On Some Lacunary Power Series with Algebraic Coefficients for Liouville Number Arguments

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Abstract. In this work, some lacunary power series with algebraic coefficients are considered and it is shown that under some conditions the values of the given series for some Liouville number arguments belong to either a certain algebraic number field or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of complex numbers, where m denotes the degree of the algebraic number field to which the coefficients of the series belong.

Keywords: Lacunary power series, $U-$ numbers.

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1. INTRODUCTION

A power series $F(z)=\sum_{i=0}^{\infty}\gamma_{n_i} z^{n_i}$ $\left(\gamma_{n_i}\in\mathbb{C},\, \gamma_{n_i}\neq 0,\, i=0,1,2,...\right)$ with a positive radius of convergence, where $\{n_i\}_{i=0}^{\infty}$ is a strictly increasing sequence of non-negative rational integers with $\lim_{i\to\infty}\frac{n_{i+1}}{n_i}=\infty$, is called a lacunary power series.

Cohn $[1]$, in 1946, showed that a lacunary power series with rational coefficients, under some conditions, takes transcendental values for non-zero algebraic number arguments.

Zeren $[11]$, in 1980, proved the following:

a) A lacunary power series with rational coefficients, under some conditions, takes values belonging to the subclass U_m in Mahler's classification of complex numbers for non-zero algebraic number arguments of degree m.

b) A lacunary power series with algebraic coefficients from a certain algebraic number field of degree m, under some conditions, takes values belonging to the subclass U_m in Mahler's classification of complex numbers for non-zero rational number arguments.

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In the present work, we will consider some lacunary power series with algebraic coefficients from a certain algebraic number field K of degree m, treated by Zeren $[11]$, and show that under some conditions these series take values belonging to either the algebraic number field K or $\bigcup_{i=1}^{m} U_i$ in Mahler's classification of complex numbers for some Liouville number arguments.

2. BACKGROUND

Mahler $[6]$, in 1932, divided the complex numbers into four classes, and called numbers in these classes $A-$numbers, $S-$numbers, $T-$numbers and $U-$numbers as follows.

We shall be concerned with polynomials $P(z) = a_n z^n + \cdots + a_0$ with rational integral coefficients. The height $H(P)$ of P is defined by $H(P) = \max(|a_n|, ..., |a_0|)$, and we shall denote the degree of P by $\deg(P)$.

Given a complex number ξ and natural numbers[2] n, H, Mahler $[6]$ puts

$$w_n(H,\xi) = \min_{\substack{\deg(P)\le n \\ H(P)\le H,\, P(\xi)\ne 0}} |P(\xi)| \quad .$$

The polynomial $P(z) \equiv 1$ is one of the polynomials which lie in the minimum, and so we have $0 < w_n(H,\xi) \le 1$. $w_n(H,\xi)$ is a non-increasing function of both n, H. Next, Mahler puts

$$w_n(\xi) = \overline{\lim_{H\to\infty}} \frac{-\log w_n(H,\xi)}{\log H}$$

and

$$w(\xi) = \overline{\lim_{n\to\infty}} \frac{w_n(\xi)}{n} \quad .$$

[2] A natural number means a positive rational integer.

$w_n(\xi)$ is a non-decreasing function of n. Furthermore, the inequalities $0 \le w_n(\xi) \le \infty$ and $0 \le w(\xi) \le \infty$ hold. If $w_n(\xi) = \infty$ for some integer n, let $\mu(\xi)$ be the smallest of such integers. In this case, we have $w_n(\xi) < \infty$ for $n < \mu(\xi)$; $w_n(\xi) = \infty$ for $n \ge \mu(\xi)$. If $w_n(\xi) < \infty$ for every n, put $\mu(\xi) = \infty$. So $\mu(\xi)$ and $w(\xi)$ are uniquely determined, and are never finite simultaneously, for the finiteness of $\mu(\xi)$ implies that there is an $n < \infty$ such that $w_n(\xi) = \infty$, whence $w(\xi) = \infty$. Therefore there are the following four possibilities for ξ. ξ is called

an $A-$number	if	$w(\xi) = 0$,	$\mu(\xi) = \infty$,
an $S-$number	if	$0 < w(\xi) < \infty$,	$\mu(\xi) = \infty$,
a $T-$number	if	$w(\xi) = \infty$,	$\mu(\xi) = \infty$,
a $U-$number	if	$w(\xi) = \infty$,	$\mu(\xi) < \infty$

(for more information see Mahler $[6]$). Every complex number ξ is of precisely one of these four types. $A-$numbers are precisely the algebraic numbers (see Schneider$[8]$). So the transcendental numbers are distributed into the three (disjoint) classes S, T, U. Let ξ be a $U-$number such that $\mu(\xi) = m$ and let U_m denote the set of all such numbers, i.e. $U_m = \{\xi \in U : \mu(\xi) = m\}$. Obviously, the set U_m $(m = 1, 2, 3, ...)$ is a subclass of U and U is the union of all the disjoint sets U_m. Leveque $[4]$ showed that U_m is not empty for any $m \ge 1$.

Koksma $[3]$, in 1939, set up another classification of the complex numbers. He divided the complex numbers into four classes A^*, S^*, T^*, U^* as follows.

Suppose that α is an algebraic number and $P(z)$ is the minimal defining polynomial of α such that its coefficients are rational integers, relatively prime and its highest coefficient is positive. Then the height $H(\alpha)$ of α is defined by $H(\alpha) = H(P)$ and the degree $\deg(\alpha)$ of α is defined as the degree of P.

Given a complex number ξ and natural numbers n, H, Koksma $[3]$ puts

$$w_n^*(H,\xi) = \min_{\substack{\alpha \text{ is algebraic} \\ \deg(\alpha) \le n}} |\xi - \alpha| \quad ,$$

$$H(\alpha) \le H,\ \alpha \ne \xi$$

$$w_n^*(\xi) = \overline{\lim_{H\to\infty}} \frac{-\log\left(H\, w_n^*\left(H,\xi\right)\right)}{\log H} ,$$

and

$$w^*(\xi) = \overline{\lim_{n\to\infty}} \frac{w_n^*(\xi)}{n} .$$

$w_n^*\left(H,\xi\right)$ is a non-increasing function of both n, H , and so $w_n^*(\xi)$ is a non-decreasing function of n. The functions $w_n^*(\xi)$ and $w^*(\xi)$ satisfy the respective inequalities $0 \le w_n^*(\xi) \le \infty$, $0 \le w^*(\xi) \le \infty$. If $w_n^*(\xi) = \infty$ for some integer n, let $\mu^*(\xi)$ be the smallest of such integers. In this case, we have $w_n^*(\xi) < \infty$ for $n < \mu^*(\xi)$; $w_n^*(\xi) = \infty$ for $n \ge \mu^*(\xi)$. If $w_n^*(\xi) < \infty$ for every n, put $\mu^*(\xi) = \infty$. So $\mu^*(\xi)$ and $w^*(\xi)$ are uniquely determined, and are never finite simultaneously. Therefore there are the following four possibilities for ξ. ξ is called

an A^*-number	if	$w^*(\xi) = 0$,	$\mu^*(\xi) = \infty$,
an S^*-number	if	$0 < w^*(\xi) < \infty$,	$\mu^*(\xi) = \infty$,
a T^*-number	if	$w^*(\xi) = \infty$,	$\mu^*(\xi) = \infty$,
a U^*-number	if	$w^*(\xi) = \infty$,	$\mu^*(\xi) < \infty$

(for more information see Koksma $[3]$). Every complex number ξ is of precisely one of these four types. Hence, the complex numbers are distributed into the four (disjoint) classes A^*, S^*, T^*, U^*. Let ξ be a U^*-number such that $\mu^*(\xi) = m$ and let U_m^* denote the set of all such numbers, i.e. $U_m^* = \left\{\xi \in U^* : \mu^*(\xi) = m\right\}$. Obviously, the set U_m^* $\left(m = 1,2,3,...\right)$ is a subclass of U^* and U^* is the union of all the disjoint sets U_m^* .

Wirsing $[9]$ proved that both classifications are equivalent, i.e. $A-$, $S-$, $T-$, $U-$numbers are the same as A^*-, S^*-, T^*-, U^*-numbers respectively. Moreover, $U_m = U_m^*$ $\left(m = 1,2,3,...\right)$.

A real number ξ is called a Liouville number if to each natural number n there exists a rational number $\frac{p_n}{q_n}$ $(p_n, q_n \in \mathbb{Z})$ such that the inequalities

$$q_n > 1, \quad 0 < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^n}$$

hold. We deduce from the definition that a Liouville number is an irrational number. The set of Liouville numbers is identical with the subclass U_1 in Mahler's classification (for more information about Liouville numbers see Perron [7], p. 178-190 and Schneider [8] , Kapitel I).

We need the following two lemmas to prove the main result of this paper.

Lemma 1. Let $\alpha_1,...,\alpha_k$ $(k \geq 1)$ be algebraic numbers which belong to an algebraic number field K of degree m, and let $F(y, x_1,...,x_k)$ be a polynomial with rational integral coefficients and with degree at least 1 in y. If η is any algebraic number such that $F(\eta, \alpha_1,...,\alpha_k) = 0$, then

$$\deg(\eta) \leq dm$$

and

$$H(\eta) \leq 3^{2dm+(l_1+\cdots+l_k)m} H^m H(\alpha_1)^{l_1 m} ... H(\alpha_k)^{l_k m} ,$$

where H is the height of the polynomial F, d is the degree of F in y, l_i $(i = 1,...,k)$ is the degree of F in x_i $(i = 1,...,k)$.

Proof. See İçen [2].

Lemma 2. Let α be an algebraic number of degree m and $\alpha^{(1)} = \alpha, \alpha^{(2)},...,\alpha^{(m)}$ be its conjugates. Then

$$\left|\overline{\alpha}\right| \leq 2H(\alpha),$$

where $\left|\overline{\alpha}\right| = \max\left(\left|\alpha^{(1)}\right|, \left|\alpha^{(2)}\right|,...,\left|\alpha^{(m)}\right|\right)$.

Proof. See Leveque $[4]$.

3. THE MAIN RESULT

Theorem. Let K be an algebraic number field of degree m, and

$$F(z)=\sum_{i=0}^{\infty}\gamma_{n_i}z^{n_i} \qquad \left(\gamma_{n_i}\in K,\ \gamma_{n_i}\neq 0,\ i=0,1,2,\ldots\right) \tag{1}$$

be a power series which satisfies the following conditions:

$$\lim_{i\to\infty}\frac{n_{i+1}}{n_i}=\infty\ , \tag{2}$$

$$\overline{\lim_{i\to\infty}}\frac{\log H(\gamma_{n_i})}{n_i}<\infty\ , \tag{3}$$

where $\{n_i\}_{i=0}^{\infty}$ is a strictly increasing sequence of positive rational integers.

Suppose that the radius of convergence R of the series $\sum_{i=0}^{\infty}H(\gamma_{n_i})\,z^{n_i}$ is positive (R may be finite or infinite). Moreover, let ξ be a Liouville number satisfying the following inequalities:

$$\left|\xi-\frac{p_{n_i}}{q_{n_i}}\right|\le\frac{1}{q_{n_i}^{n_i s_{n_i}}}\ \left(p_{n_i},q_{n_i}\in\mathbb{Z},\ q_{n_i}>1,\ s_{n_i}=\frac{n_{i+1}}{n_i\log q_{n_i}}\ (i=1,2,3,\ldots);\ \lim_{i\to\infty}s_{n_i}=\infty\right), \tag{4}$$

$$|\xi|<R\ . \tag{5}$$

Then either $F(\xi)$ is an algebraic number in K or $F(\xi)\in\bigcup_{i=1}^{m}U_i$.

Proof. We shall prove the theorem in four steps.

1) (1) is convergent for $z=\xi$. For since, by Lemma2, $\left|\gamma_{n_i}\right|\le\left|\overline{\gamma_{n_i}}\right|\le 2H(\gamma_{n_i})\ \ (i=0,1,2,\ldots)$, the radius of convergence of (1) is $\ge R$.

2) Let us consider the polynomials

$$F_k(z)=\sum_{i=0}^{k}\gamma_{n_i} z^{n_i} \qquad (k=1,2,3,...)\,. \tag{6}$$

Define the algebraic numbers

$$\eta_{n_k}=F_k\left(\frac{p_{n_k}}{q_{n_k}}\right)=\sum_{i=0}^{k}\gamma_{n_i}\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i}\in K \qquad (k=1,2,3,...)\,. \tag{7}$$

Since $\eta_{n_k}\in K \quad (k=1,2,3,...)$, $\deg(\eta_{n_k})\le m \quad (k=1,2,3,...)$. By multiplying both sides of the equality

$$\eta_{n_k}=\gamma_{n_0}\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_0}+\gamma_{n_1}\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_1}+...+\gamma_{n_k}\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_k}$$

by $q_{n_k}^{n_k}$, we obtain the equality

$$q_{n_k}^{n_k}\eta_{n_k}=\left(q_{n_k}^{n_k-n_0}p_{n_k}^{n_0}\right)\gamma_{n_0}+\left(q_{n_k}^{n_k-n_1}p_{n_k}^{n_1}\right)\gamma_{n_1}+...+p_{n_k}^{n_k}\gamma_{n_k}\,.$$

We can assume that $p_{n_k}\neq 0 \quad (k=1,2,3,...)$, since ξ is a Liouville number. Then the polynomial

$$P(y,x_0,x_1,...,x_k)=q_{n_k}^{n_k}y-\left(q_{n_k}^{n_k-n_0}p_{n_k}^{n_0}\right)x_0-\left(q_{n_k}^{n_k-n_1}p_{n_k}^{n_1}\right)x_1-...-p_{n_k}^{n_k}x_k$$

has rational integral coefficients and is of degree 1 in each $y,x_0,x_1,...,x_k$. Moreover, $P(\eta_{n_k},\gamma_{n_0},\gamma_{n_1},...,\gamma_{n_k})=0$. Denote the height of the polynomial $P(y,x_0,x_1,...,x_k)$ by H . Then, by Lemma 1, we obtain that

$$H(\eta_{n_k})\le 3^{2m+(k+1)m}H^m H(\gamma_{n_0})^m H(\gamma_{n_1})^m ... H(\gamma_{n_k})^m \qquad (k=1,2,3,...)\,. \tag{8}$$

By (4),

$$\left|\frac{p_{n_k}}{q_{n_k}}\right|<|\xi|+1 \qquad (k=1,2,3,...)\,, \tag{9}$$

and hence,

$$\left|p_{n_k}\right| < q_{n_k}\left(|\xi|+1\right) \qquad (k=1,2,3,\ldots). \tag{10}$$

Now we shall examine H, the height of the polynomial $P(y, x_0, x_1, \ldots, x_k)$.

$$H = \max\left(\left|q_{n_k}^{n_k}\right|, \left|q_{n_k}^{n_k-n_0} p_{n_k}^{n_0}\right|, \left|q_{n_k}^{n_k-n_1} p_{n_k}^{n_1}\right|, \ldots, \left|p_{n_k}^{n_k}\right|\right) \le \left|p_{n_k}\right|^{n_k} q_{n_k}^{n_k} \quad . \tag{11}$$

By (10) and (11),

$$H \le q_{n_k}^{2n_k}\left(|\xi|+1\right)^{n_k} \tag{12}$$

is obtained. Since ξ is a Liouville number, we can assume that $\lim_{k\to\infty} q_{n_k} = \infty$, and shall do so. For $\left(|\xi|+1\right)$ is independent of q_{n_k}, we have

$$|\xi|+1 \le q_{n_k} \tag{13}$$

for sufficiently large k. Hence, by (12) and (13),

$$H \le q_{n_k}^{3n_k} \tag{14}$$

for sufficiently large k. Since (3), the sequence $\left\{\frac{\log H(\gamma_{n_i})}{n_i}\right\}_{i=0}^{\infty}$ is bounded above. Then there exists a real number $M^* > 0$ such that

$$\frac{\log H(\gamma_{n_i})}{n_i} \le M^* \qquad (i=0,1,2,\ldots). \tag{15}$$

From (15) it follows that

$$H(\gamma_{n_i}) \le A^{n_i} \qquad (i=0,1,2,\ldots), \tag{16}$$

where $A = e^{M^*} > 1$ is a positive real constant independent of n_i. From (8), (14) and (16), we have for sufficiently large k

$$H(\eta_{n_k}) \le 3^{2m+(k+1)m} q_{n_k}^{3n_k m} A^{n_0 m} A^{n_1 m} \ldots A^{n_k m} \le 3^{4n_k m} q_{n_k}^{3n_k m} A^{\left(n_0+n_1+\ldots+n_k\right)m} \quad . \tag{17}$$

By (2), we have $\lim_{k\to\infty}\frac{n_k}{n_{k-1}}=\infty$, and therefore there is a natural number k^* such that

$$\frac{n_k}{n_{k-1}}>2 \tag{18}$$

for $k \ge k^*$. From (18), it follows that

$$2n_{k-1}<n_k \qquad (k \ge k^*). \tag{19}$$

Using (19), by induction,

$$n_{k^*}+n_{k^*+1}+\ldots+n_k<2n_k \qquad (k\ge k^*) \tag{20}$$

is obtained. By (17) and (20), we have for sufficiently large k

$$H(\eta_{n_k}) \le 3^{4n_k m} q_{n_k}^{3n_k m} A^{\left(n_0+\ldots+n_{k^*-1}\right)m} A^{\left(n_{k^*}+\ldots+n_k\right)m}$$

$$\le 3^{4n_k m} q_{n_k}^{3n_k m} A^{\left(n_0+\ldots+n_{k^*-1}\right)n_k m} A^{2n_k m} = c_0^{n_k} q_{n_k}^{3n_k m}\ , \tag{21}$$

where $c_0 = 3^{4m} A^{\left(n_0+\ldots+n_{k^*-1}\right)m} A^{2m}>0$ is a real constant independent of n_k and q_{n_k}. Since $c_0 \le q_{n_k}$ for sufficiently large k and (21), we obtain

$$H(\eta_{n_k}) \le q_{n_k}^{(3m+1)n_k} \tag{22}$$

for sufficiently large k.

3) We have

$$\left|F(\xi)-\eta_{n_k}\right| \le \left|F(\xi)-F_k(\xi)\right| + \left|F_k(\xi)-\eta_{n_k}\right| \qquad (k=1,2,3,\ldots). \tag{23}$$

Now we shall determine an upper bound for $\left|F(\xi)-F_k(\xi)\right|$ and $\left|F_k(\xi)-\eta_{n_k}\right|$.

$$\left|F(\xi)-F_k(\xi)\right| = \left|\sum_{i=k+1}^{\infty}\gamma_{n_i}\xi^{n_i}\right| \le \sum_{i=k+1}^{\infty}\left|\gamma_{n_i}\right|\left|\xi\right|^{n_i}\ . \tag{24}$$

Let us choose a real number ρ which satisfies the inequality

$$0<|\xi|<\rho<R \tag{25}$$

(If $R=\infty$, then ρ is chosen as $\rho>|\xi|$). Since the radius of convergence of $F(z)$ is $\geq R$, the series $F(\rho)=\sum_{i=0}^{\infty}\gamma_{n_i}\rho^{n_i}$ is convergent. Thus $\lim_{i\to\infty}\gamma_{n_i}\rho^{n_i}=0$, and so the sequence $\left\{\gamma_{n_i}\rho^{n_i}\right\}_{i=0}^{\infty}$ is bounded, and therefore there is a real number $M>0$ such that

$$\left|\gamma_{n_i}\right|\leq\frac{M}{\rho^{n_i}} \qquad (i=0,1,2,...). \tag{26}$$

From (24), (25) and (26), it follows that

$$\left|F(\xi)-F_k(\xi)\right|\leq\sum_{i=k+1}^{\infty}\frac{M}{\rho^{n_i}}|\xi|^{n_i}=M\sum_{i=k+1}^{\infty}\left(\frac{|\xi|}{\rho}\right)^{n_i}$$

$$=M\left(\frac{|\xi|}{\rho}\right)^{n_{k+1}}\left(1+\left(\frac{|\xi|}{\rho}\right)^{n_{k+2}-n_{k+1}}+\left(\frac{|\xi|}{\rho}\right)^{n_{k+3}-n_{k+1}}+\dots\right)$$

$$\leq M\left(\frac{|\xi|}{\rho}\right)^{n_{k+1}}\left(1+\frac{|\xi|}{\rho}+\left(\frac{|\xi|}{\rho}\right)^{2}+\dots\right)=\left(\frac{|\xi|}{\rho}\right)^{n_{k+1}}\frac{M}{1-\frac{|\xi|}{\rho}}\ . \tag{27}$$

By (27), we have

$$\left|F(\xi)-F_k(\xi)\right|\leq\frac{c_1}{c_2^{n_{k+1}}} \qquad (k=1,2,3,...), \tag{28}$$

where $c_1=\dfrac{M}{1-\frac{|\xi|}{\rho}}>0$, $c_2=\dfrac{\rho}{|\xi|}>1$ are real numbers independent of n_k and q_{n_k}.

$$\left|F_k(\xi)-\eta_{n_k}\right|=\left|\sum_{i=0}^{k}\gamma_{n_i}\left(\xi^{n_i}-\left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i}\right)\right|$$

$$\leq \sum_{i=0}^{k} \left|\gamma_{n_i}\right| \left|\xi - \frac{p_{n_k}}{q_{n_k}}\right| \left|\xi^{n_i-1} + \xi^{n_i-2}\frac{p_{n_k}}{q_{n_k}} + \ldots + \left(\frac{p_{n_k}}{q_{n_k}}\right)^{n_i-1}\right| . \qquad (29)$$

By (26),

$$\left|\gamma_{n_i}\right| \leq \frac{M}{\rho^{n_i}} \leq MB^{n_i} \leq M_1 B^{n_i} \leq M_1^{n_i} B^{n_i} = M_2^{n_i} \qquad (i = 0,1,2,\ldots) , \qquad (30)$$

where $B = \max\left(1, \frac{1}{\rho}\right) \geq 1,\ M_1 = \max(1, M) \geq 1,\ M_2 = M_1 B \geq 1$. Since (4), (9), (29), (30) and the fact that $|\xi| < |\xi| + 1$, it follows

$$\left|F_k(\xi) - \eta_{n_k}\right| \leq \sum_{i=0}^{k} M_2^{n_i} \frac{1}{q_{n_k}^{n_k s_{n_k}}} \left(|\xi|^{n_i-1} + |\xi|^{n_i-2}\left|\frac{p_{n_k}}{q_{n_k}}\right| + \ldots + \left|\frac{p_{n_k}}{q_{n_k}}\right|^{n_i-1} \right)$$

$$\leq \frac{1}{q_{n_k}^{n_k s_{n_k}}} \sum_{i=0}^{k} M_2^{n_k} n_i \left(|\xi|+1\right)^{n_i-1} \leq \frac{1}{q_{n_k}^{n_k s_{n_k}}} \sum_{i=0}^{k} M_2^{n_k} n_k \left(|\xi|+1\right)^{n_k}$$

$$= \frac{1}{q_{n_k}^{n_k s_{n_k}}} (k+1) M_2^{n_k} n_k \left(|\xi|+1\right)^{n_k} \leq \frac{1}{q_{n_k}^{n_k s_{n_k}}} n_k^2 M_2^{n_k} \left(|\xi|+1\right)^{n_k} . \qquad (31)$$

On the other hand, for $\lim_{k\to\infty} n_k = \infty$, $\lim_{k\to\infty} \sqrt[n_k]{n_k} = 1$, and so $\lim_{k\to\infty} \sqrt[n_k]{n_k^2} = 1$. Then there is a real number $c_3 > 1$ such that

$$n_k^2 \leq c_3^{n_k} \qquad (32)$$

for sufficiently large k . From (31) and (32),

$$\left|F_k(\xi) - \eta_{n_k}\right| \leq \frac{1}{q_{n_k}^{n_k s_{n_k}}} c_3^{n_k} M_2^{n_k} \left(|\xi|+1\right)^{n_k} = \frac{1}{q_{n_k}^{n_k s_{n_k}}} c_4^{n_k} , \qquad (33)$$

where $c_4 = c_3 M_2 \left(|\xi|+1\right) > 1$ is a real constant independent of n_k and q_{n_k} . For sufficiently large k ,

$$c_4 \leq q_{n_k} . \qquad (34)$$

By (33) and (34),

$$\left|F_k(\xi)-\eta_{n_k}\right| \le \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)}} \tag{35}$$

for sufficiently large k.

Now let us turn to (28). Let λ be a positive real number which will be specified later. For sufficiently large k, the inequality

$$\frac{c_1}{c_2^{n_{k+1}}} \le \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \tag{36}$$

holds. For otherwise the inequality

$$\frac{c_1}{c_2^{n_{k+1}}} > \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \tag{37}$$

would hold for infinitely many k. Thus, if we denote the subsequence of natural numbers formed by the numbers k satisfying (37) by $\left\{k_j\right\}_{j=1}^{\infty}$,

$$\frac{c_1}{c_2^{n_{k_j+1}}} > \frac{1}{q_{n_{k_j}}^{n_{k_j}\left(s_{n_{k_j}}-1\right)\lambda}} \qquad (j=1,2,3,...)$$

is obtained. From this and (4),

$$\log c_1 - n_{k_j+1}\log c_2 > -n_{k_j}(s_{n_{k_j}}-1)\lambda\log q_{n_{k_j}}$$

$$\Rightarrow \log c_1 + n_{k_j}(s_{n_{k_j}}-1)\lambda\log q_{n_{k_j}} > n_{k_j+1}\log c_2$$

$$\Rightarrow \log c_1 + n_{k_j}\left(\frac{n_{k_j+1}}{n_{k_j}\log q_{n_{k_j}}}-1\right)\lambda\log q_{n_{k_j}} > n_{k_j+1}\log c_2$$

$$\Rightarrow \log c_1 + \left(n_{k_j+1} - n_{k_j}\log q_{n_{k_j}}\right)\lambda > n_{k_j+1}\log c_2$$

$$\Rightarrow \frac{\log c_1}{n_{k_j+1}} + \left(1 - \frac{n_{k_j} \log q_{n_{k_j}}}{n_{k_j+1}}\right)\lambda > \log c_2$$

$$\Rightarrow \lim_{j\to\infty}\left(\frac{\log c_1}{n_{k_j+1}} + \left(1 - \frac{n_{k_j} \log q_{n_{k_j}}}{n_{k_j+1}}\right)\lambda\right) \geq \lim_{j\to\infty} \log c_2$$

$$\Rightarrow \lambda \geq \log c_2 > 0 \ .$$

Then if we choose the real number λ as $0 < \lambda < \log c_2$, the inequality (36) holds for sufficiently large k . On the other hand, for $0 < \lambda < 1$, the inequality

$$\frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)}} \leq \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \tag{38}$$

holds for sufficiently large k. Finally, if the real number λ is chosen as $0 < \lambda < \min\left(1, \log c_2\right)$, it follows from (28) and (36) that

$$\left|F(\xi) - F_k(\xi)\right| \leq \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \tag{39}$$

for sufficiently large k , and from (35) and (38) that

$$\left|F_k(\xi) - \eta_{n_k}\right| \leq \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \tag{40}$$

for sufficiently large k . By (23), (39) and (40),

$$\left|F(\xi) - \eta_{n_k}\right| \leq \frac{2}{q_{n_k}^{n_k\left(s_{n_k}-1\right)\lambda}} \leq \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-2\right)\lambda}} \tag{41}$$

for sufficiently large k . We deduce from (41) that $\left\{\left|F(\xi) - \eta_{n_k}\right|\right\}$ is a null sequence, for $\lim_{k\to\infty} \frac{1}{q_{n_k}^{n_k\left(s_{n_k}-2\right)\lambda}} = 0$. In other words,

$$\lim_{k\to\infty} \eta_{n_k} = F(\xi) . \tag{42}$$

It follows from (22) and (41) that

$$\left| F(\xi) - \eta_{n_k} \right| \le \frac{1}{H(\eta_{n_k})^{\frac{(s_{n_k}-2)\lambda}{3m+1}}} \tag{43}$$

for sufficiently large k. Note that $\lim_{k\to\infty} \frac{(s_{n_k}-2)\lambda}{3m+1} = \infty$, since $\lim_{k\to\infty} s_{n_k} = \infty$, $\lambda > 0$ and $3m+1>0$. Put $\omega_{n_k} = \frac{(s_{n_k}-2)\lambda}{3m+1}$ $(k = 1,2,3,...)$. Then from (43),

$$\left| F(\xi) - \eta_{n_k} \right| \le \frac{1}{H(\eta_{n_k})^{\omega_{n_k}}} \qquad (\lim_{k\to\infty} \omega_{n_k} = \infty) \tag{44}$$

for sufficiently large k.

4) There exist the following two cases for the null sequence $\left\{ \left| F(\xi) - \eta_{n_k} \right| \right\}$:

a) $\left| F(\xi) - \eta_{n_k} \right| = 0$ from some k onward :

In this case $\eta_{n_k} = F(\xi)$ from some k onward, that is, $\left\{ \eta_{n_k} \right\}$ is a constant sequence. Since $\eta_{n_k} \in K$ $(k = 1,2,3,...)$, in the case a) it is obtained that $F(\xi)$ is an algebraic number in K.

b) $\left| F(\xi) - \eta_{n_k} \right| \ne 0$ for infinitely many k:

In this case the sequence $\left\{ \eta_{n_k} \right\}$ has an infinite number of different terms. For otherwise $\left\{ \eta_{n_k} \right\}$ would have a finite number of different terms, and so the sequence $\left\{ \left| F(\xi) - \eta_{n_k} \right| \right\}$ would have a finite number of different terms. Since $\left| F(\xi) - \eta_{n_k} \right| \ne 0$ for an infinite number of k, there is a non-zero term in the sequence $\left\{ \left| F(\xi) - \eta_{n_k} \right| \right\}$. Then

$\left\{\left|F(\xi)-\eta_{n_k}\right|\right\}$ would have only a finite number of different terms which are not zero. Hence, let us denote the different and non-zero terms in the sequence $\left\{\left|F(\xi)-\eta_{n_k}\right|\right\}$ by $u_1, u_2, ..., u_t \quad (t \geq 1)$. Put $c = \min(u_1, u_2, ..., u_t)$. Note that c is a positive real number, since all the $u_i \quad (i = 1, 2, ..., t)$ are positive real numbers. Thus, for any natural number k

$$\text{either } \left|F(\xi)-\eta_{n_k}\right| = 0 \text{ or } \left|F(\xi)-\eta_{n_k}\right| \geq c. \tag{45}$$

Since $\left\{\left|F(\xi)-\eta_{n_k}\right|\right\}$ is a null sequence, there exists a natural number k_0 such that

$$\left|F(\xi)-\eta_{n_k}\right| < c \tag{46}$$

for $k \geq k_0$. However, since $\left|F(\xi)-\eta_{n_k}\right| \neq 0$ for an infinite number of k, there exists a natural number $\bar{k} > k_0$ for which $\left|F(\xi)-\eta_{n_{\bar{k}}}\right| \neq 0$. From (45), we have $\left|F(\xi)-\eta_{n_{\bar{k}}}\right| \geq c$ which contradicts (46). Therefore $\left\{\eta_{n_k}\right\}$ must have an infinite number of different terms.

The sequence $\left\{H(\eta_{n_k})\right\}$ of natural numbers, formed by the heights of the algebraic numbers η_{n_k}, is unbounded above. For otherwise there would be a real number $M_3 > 0$ such that $H(\eta_{n_k}) \leq M_3$ for $k = 1, 2, 3, ...$. Then since also $\deg(\eta_{n_k}) \leq m \quad (k = 1, 2, 3, ...)$, the sequence $\left\{\eta_{n_k}\right\}$ would have a finite number of different terms, contrary to the fact that $\left\{\eta_{n_k}\right\}$ has an infinite number of different terms. Thus $\overline{\lim_{k\to\infty}} H(\eta_{n_k}) = \infty$, for $\left\{H(\eta_{n_k})\right\}$ is unbounded above. Since $\overline{\lim_{k\to\infty}} H(\eta_{n_k}) = \infty$, the sequence $\left\{H(\eta_{n_k})\right\}$ of natural numbers has a subsequence $\left\{H(\eta_{n_{k_j}})\right\}_{j=1}^{\infty}$ such that

$$1 < H(\eta_{n_{k_1}}) < H(\eta_{n_{k_2}}) < H(\eta_{n_{k_3}}) < ..., \qquad \lim_{j\to\infty} H(\eta_{n_{k_j}}) = \infty. \tag{47}$$

Since (47), the terms of the sequence $\left\{\eta_{n_{k_j}}\right\}_{j=1}^{\infty}$ are all different, i.e. if $i \neq j$, then $\eta_{n_{k_i}} \neq \eta_{n_{k_j}}$. So the sequence $\left\{\eta_{n_{k_j}}\right\}_{j=1}^{\infty}$ may have at most one term equal to $F(\xi)$. If there is a term equal to $F(\xi)$ among the terms $\eta_{n_{k_j}}$ $(j = 1,2,3,...)$, i.e. if there exists a natural number j_0 for which $\eta_{n_{k_{j_0}}} = F(\xi)$, then we throw away the first j_0 terms $\eta_{n_{k_1}}, \eta_{n_{k_2}}, ..., \eta_{n_{k_{j_0}}}$ and renumber the terms of the sequence $\left\{\eta_{n_{k_j}}\right\}$ $(j_0 + 1 \to 1, j_0 + 2 \to 2, ...)$, and so all the terms of the sequence $\left\{\eta_{n_{k_j}}\right\}$ are now different from $F(\xi)$. To summarize, the sequence $\left\{\eta_{n_k}\right\}_{k=1}^{\infty}$ has a subsequence $\left\{\eta_{n_{k_j}}\right\}_{j=1}^{\infty}$ for which the following properties hold:

i) $\eta_{n_{k_j}} \neq F(\xi) \qquad (j = 1,2,3,...)$,

ii) $1 < H(\eta_{n_{k_1}}) < H(\eta_{n_{k_2}}) < H(\eta_{n_{k_3}}) < ..., \qquad \lim_{j\to\infty} H(\eta_{n_{k_j}}) = \infty$,

iii) $\deg(\eta_{n_{k_j}}) \leq m \quad (j = 1,2,3,...)$, for $\eta_{n_{k_j}} \in K \quad (j = 1,2,3,...)$.

From (44) and i), we obtain for sufficiently large j that

$$0 < \left|F(\xi) - \eta_{n_{k_j}}\right| \leq \frac{1}{H(\eta_{n_{k_j}})^{\omega_{n_{k_j}}}} \qquad (\lim_{j\to\infty} \omega_{n_{k_j}} = \infty). \tag{48}$$

Put $H_j = H(\eta_{n_{k_j}}) > 1 \quad (j = 1,2,3,...)$. Since ii), $\left\{H_j\right\}_{j=1}^{\infty}$ is a strictly increasing subsequence of natural numbers. By i), ii), iii) and (48),

$$w_m^*\left(H_j, F(\xi)\right) = \min_{\substack{\alpha \text{ is algebraic} \\ \deg(\alpha) \leq m \\ H(\alpha) \leq H_j,\, \alpha \neq F(\xi)}} \left|F(\xi) - \alpha\right| \leq \left|F(\xi) - \eta_{n_{k_j}}\right|$$

$$\leq \frac{1}{H(\eta_{n_{k_j}})^{\omega_{n_{k_j}}}} = \frac{1}{H_j^{\omega_{n_{k_j}}}} \qquad \text{(for sufficiently large } j \text{)}$$

$$\Rightarrow 0 < w_m^*\left(H_j, F(\xi)\right) \leq \frac{1}{H_j^{\omega_{n_{k_j}}}} \qquad \text{(for sufficiently large } j \text{)}$$

$$\Rightarrow 0 < H_j \; w_m^*\left(H_j, F(\xi)\right) \leq \frac{1}{H_j^{\omega_{n_{k_j}}-1}} \qquad \text{(for sufficiently large } j \text{)}$$

$$\Rightarrow \frac{1}{H_j \; w_m^*\left(H_j, F(\xi)\right)} \geq H_j^{\omega_{n_{k_j}}-1} > 0 \qquad \text{(for sufficiently large } j \text{)}$$

$$\Rightarrow \frac{\log \dfrac{1}{H_j \; w_m^*\left(H_j, F(\xi)\right)}}{\log H_j} \geq \omega_{n_{k_j}} - 1 \qquad \text{(for sufficiently large } j \text{)}$$

$$\Rightarrow \lim_{j\to\infty} \frac{\log \dfrac{1}{H_j \; w_m^*\left(H_j, F(\xi)\right)}}{\log H_j} = \infty \text{, since } \lim_{j\to\infty} \omega_{n_{k_j}} = \infty .$$

$$\Rightarrow \overline{\lim_{H\to\infty}} \frac{-\log\left(H \; w_m^*\left(H, F(\xi)\right)\right)}{\log H} = \infty$$

$$\Rightarrow w_m^*\left(F(\xi)\right) = \infty$$

$$\Rightarrow F(\xi) \in U^* \text{ and } \mu^*\left(F(\xi)\right) \leq m$$

$$\Rightarrow F(\xi) \in \bigcup_{i=1}^{m} U_i^* \; . \qquad (49)$$

From (49), we obtain that $F(\xi) \in \bigcup_{i=1}^{m} U_i$, since U_m^* is identical with U_m for any natural number m. Hence, in the case b) $F(\xi) \in \bigcup_{i=1}^{m} U_i$.

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