On Transcendence of Values of Some Generalized Lacunary Power Series with Algebraic Coefficients for Some Algebraic Arguments in *p*-Adic Domain I•

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Abstarct. In this paper transcendence of values of some generalized lacunary power series with algebraic coefficients for some algebraic arguments are studied, and these values were also analyzed of which $U_m\,(m \geq 1)$ subclass of U class in p-adic domain should be in.

Keywords: The Field of p-Adic Numbers, Generalized Lacunary Power Series, Algebraic Numbers, Transcendence, Koksma Classification.

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1. INTRODUCTION

Let $\{r_n\}$ and $\{s_n\}$ be two infinite sequences of non-negative integers satisfying the following condition

$$0 \leq s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \dots .$$

Let's consider the power series

$$F(z) = \sum_{h=0}^{\infty} c_h z^h ,$$

where

$$\begin{aligned} c_h &= 0, \quad r_n < h < s_n \qquad n = 1, 2, \dots, \\ c_h &\neq 0, \quad h = r_n \qquad\qquad n = 1, 2, \dots, \\ c_h &\neq 0, \quad h = s_n \qquad\qquad n = 0, 1, \dots . \end{aligned}$$

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This power series can also be written as

$$F(z)=\sum_{k=0}^{\infty} P_k(z),$$

such that

$$P_k(z)=\sum_{h=s_k}^{r_{k+1}} c_h z^h \quad (k=0,1,\ldots).$$

A power series with these properties is called generalized lacunary power series. If $s_k = r_{k+1}$ $(k=0,1,2,\ldots)$, then $P_k(z)=c_{r_{k+1}} z^{r_{k+1}}$ $(k=0,1,2,\ldots)$, and we can write

$$F(z)=\sum_{n=0}^{\infty} c_{r_{n+1}} z^{r_{n+1}} \quad (c_{r_{n+1}} \neq 0) .$$

A power series with these properties is called simple lacunary power series for it separates from generalized lacunary power series.

In this paper we shall be concerned with generalized lacunary power series.

Chon [1] in 1946 showed that the values of simple lacunary power series with rational coefficients are transcendental for some algebraic arguments. The paper was transferred to p-adic domain by Şenkon [2] in 1977. In 1946, Mahler [3] obtained that the values of generalized lacunary power series with rational coefficients are transcendental for some algebraic arguments. In 1977, Braune [4] obtained further results about the values of generalized lacunary power series with algebraic coefficients. In 1980, Zeren [5] showed that the values of simple lacunary power series with rational coefficient for some algebraic arguments and the values of simple lacunary power series with algebraic coefficient for some rational arguments belong to the Mahler's U_m-subclass $(m\geq 1)$. In the same paper, the results were transferred to p-adic domain. In 1988, Zeren [6] obtained that the values of generalized lacunary power series with algebraic coefficient for some algebraic arguments belong to Mahler's U_m-subclass $(m\geq 1)$.

With this paper, it is proved that the values of generalized lacunary power series with algebraic coefficients for some algebraic arguments belong to the p-adic U_m-subclass $(m \geq 1)$ in p-adic domain, and so Zeren's [6] paper is transferred to p-adic domain by using Koksma classification in p-adic domain.

2. PRELIMINARIES

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and p denotes natural numbers, integer numbers, rational numbers and a given prime number respectively. $|.|_p$ and $\mathbb{Q}_p$ denotes p-adic valuation on $\mathbb{Q}$ and the field of p-adic numbers respectively.

2.1. Mahler's Classification in $\mathbb{Q}_p$ [2]

Let n be a natural number. The height of the polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x], \quad a_n \neq 0,$$

denoted by $H(P)$, is the form

$$H(P) = \max\left(|a_n|, \ldots, |a_1|, |a_0| \right).$$

The degree of the polynomial $P(x)$ is denoted with $\deg(P)$. Let ξ be an element of $\mathbb{Q}_p$. For given positive integer n and real number $H\ (\geq 1)$, we define the quantity

$$w_n(H, \xi) := \min_{\substack{P(x)\in\mathbb{Z}[x] \\ H(P)\leq H \\ \deg(P)\leq n \\ P(\xi)\neq 0}} \left\{ |P(\xi)|_p \right\}.$$

It is clear that

$$0 < w_n(H, \xi) \leq 1,$$

[2] See [7].

since $\left|P(\xi)\right|_p=1$ for $P(x)=1$. $w_n(H,\xi)$ is a non-increasing function of both n and H. Then we set

$$w_n(\xi)=\overline{\lim_{H\to+\infty}}\frac{\log\dfrac{1}{w_n(H,\xi)}}{\log H} \qquad \text{and} \qquad w(\xi)=\overline{\lim_{n\to+\infty}}\frac{w_n(\xi)}{n}.$$

$w_n(\xi)$ as a function of n is non-decreasing. The inequalities $0\le w_n(\xi)\le+\infty$ and $0\le w(\xi)\le+\infty$ ($n\ge1$, $H\ge1$) hold.

If $w_n(\xi)=+\infty$ for some integers n, let $\mu(\xi)$ $(=\mu)$ be the smallest of such integers, and if $w_n(\xi)<+\infty$ for every n, put $\mu(\xi)=+\infty$. The two quantities $\mu(\xi)$, $w(\xi)$ are never finite simultaneously. Then the number ξ is called an

A-number if $w(\xi)=0$, $\mu(\xi)=+\infty$,

S-number if $0<w(\xi)<+\infty$, $\mu(\xi)=+\infty$,

T-number if $w(\xi)=+\infty$, $\mu(\xi)=+\infty$,

U-number if $w(\xi)=+\infty$, $\mu(\xi)<+\infty$.

All p-adic numbers are distributed into the four classes A, S, T, U. With this classification:

1) A-numbers are exactly algebraic numbers[3].

2) If two p-adic numbers are algebraically dependent, then they belong to the same class[4].

[3] See [7].

[4] See [7].

Let ξ be a U-number such that $\mu(\xi)=m$, and let U_m denotes the set of all such numbers. For every natural m, U_m-class is a subclass of U, and $U_m \cap U_n = \varnothing$ if $m \neq n$. Therefore we have the partition $U = \bigcup_{m=1}^{\infty} U_m$.

2.2. Koksma's Classification in $\mathbb{Q}_p$[5]:

Let α be a p-adic number. The height of the p-adic number α, denoted by $H(\alpha)$, is the height of its minimal polynomial over $\mathbb{Z}$. The degree of the p-adic number α is denoted by $\deg(\alpha)$.

Let ξ be a p-adic number. For given positive integer n and real number $H\ (\geq 1)$, we define the quantity

$$w_n^*\ (H, \xi) := \min_{\substack{\alpha \text{ algebraic number} \\ H(\alpha) \leq H \\ \deg(\alpha) \leq n \\ \alpha \neq \xi}} \left\{ \left| \xi - \alpha \right|_p \right\}.$$

Then we set

$$w_n^*(\xi) = \overline{\lim_{H \to +\infty}} \frac{\log \dfrac{1}{w_n^*(H, \xi)}}{\log H} \qquad \text{and} \qquad w^*(\xi) = \overline{\lim_{n \to +\infty}} \frac{w_n^*(\xi)}{n}.$$

The inequalities $0 \leq w_n^*(\xi) \leq +\infty$ and $0 \leq w^*(\xi) \leq +\infty$ hold.

If $w_n^*(\xi) = +\infty$ for some integers n, let $\mu^*(\xi)\ (=\mu^*)$ be the smallest of such integers, if $w_n^*(\xi) < +\infty$ for every n, put $\mu^*(\xi) = +\infty$. The two quantities $\mu^*(\xi)$, $w^*(\xi)$ are never finite simultaneously. Then the number ξ is called an

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[5] See [8].

A^*-number if $w^*(\xi)=0$, $\mu^*(\xi)=+\infty$,

S^*-number if $0<w^*(\xi)<+\infty$, $\mu^*(\xi)=+\infty$,

T^*-number if $w^*(\xi)=+\infty$, $\mu^*(\xi)=+\infty$,

U^*-number if $w^*(\xi)=+\infty$, $\mu^*(\xi)<+\infty$.

Hence, all p-adic numbers are distributed into the four classes A^*, S^*, T^*, U^*.

Let ξ be a U^*-number such that $\mu^*(\xi)=m$, and let U_m^* denotes the set of all such numbers. For every natural m, U_m^*-class is a subclass of U^*, and $U_m^* \cap U_n^* = \varnothing$ if $m \neq n$. Therefore we have the partition $U^* = \bigcup_{m=1}^{\infty} U_m^*$.

Schlikewei [8] proved that p-adic T-numbers exist. It follows from the results of Schlikewei [8] and Long [9] that the classifications of Mahler and of Koksma are equivalent.

Let ξ be a p-adic number and let m be a positive integer. The number ξ is called a U_m^*-number if $\mu^*(\xi)=m$, and $\mu^*(\xi)=m$ if the following conditions are satisfied:

i) For every $\omega>0$, if there are infinitely many algebraic numbers η of degree m such that

$$0<\left|\xi-\eta\right|_p \leq c\,H(\eta)^{-\omega},$$

then

$$\mu^*(\xi)\leq m \qquad \text{(that is } \xi \in U_1^* \cup U_2^* \cup \ldots \cup U_m^*\text{)},$$

where the positive constant c is independent of $H(\eta)$.

ii) If there exists constants $c'>0$ and s depending only on ξ and m such that the relation

$$|\xi-\beta|_p > c' H(\beta)^{-s}$$

holds for every algebraic number β of degree $<m$, then

$$\mu^*(\xi)\geq m \qquad \text{(that is } \xi\notin U_1^*\cup U_2^*\cup\ldots\cup U_{m-1}^*\text{)}.$$

For the proof of main result, we shall need the following lemmas.

Lemma 2.1 Let $\alpha_1,\ldots,\alpha_k$ $(k\geq 1)$ be algebraic numbers in $\mathbb{Q}_p$ with $[\mathbb{Q}(\alpha_1,\ldots,\alpha_k):\mathbb{Q}]=g$ and let $F(y,x_1,\ldots,x_k)$ be a polynomial with integral coefficients, whose degree in y is at least one. If η is an algebraic number such that $F(\eta,\alpha_1,\ldots,\alpha_k)=0$, then the degree of $\eta\ \leq dg$ and

$$H(\eta)\leq 3^{2dg+(l_1+\ldots+l_k)g}\,H^g\,H(\alpha_1)^{l_1 g}\ldots H(\alpha_k)^{l_k g},$$

where $H(\eta)$ is the height of η, $H(\alpha_i)$ is the height of α_i $(i=1,\ldots,k)$, H is the maximum of the absolute values of the coefficients of F, l_i is the degree of F in x_i $(i=1,\ldots,k)$ and d is the degree of F in y.

Proof. See [10].

Lemma 2.2 Let $P(x)$ be a polynomial of degree n with rational integer coefficients, and let $H(P)$ denote the height of $P(x)$ and let α_i $(i=1,\ldots,n)$ denote the roots of $P(x)$. Then for $\alpha_i\neq\alpha_j$ $(i\neq j)$

$$|\alpha_i-\alpha_j|_p > \frac{c}{H(P)^{n-1}},$$

where c is a positive constant depending on n but not on $H(P)$.

Proof. See [11].

Lemma 2.3 Let α, β be two p-adic algebraic numbers such that they have different minimal polynomials and let t, k be degrees of α, β respectively. Then for $|\alpha|_p = p^{-h}$ and $r = \min(0, h)$

$$|\alpha - \beta|_p > \frac{c}{H(\alpha)^{M-1} H(\beta)^M},$$

where $M > \max(t, k)$ and $c = \dfrac{p^{(M-1)r - M(|h|+1)}}{(2M)!}$.

Proof. See [8].

3. MAIN RESULT

Theorem Let $\{r_n\}$, $\{s_n\}$ be two infinite sequence of integers satisfying

$$0 \le s_0 \le r_1 < s_1 \le r_2 < s_2 \le r_3 < s_3 \le \ldots .$$

Let

$$\begin{aligned} F(z) &= \sum_{h=0}^{\infty} c_h z^h = \sum_{k=0}^{\infty} P_k(z) \\ P_k(z) &= \sum_{h=s_k}^{r_{k+1}} c_h z^h \end{aligned} \tag{3.1}$$

be a generalized lacunary power series such that

$$\begin{aligned} &c_h = 0, \quad r_n < h < s_n \qquad n = 1, 2, \ldots, \\ &c_h \ne 0, \quad h = r_n \qquad n = 1, 2, \ldots, \\ &c_h \ne 0, \quad h = s_n \qquad n = 0, 1, \ldots, \end{aligned}$$

where the coefficients c_h are algebraic numbers in a constant number field $\mathrm{K} = \mathrm{K}(\theta)$ such that $[\mathrm{K} : \mathbb{Q}] = c$, and $c_h = 0$ if $r_n < h < s_n$, but $c_{r_n} \ne 0$, $c_{s_n} \ne 0$ $(n = 1, 2, \ldots)$, and let

$$\overline{\lim_{i\to\infty}} \sqrt[i]{\left| c_i^{\{j\}} \right|_p} < +\infty \quad (j=1,\dots,c), \tag{3.2}$$

where $c_i^{\{j\}}$ $(j=1,\dots,c)$ denote the conjugates of c_i over K. Furthermore, suppose that the following conditions hold:

$$\lim_{n\to\infty} \frac{s_n}{r_n} = +\infty, \tag{3.3}$$

$$\overline{\lim_{n\to\infty}} \frac{r_n}{s_{n-1}} := \tau < +\infty, \tag{3.4}$$

$$\overline{\lim_{n\to\infty}} \frac{\log A_n}{n} := \sigma < +\infty, \tag{3.5}$$

$$\overline{\lim_{n\to\infty}} \frac{\log h_n}{n} := l < +\infty \quad \left(h_n = H(c_n)\right), \tag{3.6}$$

where a_v is a suitable natural number such that $a_v c_v$ is an algebraic integer and $A_v = [a_0,\dots,a_v]$ is the least common multiple of $a_0,\dots,a_v$. Let α be an algebraic numbers of degree m satisfying $0 < \left|\overline{\alpha}\right|_p < R$ such that $\left|\overline{\alpha}\right|_p = \max \left|\alpha^{(i)}\right|_p$ and $R = \min_{j=1}^{c} \frac{1}{\overline{\lim_{i\to\infty}} \sqrt[i]{\left| c_i^{\{j\}} \right|_p}}$. Then $F(\alpha) \in U_t$ for $z=\alpha$, where t is the maximum of the degrees of the partial sums $F_n(\alpha) = \sum_{k=0}^{n-1} P_k(\alpha)$ and $t \le [\mathbb{Q}(\theta,\alpha):\mathbb{Q}] := g \le cm$. Also, assume that $P_n(\alpha) \ne 0$ for infinitely many integers n.

Proof. 1°) The radius of convergence of (3.1) is $\ge R$ from the condition (3.2), since the series $\sum_{h=0}^{\infty} \left| c_h \right|_p z^h$ is majorant of the series $\sum_{h=0}^{\infty} c_h z^h$.

2°) Let's take $F(\alpha) = \beta$. We can write

$$\beta = \beta_n + \rho_n, \tag{3.7}$$

such that

$$\beta_n = \sum_{k=0}^{n-1} P_k(\alpha) = \sum_{v=s_0}^{r_n} c_v(\theta)\alpha^v, \tag{3.8}$$

$$\rho_n = \sum_{k=n}^{\infty} P_k(\alpha) = \sum_{v=s_n}^{\infty} c_v(\theta)\alpha^v. \tag{3.9}$$

Now, we will find an upper bound for the height $H(\beta_n)$ of β_n.

If both sides of the equality (3.8) are multiplied by A_{r_n}, then we have

$$A_{r_n}\beta_n - \sum_{v=s_0}^{r_n} A_{r_n} c_v(\theta)\alpha^v = 0, \tag{3.10}$$

where $A_{r_n} c_v(\theta)$ $(v = s_0, \dots, r_n)$ is an algebraic integer since a_n is a suitable natural number such that $a_n c_n(\theta)$ is an algebraic integer, and $A_{r_n} = \left[a_0, \dots, a_{r_n} \right]$. Also, $A_{r_n} c_v(\theta)$ is an element in $K = \mathbb{Q}(\theta)$. Therefore we can write that

$$A_{r_n} c_v(\theta) = \frac{\xi_0^{(v)}}{D} + \frac{\xi_1^{(v)}}{D}\theta + \dots + \frac{\xi_{c-1}^{(v)}}{D}\theta^{c-1}, \tag{3.11}$$

where the numbers $D = \left| \Delta^2(1, \theta, \dots, \theta^{c-1}) \right|$, $\xi_0^{(v)}, \xi_1^{(v)}, \dots, \xi_{c-1}^{(v)}$ are rational integers and $D \ge 1$.

By using (3.10) and (3.11), it follows that

$$D A_{r_n}\beta_n - \sum_{\mu=0}^{c-1}\sum_{v=s_0}^{r_n} \xi_\mu^{(v)}\theta^\mu\alpha^v = 0, \tag{3.12}$$

where $D A_{r_n}$ and $\xi_\mu^{(v)}$ $(\mu = 0, 1, \dots, c-1)$ are rational integers.

Let’s consider the polynomial

$$P(y,x_1,x_2)=D\,A_{r_n}\,y-\sum_{\mu=0}^{c-1}\sum_{v=s_0}^{r_n}\xi_{\mu}^{(v)}\,x_1^{\mu}\,x_2^{v}\ . \tag{3.13}$$

The value of the polynomial $P(y,x_1,x_2)$ for $y=\beta_n$, $x_1=\theta$, $x_2=\alpha$ is the left side of the equality (3.12).

Now, we would like to obtain the quantity $\left|\,\xi_{\mu}^{(v)}\,\right|$. We put

$$D\,A_{r_n}\,c_v\,(\theta)=\delta\,. \tag{3.14}$$

Then, from (3.11), we have

$$\delta=\xi_0^{(v)}+\xi_1^{(v)}\,\theta+\ldots+\xi_{c-1}^{(v)}\,\theta^{c-1}\,.$$

For the conjugates of θ, we have the linear equation system

$$\delta^{\{j\}}=\xi_0^{(v)}+\xi_1^{(v)}\,\theta^{\{j\}}+\ldots+\xi_{c-1}^{(v)}\,(\theta^{\{j\}})^{c-1}\quad(j=1,\,2,\,\ldots,c)\,. \tag{3.15}$$

The solution of this equation system is

$$\xi_{\mu}^{(v)}=\sum_{j=1}^{c}\frac{\Delta_{\mu_j}}{\Delta}\,\delta^{\{j\}}\quad(\mu=0,\,1,\,\ldots,c-1)\,, \tag{3.16}$$

where Δ and Δ_{μ_j} depends only on θ and on the conjugates of θ but not on δ (and so n and v).

Since D and A_{r_n} are rational integers, by (3.14), we have

$$\left|\overline{\delta}\right|\le D\,A_{r_n}\left|\overline{c_v}\right|. \tag{3.17}$$

Hence, combining the relations (3.16) and (3.17), we obtain

$$\left|\,\xi_{\mu}^{(v)}\,\right|=\left|\sum_{j=1}^{c}\frac{\Delta_{\mu_j}}{\Delta}\,\delta^{\{j\}}\right|\le\hat{C}(K)\,A_{r_n}\left|\overline{c_v}\right|\quad(\mu=0,\,1,\,\ldots,c-1)\,, \tag{3.18}$$

where $\hat{C}(\mathrm{K})$ is a positive constant in the number field K.

By using (3.4) and (3.5), we have the relations

$$\frac{r_n}{s_{n-1}} < \tau_1 , \qquad (3.4') \qquad \text{and} \qquad \frac{\log A_{r_n}}{r_n} < \sigma_1 , \qquad (3.5')$$

for $n > N_0$ ($N_0 = N_0(\tau_1, \sigma_1)$), where $\tau_1 > \tau$ and $\sigma_1 > \sigma$. From (3.6), we can write $h_n < B^n$ for all sufficiently large n, where $B > 1$. Moreover there is a number $K > 1$ such that

$$h_n \le K B^n \qquad (n = 0, 1, \ldots), \qquad (3.6')$$

Since $\left|\overline{c_n}\right| \le 2 h_n$, it follows from $(3.6')$ that

$$\left|\overline{c_n}\right| \le 2 K B^n \quad (n = 0, 1, \ldots). \qquad (3.6'')$$

Hence we see from (3.13), (3.18) and $(3.6'')$ that

$$H = \max_{\mu, \nu} \left(\left| D A_{r_n} \right|, \left| \xi_{\mu}^{(\nu)} \right| \right) \le C(K) A_{r_n} B^{r_n} , \qquad (3.19)$$

where $C(K) = \max(D, \hat{C}(K)) 2K$ (>1) is a positive constant in the number field K (H denotes the height of the polynomial P).

We can use Lemma 2.1 with $l_1 = c - 1$, $l_2 = r_n$, $d = 1$, from (3.19) and $(3.5')$, we obtain

$$\begin{aligned} H(\beta_n) &\le 3^{2g + (c-1+r_n)g} H^g H(\theta)^{(c-1)g} H(\alpha)^{r_n g} \\ &= \left[(3 e^{\sigma_1} B H(\alpha))^{r_n} 3^{c+1} C(\mathrm{K}) H(\theta)^{(c-1)} \right]^g , \end{aligned}$$

or putting $c_0 = \left[3^{c+2} e^{\sigma_1} B H(\alpha) C(K) H(\theta)^{(c-1)} \right]^g$

$$H(\beta_n) \le c_0^{r_n} \quad (n > N_0), \qquad (3.20)$$

where $c_0 > 1$ since $\sigma_1 > \sigma \ge 0$, $B > 1$ and $C(K) > 1$.

3°) Now we shall give an upper bound for $|\beta-\beta_n|_p=|\rho_n|_p$. Let's $0<R<+\infty$ ($R=\min\limits_{1\le j\le c}\dfrac{1}{\overline{\lim\limits_{i\to\infty}}\sqrt[i]{\left|c_i^{\{j\}}\right|_p}}$). If $\rho_j=\dfrac{1}{\overline{\lim\limits_{i\to\infty}}\sqrt[i]{\left|c_i^{\{j\}}\right|_p}}$ $(j=1,\ldots,c)$, then $R\le\rho_j$ $(j=1,\ldots,c)$ and so $R=\min\limits_{1\le j\le c}\rho_j$.

We easily see that for all sufficiently large i

$$\sqrt[i]{\left|c_i^{\{j\}}\right|_p}<\frac{1}{\rho_j-\varepsilon}\le\frac{1}{R-\varepsilon}\quad (j=1,\ldots,c),$$

and so

$$\left|c_i^{\{j\}}\right|_p<\frac{1}{(R-\varepsilon)^i}\quad (j=1,\ldots,c),$$

where ε is any sufficiently small positive number. Also, there is $M_1>0$ such that

$$\left|c_i^{\{j\}}\right|_p\le\frac{M_1}{(R-\varepsilon)^i}\quad (j=1,\ldots,c)\ (i=0,1,\ldots). \tag{3.21}$$

If we choose a sufficiently small positive number ε, then we have $R-\varepsilon>\left|\overline{\alpha}\right|_p$ (e.g. $\varepsilon=\dfrac{R-\left|\overline{\alpha}\right|_p}{2}$). Thus $\dfrac{\left|\overline{\alpha}\right|_p}{R-\varepsilon}<1$.

By using (3.9) and (3.21), we have

$$|\beta-\beta_n|_p=|\rho_n|_p=\left|\sum_{v=s_n}^{\infty}c_v\,\alpha^v\right|_p\le M_1\left(\frac{\left|\overline{\alpha}\right|_p}{R-\varepsilon}\right)^{s_n}\quad (n=1,2,\ldots). \tag{3.22}$$

If $R=+\infty$, then we can choose any arbitrary number ρ which is $\rho>\left|\overline{\alpha}\right|_p$. Because the series $\sum_{n=0}^{\infty} c_n^{\{j\}}\rho^n$ $(j=1, 2, \ldots, c)$ converges, and so there exists a positive M_2 such that

$$\left|c_n^{\{j\}}\right|_p \leq \frac{M_2}{\rho^n} \quad (j=1,\ldots, c)\ (n=0, 1,\ldots). \tag{3.23}$$

Here $\frac{\left|\overline{\alpha}\right|_p}{\rho}<1$. Combining (3.9) and (3.23), we have

$$|\beta-\beta_n|_p=|\rho_n|_p=\left|\sum_{v=s_n}^{\infty} c_v\,\alpha^v\right|_p \leq M_2\left(\frac{\left|\overline{\alpha}\right|_p}{\rho}\right)^{s_n} \quad (n=1, 2, \ldots) \tag{3.24}$$

Putting $\max(M_1, M_2)=c_1$ and $\min(\rho, R-\varepsilon)=\rho^*$, we write from (3.21) and (3.23)

$$\left|c_n^{\{j\}}\right|_p \leq \frac{c_1}{(\rho^*)^n} \quad (j=1,\ldots, c)\ (n=0, 1,\ldots), \tag{3.25}$$

and from (3.22) and (3.24)

$$|\beta-\beta_n|_p=|\rho_n|_p \leq c_1\left(\frac{\left|\overline{\alpha}\right|_p}{\rho^*}\right)^{s_n} \quad (n=1, 2, \ldots). \tag{3.26}$$

Using $\frac{\rho^*}{\left|\overline{\alpha}\right|_p}=c_2$ $(c_2>1)$ in the relation (3.26), we obtain

$$|\beta-\beta_n|_p \leq \frac{c_1}{c_2^{s_n}} \quad (n=1, 2, \ldots). \tag{3.27}$$

It follows from (3.20) and (3.27) that

$$|\beta-\beta_n|_p \leq \frac{c_1}{(H(\beta_n))^{\frac{s_n}{r_n}c_3}} \quad (n>N_0), \tag{3.28}$$

where $c_3 = \dfrac{\log c_2}{\log c_0}$ (>0). Hence from (3.3), we see that

$$\lim_{n\to\infty} \frac{s_n}{r_n} c_3 = +\infty . \tag{3.29}$$

4°) Now, we will examine the sequence of height $\{H(\beta_n)\}$ and the sequence of degree $\{d(\beta_n)\}$ of the algebraic numbers β_n. These sequences provide the following conditions A, B, C.

A) $\{H(\beta_n)\}$ is not bounded from above.

Proof. If $\{H(\beta_n)\}$ were bounded from above, then $\{\beta_n\}$ would contain only finitely many different elements since the degrees of the algebraic numbers β_n are bounded from above with g. But the sequence $\{\beta_n\}$ contains infinitely many different elements: Since there are infinitely many non-zero polynomials $P_k(\alpha)$, we have

$$|\beta_{n+1} - \beta_n|_p = |P_n(\alpha)|_p \neq 0$$

for infinitely many integers n. In this case, the sequence $\{\beta_n\}$ contains at least two different elements. If the sequence $\{\beta_n\}$ were contained only finitely many different elements, then either it would be $|\beta_{n+1} - \beta_n|_p = 0$ or there could be a positive number k such that $|\beta_{n+1} - \beta_n|_p \geq k$ (If the values of finitely many different β_n were $v_1, \ldots, v_p$ $(p \geq 2)$, then it could be $k = \min\limits_{\substack{r,s=1,\ldots,p \\ r\neq s}} |v_r - v_s|_p$). Since the series $F(\alpha) = \sum_{h=0}^{\infty} c_h \alpha^h = \sum_{k=0}^{\infty} P_k(\alpha)$ converges, the value $|\beta_{n+1} - \beta_n|_p = |P_n(\alpha)|_p$ is sufficiently small for all sufficiently large n. Since $|P_n(\alpha)|_p \neq 0$ for infinitely many integers n, for a sufficiently large $\bar{n}$, we have $0 < |P_{\bar{n}}(\alpha)|_p < k$, that is $0 < |\beta_{\bar{n}+1} - \beta_{\bar{n}}|_p < k$, which is a contradiction to the abovementioned case. Hence, the sequence $\{\beta_n\}$ contains infinitely many different elements.

B) Starting from a suitable n, $\{d(\beta_n)\}$ is a constant sequence.

Proof. There are two different cases:

a) Let $d(\beta_n)=1$ as starting from a suitable n. Then the condition B) is satisfied.

b) Let $d(\beta_n)>1$ for infinitely many integers n. If $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ for a fixed pair (i,j) ($i \neq j$) and for any sufficiently large n, then $\beta_{n+1}^{\{i\}} \neq \beta_{n+1}^{\{j\}}$: from (3.8), we write

$$\beta_{n+1} = \beta_n + c_{s_n}\alpha^{s_n} + \ldots + c_{r_{n+1}}\alpha^{r_{n+1}}. \tag{3.30}$$

Therefore

$$\beta_{n+1}^{\{i\}} = \beta_n^{\{i\}} + c_{s_n}^{\{i\}}\left(\alpha^{\{i\}}\right)^{s_n} + \ldots + c_{r_{n+1}}^{\{i\}}\left(\alpha^{\{i\}}\right)^{r_{n+1}},$$
$$\beta_{n+1}^{\{j\}} = \beta_n^{\{j\}} + c_{s_n}^{\{j\}}\left(\alpha^{\{j\}}\right)^{s_n} + \ldots + c_{r_{n+1}}^{\{j\}}\left(\alpha^{\{j\}}\right)^{r_{n+1}}$$

for $i \neq j$. Hence

$$\begin{aligned}\left|\beta_{n+1}^{\{i\}} - \beta_{n+1}^{\{j\}}\right|_p = \Big|&(\beta_n^{\{i\}} - \beta_n^{\{j\}}) + (c_{s_n}^{\{i\}}(\alpha^{\{i\}})^{s_n} - c_{s_n}^{\{j\}}(\alpha^{\{j\}})^{s_n}) \\ &+ \ldots + (c_{r_{n+1}}^{\{i\}}(\alpha^{\{i\}})^{r_{n+1}} - c_{r_{n+1}}^{\{j\}}(\alpha^{\{j\}})^{r_{n+1}})\Big|_p.\end{aligned} \tag{3.31}$$

If

$$\left|\beta_n^{\{i\}} - \beta_n^{\{j\}}\right|_p > \left|(c_{s_n}^{\{i\}}(\alpha^{\{i\}})^{s_n} - c_{s_n}^{\{j\}}(\alpha^{\{j\}})^{s_n}) + \ldots + (c_{r_{n+1}}^{\{i\}}(\alpha^{\{i\}})^{r_{n+1}} - c_{r_{n+1}}^{\{j\}}(\alpha^{\{j\}})^{r_{n+1}})\right|_p, \tag{3.32}$$

then

$$\left|\beta_{n+1}^{\{i\}} - \beta_{n+1}^{\{j\}}\right|_p = \left|\beta_n^{\{i\}} - \beta_n^{\{j\}}\right|_p. \tag{3.33}$$

Now, we shall show that the inequality (3.32) is valid for all sufficiently large n.

If the degree of β_n is denoted by q, then $2 \leq q \leq g$. Using Lemma 2.2, we have

$$\left| \beta_n^{\{i\}} - \beta_n^{\{j\}} \right|_p > \frac{c_4}{H(\beta_n)^{q-1}},$$

where c_4 is a positive constant independent of $H(\beta_n)$. Hence

$$\left| \beta_n^{\{i\}} - \beta_n^{\{j\}} \right|_p > \frac{c_4}{H(\beta_n)^{g-1}}. \tag{3.34}$$

Combining the relations (3.20) and (3.34), we obtain

$$\left| \beta_n^{\{i\}} - \beta_n^{\{j\}} \right|_p > \frac{c_4}{c_0^{r_n(g-1)}} \quad (n > N_0). \tag{3.35}$$

Now, let's consider the right side of the inequality (3.32). From (3.25), we obtain

$$\begin{aligned} &\left| (c_{s_n}^{\{i\}} (\alpha^{\{i\}})^{s_n} - c_{s_n}^{\{j\}} (\alpha^{\{j\}})^{s_n}) + \ldots + (c_{r_{n+1}}^{\{i\}} (\alpha^{\{i\}})^{r_{n+1}} - c_{r_{n+1}}^{\{j\}} (\alpha^{\{j\}})^{r_{n+1}}) \right|_p \\ &\qquad \le c_1 \left(\frac{\left|\overline{\alpha}\right|_p}{\rho^*} \right)^{s_n} = \frac{c_1}{c_2^{s_n}}, \end{aligned} \tag{3.36}$$

where $c_2 = \dfrac{\rho^*}{\left|\overline{\alpha}\right|_p} \; (>1)$. For a sufficiently large n_0, we have

$$\frac{c_1}{c_2^{s_n}} < \frac{c_4}{c_0^{r_n(g-1)}} \quad (n > n_0). \tag{3.37}$$

Then we see from (3.35), (3.36) and (3.37) that

$$\begin{aligned} \left| \beta_n^{\{i\}} - \beta_n^{\{j\}} \right|_p &> \frac{c_4}{c_0^{r_n(g-1)}} > \frac{c_1}{c_2^{s_n}} \\ &\ge \left| (c_{s_n}^{\{i\}} (\alpha^{\{i\}})^{s_n} - c_{s_n}^{\{j\}} (\alpha^{\{j\}})^{s_n}) + \ldots + (c_{r_{n+1}}^{\{i\}} (\alpha^{\{i\}})^{r_{n+1}} - c_{r_{n+1}}^{\{j\}} (\alpha^{\{j\}})^{r_{n+1}}) \right|_p \end{aligned}$$

for all sufficiently large n. Finally, the inequality (3.32) is satisfied for all sufficiently large n. In this case we have $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ for all n which are larger than a suitable n. This is exactly valid, because $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ is satisfied for at least a pair (i, j) and for infinitely many integers n from the hypothesis b). This case can also be provided for all pairs (i, j). Hence we have

$$d(\beta_{N_1}) \le d(\beta_{N_1+1}) \le d(\beta_{N_1+2}) \le \ldots$$

for a sufficiently large N_1. Since $d(\beta_n) \le g$, for a sufficiently large N_2 we can write that

$$d(\beta_{N_2}) = d(\beta_{N_2+1}) = d(\beta_{N_2+2}) = \ldots,$$

such that $N_2 \ge N_1$. If the common value is shown by t, then

$$d(\beta_n) = t, \quad n \ge N_2. \tag{3.38}$$

C) We can choose a subsequence of the sequence $\{\beta_n\}$ such that

0) $\beta_{n_j} \ne \beta \quad (j = 1, 2, \ldots)$.

1) $\left\{ H(\beta_{n_j}) \right\}$ is the monotone increasing sequence of natural numbers, hence it is diverges to $+\infty$.

2) $\left\{ d(\beta_{n_j}) \right\}$ is a constant sequence.

Proof. The proof is obtained from the properties A) and B); also the constant value of $d(\beta_{n_j})$ is t **.**

5°) We shall show that the number β is a U^*-number. To show this, we will use the subsequence $\left\{ \beta_{n_j} \right\}$ defined in C).

Putting $H_{n_j} = H(\beta_{n_j})$, from C)-0) and (3.28), we can write

$$\begin{aligned} w_t^*(H_{n_j}, \beta) &:= \min_{\substack{\deg(\eta) \le t \\ H(\eta) \le H_{n_j} \\ \eta \ne \beta}} \left\{ \left| \beta - \eta \right|_p \right\} \\ &\le \left| \beta - \beta_{n_j} \right|_p < \frac{1}{H_{n_j}^{\frac{s_{n_j}}{r_{n_j}} c_3}} \quad , \quad n_j > \max(N_0, N_2). \end{aligned} \tag{3.39}$$

We have from (3.39)

$$\frac{-\log(w_t^*(H_{n_j},\beta))}{\log H_{n_j}} > \frac{s_{n_j}}{r_{n_j}} c_3 , \tag{3.40}$$

and from (3.29) and (3.40)

$$\lim_{n_j\to\infty} \frac{-\log(w_t^*(H_{n_j},\beta))}{\log H_{n_j}} = +\infty . \tag{3.41}$$

Finally, we obtain

$$w_t^*(\beta) = \overline{\lim_{n_j\to\infty}} \frac{-\log(w_t^*(H_{n_j},\beta))}{\log H_{n_j}} = +\infty . \tag{3.42}$$

Thus, it follows from the definition of $\mu^*(\beta)$ that

$$\mu^*(\beta) \le t . \tag{3.43}$$

This shows that the number β is a U^*-number.

6°) Now, we will show that $\mu^*(\beta) = t$.

a) If $t=1$, then $\mu^*(\beta)=1$ from (3.43). In this case $\beta \in U_1^*$.

b) If $t>1$, then we shall show that

$$w_{t-1}^*(\beta) < +\infty . \tag{3.44}$$

Consider

$$|\beta-\gamma|_p \ge |\beta_l-\gamma|_p - |\beta_l-\beta|_p , \tag{3.45}$$

where γ is a algebraic number of degree $<t$. We would like to find an upper bound for $|\beta_l-\beta|_p$ and a lower bound for $|\beta_l-\gamma|_p$.

Let $H(\beta_l)$ be the height of β_l, and let $H(\gamma)$ and s be the height and the degree of γ respectively. It is satisfied the inequality $1\le s\le t-1$. The degree of β_l is exactly t for $l\ge N_2$. Therefore we can use Lemma 2.3, and so we obtain

$$|\beta_l-\gamma|_p>\frac{c_5}{H(\beta_l)^{M-1}H(\gamma)^M} \tag{3.46}$$

for $M>t$ and $l\ge N_2$, where c_5 is a positive constant independent of γ. For $l>\max(N_0, N_2)$, we have from (3.20), (3.46)

$$|\beta_l-\gamma|_p>\frac{c_5}{H(\gamma)^M c_0^{r_l(M-1)}},$$

and from $(3.4')$

$$|\beta_l-\gamma|_p>\frac{c_5}{H(\gamma)^M c_0^{s_{l-1}\tau_1(M-1)}} \tag{3.47}$$

since $c_0>1$. From (3.27), (3.45) and (3.47), we have

$$|\beta-\gamma|_p\ge\frac{c_5}{H(\gamma)^M c_0^{s_{l-1}\tau_1(M-1)}}-\frac{c_1}{c_2^{s_l}},$$

and so

$$|\beta-\gamma|_p\ge\frac{c_5}{H(\gamma)^M c_0^{s_{l-1}\tau_1(M-1)}}-\frac{c_1}{c_0^{s_l c_3}}. \tag{3.48}$$

Let's take a number λ such that

$$\lambda>1 \tag{3.49}$$

(the value of λ will be announced later). Since $s_{n-1}\le r_n$, we get from (3.3)

$$\lim_{n\to+\infty}\frac{s_n}{s_{n-1}}=+\infty .$$

Therefore, for the number μ which is chosen such that

$$\mu>\lambda , \tag{3.50}$$

there exists $N_3\in\mathbb{N}$ such that

$$\frac{s_n}{s_{n-1}}>\mu \tag{3.51}$$

for $n>N_3$ (the value of μ will be announced later).

Now let's consider the inequality

$$c_0^{s_{n-1}}\le H(\gamma)<c_0^{s_n} \tag{3.52}$$

for any algebraic number γ satisfying the relation

$$H(\gamma)>H_0 , \tag{3.53}$$

such that

$$H_0=\max\left(c_0^{s_{N_0}} , c_0^{s_{N_2}} , c_0^{s_{N_3}} , \left(\frac{2c_1}{c_5}\right)^{1/c_3} \right). \tag{3.54}$$

There is exactly only one n satisfying the inequality (3.52): since $\lim_{l\to\infty}c_0^{s_l}=+\infty$, there are infinitely many indexes l such that $c_0^{s_l}>H(\gamma)$. Also, from (3.54) and (3.53) there are indexes l such that $c_0^{s_l}\le H(\gamma)$, but the number of these indexes l is finite. Let the maximum of these indexes l be $n-1$. In this case it can be seen easily that $c_0^{s_{n-1}}\le H(\gamma)<c_0^{s_n}$. Now we shall show that there is only one n satisfying the inequality (3.52): If two solutions of (3.52) were n_1, n_2, then it would have

$$\left.\begin{array}{l} c_0^{s_{n_1}-1} \le H(\gamma) < c_0^{s_{n_1}} \\ c_0^{s_{n_2}-1} \le H(\gamma) < c_0^{s_{n_2}} \end{array}\right\} \begin{array}{lll} c_0^{s_{n_1}-1} \le H(\gamma) < c_0^{s_{n_2}} & \Rightarrow n_1 - 1 < n_2 & \Rightarrow n_1 \le n_2 \\ c_0^{s_{n_2}-1} \le H(\gamma) < c_0^{s_{n_1}} & \Rightarrow n_2 - 1 < n_1 & \Rightarrow n_2 \le n_1 \end{array} \left.\vphantom{\begin{array}{l}a\\b\end{array}}\right\} \Rightarrow n_1 = n_2 .$$

From (3.50), (3.53) and (3.52), we have

$$n > \max(N_0, N_2, N_3). \tag{3.55}$$

We see that from (3.50), (3.51) and (3.55)

$$s_{n-1} < \frac{s_n}{\lambda}, \tag{3.56}$$

and from (3.49)

$$\frac{s_n}{\lambda} < s_n . \tag{3.57}$$

In this case, the interval $\left[\, c_0^{s_{n-1}}, c_0^{s_n} \right)$ can be divided into two subintervals such that these subintervals are $\left[\, c_0^{s_{n-1}}, c_0^{s_n/\lambda} \right)$ and $\left[\, c_0^{s_n/\lambda}, c_0^{s_n} \right)$. Then $H(\gamma)$ satisfying the relation (3.52) belong to one of the following two subintervals:

I) $c_0^{s_{n-1}} \le H(\gamma) < c_0^{s_n/\lambda}$,

II) $c_0^{s_n/\lambda} \le H(\gamma) < c_0^{s_n}$.

Case I) If we write the relation (3.48) with l replaced by n, then we get

$$|\beta - \gamma|_p \ge \frac{c_5}{H(\gamma)^{M+\tau_1(M-1)}} - \frac{c_1}{H(\gamma)^{\lambda c_3}} . \tag{3.58}$$

If we choose

$$\lambda := \frac{M + \tau_1(M-1)}{c_3} + 1, \tag{3.59}$$

then from (3.58)

$$|\beta-\gamma|_p \geq \frac{1}{H(\gamma)^{M+\tau_1(M-1)}}\left\{c_5-\frac{c_1}{H(\gamma)^{c_3}}\right\}; \quad H(\gamma)>H_0. \tag{3.60}$$

From (3.54) and (3.53), we have

$$H(\gamma)>\left(\frac{2c_1}{c_5}\right)^{1/c_3} \quad \Rightarrow \quad c_5-\frac{c_1}{H(\gamma)^{c_3}}>\frac{c_5}{2}>0. \tag{3.61}$$

Hence from (3.60) and (3.61), we obtain

$$|\beta-\gamma|_p \geq \frac{c_5/2}{H(\gamma)^{M+\tau_1(M-1)}} \;; \; H(\gamma)>H_0. \tag{3.62}$$

Case II) If we write the relation (3.48) with l replaced by $n+1$, then we get

$$|\beta-\gamma|_p \geq \frac{c_5}{H(\gamma)^{M+\lambda\tau_1(M-1)}}-\frac{c_1}{H(\gamma)^{\frac{s_{n+1}}{s_n}c_3}}, \tag{3.63}$$

and using the inequality (3.51), we obtain

$$|\beta-\gamma|_p \geq \frac{c_5}{H(\gamma)^{M+\lambda\tau_1(M-1)}}-\frac{c_1}{H(\gamma)^{\mu c_3}}. \tag{3.64}$$

If we choose

$$\mu := \frac{M+\lambda\tau_1(M-1)}{c_3}+1, \tag{3.65}$$

then from (3.64)

$$|\beta-\gamma|_p \geq \frac{1}{H(\gamma)^{M+\lambda\tau_1(M-1)}}\left\{c_5-\frac{c_1}{H(\gamma)^{c_3}}\right\}; \; H(\gamma)>H_0. \tag{3.66}$$

Hence we write from (3.61)

$$|\beta-\gamma|_p \geq \frac{c_5/2}{H(\gamma)^{M+\lambda\tau_1(M-1)}} \;\; ; \; H(\gamma)>H_0 . \tag{3.67}$$

The inequality (3.67) is also satisfied in case I), since $M+\lambda\tau_1(M-1)>M+\tau_1(M-1)$ from (3.49). Putting $x=M+\lambda\tau_1(M-1)$ in both cases, we have

$$|\beta-\gamma|_p \geq \frac{c_5/2}{H(\gamma)^x} \;\; ; \; H(\gamma)>H_0 , \tag{3.68}$$

since $M>t$, $x>t$.

From 5°), we have

$$w^*_{t-1}(H_0,\beta)= \min_{\substack{s\leq t-1\\ H(\gamma)\leq H_0\\ \gamma\neq\beta}} \left\{ |\beta-\gamma|_p \right\} \leq |\beta-\gamma|_p \tag{3.69}$$

for all γ which have $s\leq t-1$ and $H(\gamma)\leq H_0$. Hence we can write

$$|\beta-\gamma|_p \geq w^*_{t-1}(H_0,\beta) \geq \frac{w^*_{t-1}(H_0,\beta)}{H(\gamma)^x} \;; \quad s\leq t-1,\, H(\gamma)\leq H_0 . \tag{3.70}$$

Putting

$$c_6=\min\left(\frac{c_5}{2},\, w^*_{t-1}(H_0,\beta) \right), \tag{3.71}$$

we obtain from (3.68), (3.70)

$$|\beta-\gamma|_p \geq \frac{c_6}{H(\gamma)^x} \tag{3.72}$$

for $s\leq t-1$ and $H(\gamma)=1, 2, \ldots$. We see from (3.72) that

$$|\beta-\gamma|_p \geq \frac{c_6}{H^x} \tag{3.73}$$

for all γ which have $s\leq t-1$ and $H(\gamma)\leq H$ (where H is any positive integer). Thus

$$w_{t-1}^{*}(H,\beta) \geq \frac{c_6}{H^x} \qquad \text{for all } H, \tag{3.74}$$

since

$$w_{t-1}^{*}(H,\beta) = \min_{\substack{s \leq t-1 \\ H(\gamma)\leq H \\ \gamma\neq\beta}} \left\{ \left| \beta-\gamma \right|_p \right\}. \tag{3.75}$$

From (3.74), we obtain

$$\frac{-\log(w_{t-1}^{*}(H,\beta))}{\log H} \leq x - \frac{\log c_6}{\log H} \tag{3.76}$$

and then

$$w_{t-1}^{*}(\beta) = \overline{\lim_{H\to\infty}} \frac{-\log(w_{t-1}^{*}(H,\beta))}{\log H} \leq x. \tag{3.77}$$

Therefore it follows from definition of $\mu^{*}(\beta)$ that

$$\mu^{*}(\beta) > t-1, \quad \text{that is} \quad \mu^{*}(\beta) \geq t. \tag{3.78}$$

Finally, from (3.43) and (3.78), we have

$$\mu^{*}(\beta) = t, \quad t > 1. \tag{3.79}$$

In other words, $\beta \in U_t^{*}$. Hence we obtain $\beta \in U_t^{*}$ in both of the cases 6) a) and b), and so $\beta \in U_t$.

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