^* in the second derivative of $E[\pi(t)]$ with respect to v, we get

$$\frac{d^2E[\pi(t)]}{dv^2}\Big|_{v=v^*} = -iacte^{(-cv^*+d)t} \quad (21)$$

In the following, the value of v^*, the conditions for which $E[\pi(t)]$ has made maximum or minimum are investigated.

I. If all the product is not the subject of the final control, that is if $0 < A < 1$ then c will be positive, that is,if $\pi_1 - C_s > 0$ and for all $C_i > 0$, $a = p(\pi_1 - C_s) + C_i > 0$ and for $v = v^*$, $d^2E[\pi(t)]/dv^2 < 0$., so $E[\pi(t)]$ will be maximum in v^*. If $\pi_1 - C_s < 0$ but $a = p(\pi_1 - C_s) + C_i > 0$ then for $v = v^*$, $E[\pi(t)]$ will be maximum.

If $\pi_1 - C_s < 0$ and $a = p(\pi_1 - C_s) + C_i < 0$ then for $v = v^*$ $d^2E[\pi(t)]/dv^2 > 0$ so. $E[\pi(t)]$ will be minimum for $v = v^*$. This means that $E[\pi(t)]$ will reach to its maximum at $v = 0$ or $v = 1$. Due to this, let us compare the value of $E[\pi(t)]$ at $v = 1$ with its value at $v = 0$.

$$\frac{E[\pi(t)]|_{v=1}}{E[\pi(t)]|_{v=0}} = \frac{a+b}{b} e^{-ct}.$$

$a < 0$ and so $a + b < b$. on the other hand $v = 0$ can not avoid the profit being positive by itself and so $b > 0$ and $(a + b)/b < 1$. Since $c > 0$ and $t > 0$, $e^{-ct} < 1$. So,

$$\frac{E[\pi(t)]|_{v=1}}{E[\pi(t)]|_{v=0}} < 1$$

and $E[\pi(t)]$ will reach its maximum at $v = 0$.

The optimum value of the upper limit of the faulty products in the sample r^* was determined for $v^* = 0$, $0 < v^* < 1$, $v^* = 1$ and the following results were obtained:

II. If $v^* = 0$, then $v = P(y \le r) = 0$. In this case $P(y > r) = 1$, that is $A = 1$ and so the sample must be rejected and the whole of the product must be controlled. This means that $r = r^* = 0$. In this case, $a = c = 0$. $E[\pi(t)] = ibe^{dt}$ and is independent of v . Since the case $P(y \le r) = 0$ can not avoid the profit being positive by itself, we get the result that $b > 0$ and we will investigate it later on.

III. For $0 < v^* < 1$ when $a > 0$,

$$0 < v^* = \frac{a - bct}{act} < 1, \ ac > 0$$

and will be

$$0 < a - bct < act.$$

From $0 < a - bct$ we get $k > bl$. Here, $k = C_i + p(\pi_1 - C_s)$ and $l = (\alpha + \beta)pt$. From $a - bct < act$ we get $A < 1 - \frac{k-bl}{kl}$. Under these conditions, the optimum value r^* of the faulty products which is admissible in the sample is found from the relation

$$P(y \le r^*) = v^* = \sum_{i=0}^{r^*} C(Ax, i) p^i (1-p)^{Ax-i}.$$

IV. If $v^* = 1$ then $A = 1 - \frac{k-bl}{kl}$, $0 < A < 1$. The relation $P(y \le r^*) = v^* = 1$ means that the optimum admissible value of the faulty products in the sample is $r^* = Ax$, so $r = Ax$.

V. If $A = 1$, in other words if the whole of the product is controlled. then $a = c = 0$ and $E[\pi(t)] = ibe^{dt}$. This means that the expected value of the product is independent of v.

For this case let us investigate the path of $E[\pi(t)]$ which will be a function of the value $p = p_o$. The first derivative of $E[\pi(t)]$ with respect to p is

$$\frac{dE[\pi(t)]}{dp} = i\{-\pi_1 - [\pi_1(1-p) - C_i](\sigma - \delta)t\}exp\{[\theta + (\sigma - \delta)(1-p)]t\}.$$

Now, let us investigate the changes in $E[\pi(t)]$ for $(\sigma - \delta) > 0$ and $(\sigma - \delta) < 0$.

1^o) If $\sigma - \delta > 0$, then $dE[\pi(t)]/dp < 0$, so $E[\pi(t)]$ is monotone decreasing and reach its maximum at $p = 0$ and minimum at $p = 1$. On the other hand if $A = 1$, in other words if the value of p is so high that the control of the whole of the product is necessary, then p must be positive. The situation $p = 1$ is an extreme case which can not be encountered in practice so it can be omitted. If we assume that p can not be greater than a preassigned value p_2 and smaller than a preassigned value p_1, in other words if $0 < p_1 < p < p_2 < 1$ then

$$maxE[\pi(t)] = i[\pi_1(1 - p_1) - C_i] * exp\{[\theta + (\sigma - \delta)(1 - p_1)]t\}.$$

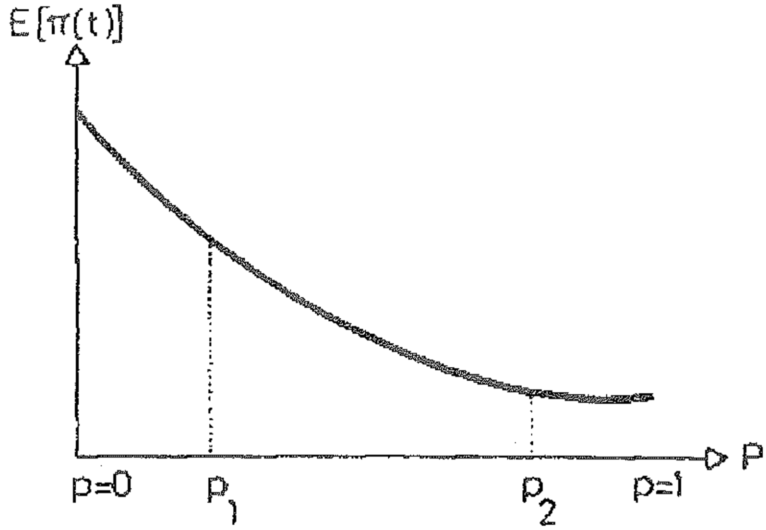
$$minE[\pi(t)] = i[\pi_1(1 - p_2) - C_i] * exp\{[\theta + (\sigma - \delta)(1 - p_2)]t\}.$$

On the other hand,

$$\frac{d^2E[\pi(t)]}{dp^2} = i(\sigma - \delta)t\{2\pi_1 + [\pi_1(1 - p) - C_i](\sigma - \delta)t\} exp\{[\theta + (\sigma - \delta)(1 - p)]t\}$$

and when $\sigma - \delta > 0$ then $d^2E[\pi(t)]/dp^2 > 0$ and $E[\pi(t)]$ is a monotone decreasing convex function(Figure 2).

Figure 2. The relation between p and $E[\pi(t)]$ for $\sigma - \delta > 0$



2^o) If $\sigma - \delta < 0$ then it is possible to solve the equation $dE[\pi(t)]/dp = 0$. From this equation,

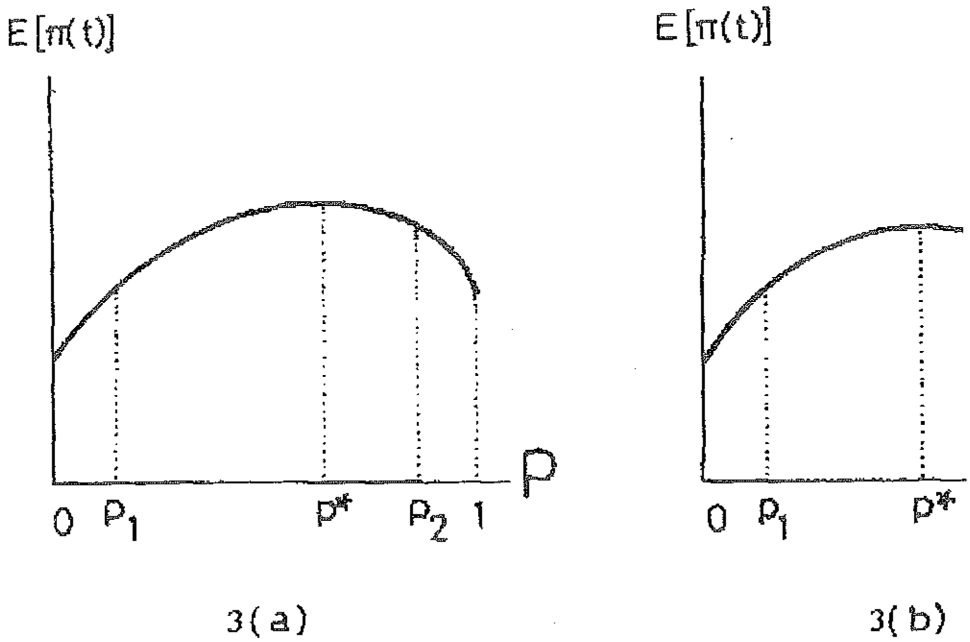
$$p^* = 1 - \frac{1}{\pi_1}[C_i - \frac{\pi_1}{(\sigma - \delta)t}].$$

Here. $0 < \frac{1}{\pi_1}[C_i - \frac{\pi_1}{(\sigma-\delta)t}] < 1$. And

$$\frac{d^2E[\pi(t)]}{dp^2}|_{p=p^*} = i\pi_1(\sigma - \delta)t * exp\{[\theta + (\sigma - \delta)(1 - p^*)]t\} < 0$$

This means that $E[\pi(t)]$ is maximum at $p = p^*$ and is minimum at one or both of the end points of the interval $(0, 1)$ (Figure 3(a) and Figure 3(b)).

Figure 3(a) and 3(b). The relation between p and $E[\pi(t)]$ for $\sigma - \delta < 0$



CONCLUSIONS

In this work, total quality cost was considered as a function of the quality level which is a function of the time variable. Net income was formulated as a function of the several components of the quality cost and was proved that optimum quality path requires an action toward zero defect point. The net income function at time t was formulated in terms of the cost $C_i[q(t)]$ caused by the inspection of a unit product and the lost $C_s[q(t)]$ caused by the use of a faulty product which was sold without final inspection. By establishing a correspondence between new customers and arrival process, lost customers and service process the problem was considered as a linear queuing problem. The rate of the faulty products p is considered as a random variable having a degenerate probability density function and the optimum boundries of the faulty products and the rate of defective items in the quality control samples was found in order to optimize the net income function.

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