On Transcendence of Values of Some Generalized Lacunary Power Series with Algebraic Coefficients for Some Algebraic Arguments in *p*-Adic Domain II[•]

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Abstarct. In this paper, the theorem in Çalışkan's [6] paper is proved by using Mahler classification in p-adic domain.

Keywords: The Field of p-Adic Numbers, Generalized Lacunary Power Series, Algebraic Numbers, Transcendence, Mahler Classification.

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1. INTRODUCTION

In this study, it is proved that the values of generalized lacunary power series with algebraic coefficients for some algebraic arguments belong to the p-adic U_m-subclass $(m \geq 1)$. This theorem was proved by using Koksma classification in Çalışkan's [6] paper, but in present paper this theorem is proved by using Mahler classification. So Zeren's [5] paper is transferred to p-adic domain by using Mahler classification. In particular, this article benefited greatly from the papers of Cohn [1] and Zeren [4].

Basic information about the subject of theorem is given in Schneider [3]. In here, it is only expressed Mahler's classification in p-adic domain, which was introduced by Mahler [2].

2. PRELIMINARIES

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and p denotes natural numbers, integer numbers, rational numbers and a given prime number respectively. $|.|_p$ and $\mathbb{Q}_p$ denotes p-adic valuation on $\mathbb{Q}$ and the field of p-adic numbers respectively.

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2.1. Mahler's Classification in $\mathbb{Q}_p$ [2]

Let n be a natural number. The height of the polynomial

$$P(x)=a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x], \;\; a_n \neq 0,$$

denoted by $H(P)$, is the form

$$H(P)=\max\left(\left|a_n\right|, \ldots, \left|a_1\right|, \left|a_0\right|\right).$$

The degree of the polynomial $P(x)$ is denoted with $\deg(P)$. Let ξ be an element of $\mathbb{Q}_p$. For given positive integer n and real number $H\ (\geq 1)$, we define the quantity

$$w_n\,(H,\,\xi):=\min_{\substack{P(x)\in\mathbb{Z}[x]\\ H(P)\leq H\\ \deg(P)\leq n\\ P(\xi)\neq 0}}\left\{\left|P(\xi)\right|_p\right\}.$$

It is clear that

$$0<w_n\,(H,\,\xi)\leq 1,$$

since $\left|P(\xi)\right|_p=1$ for $P(x)=1$. $w_n(H,\xi)$ is a non-increasing function of both n and H. Then we set

$$w_n\,(\xi)=\overline{\lim_{H\to+\infty}}\frac{\log\dfrac{1}{w_n\,(H,\xi)}}{\log H} \qquad \text{and} \qquad w(\xi)=\overline{\lim_{n\to+\infty}}\frac{w_n\,(\xi)}{n}.$$

$w_n(\xi)$ as a function of n is non-decreasing. The inequalities $0\leq w_n(\xi)\leq+\infty$ and $0\leq w(\xi)\leq+\infty$ ($n\geq 1$, $H\geq 1$) hold.

If $w_n(\xi)=+\infty$ for some integers n, let $\mu(\xi)\ (=\mu)$ be the smallest of such integers, and if $w_n(\xi)<+\infty$ for every n, put $\mu(\xi)=+\infty$. The two quantities $\mu(\xi)$, $w(\xi)$ are never finite simultaneously. Then the number ξ is called an

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[2] See [2].

A-number if $w(\xi)=0$, $\mu(\xi)=+\infty$,

S-number if $0<w(\xi)<+\infty$, $\mu(\xi)=+\infty$,

T-number if $w(\xi)=+\infty$, $\mu(\xi)=+\infty$,

U-number if $w(\xi)=+\infty$, $\mu(\xi)<+\infty$.

All p-adic numbers are distributed into the four classes A, S, T, U. With this classification:

1) A-numbers are exactly algebraic numbers[3].

2) If two p-adic numbers are algebraically dependent, then they belong to the same class[4].

Let ξ be a U-number such that $\mu(\xi)=m$, and let U_m denotes the set of all such numbers. For every natural m, U_m-class is a subclass of U, and $U_m \cap U_n = \emptyset$ if $m \neq n$. Therefore we have the partition $U = \bigcup_{m=1}^{\infty} U_m$.

Let ξ be a p-adic number and let m be a positive integer. The number ξ is called a U_m-number if $\mu(\xi)=m$, and $\mu(\xi)=m$ if the following conditions are satisfied:

i) For every $\omega>0$, if there are infinitely many polynomials P of degree m with integral coefficients such that

$$0<\left|P(\xi)\right|_p \leq cH(P)^{-\omega},$$

then

[3] See [2].

[4] See [2].

$$\mu(\xi) \leq m \quad \text{(that is } \xi \in U_1 \cup U_2 \cup \ldots \cup U_m\text{)}$$

where the positive constant c is independent of $H(P)$.

ii) If there exists constants $c' > 0$ and s depending only on ξ and m such that the relation

$$|P(\xi)|_p > c' H(P)^{-s}$$

holds for every polynomial P of degree $< m$ with integral coefficients, then

$$\mu(\xi) \geq m \quad \text{(that is } \xi \notin U_1 \cup U_2 \cup \ldots \cup U_{m-1}\text{)}.$$

Let α be a algebraic number. The height of the p-adic number α, denoted by $H(\alpha)$, is the height of its minimal polynomial over $\mathbb{Z}$. The degree of the p-adic number α denoted by $\deg(\alpha)$ is the degree of its minimal polynomial.

For the proof of main result, we shall need the following lemma.

Lemma Let $P(x)$ be a polynomial of degree m with integral coefficients and let α be a algebraic number of degree n with $P(\alpha) \neq 0$. Then the relation

$$|P(\alpha)|_p \geq \frac{p^{(n-1)t}}{(n+m)!\, H(\alpha)^m\, H(P)^n}$$

holds, where $|\alpha|_p = p^{-h}$, $t = \min(0, h)$, and $H(P)$, $H(\alpha)$ are the height of $P(x)$ and the height of the algebraic number α respectively.

Proof. See [2].

3. MAIN RESULT

Theorem Let $\{r_n\}$, $\{s_n\}$ be two infinite sequence of integers satisfying

$$0 \leq s_0 \leq r_1 < s_1 \leq r_2 < s_2 \leq r_3 < s_3 \leq \ldots.$$

Let

$$F(z)=\sum_{h=0}^{\infty} c_h z^h=\sum_{k=0}^{\infty} P_k(z)$$
$$P_k(z)=\sum_{h=s_k}^{r_{k+1}} c_h z^h \tag{3.1}$$

be a generalized lacunary power series such that

$$\begin{aligned} &c_h=0, \quad r_n<h<s_n \qquad n=1,2,\ldots, \\ &c_h\neq 0, \quad h=r_n \qquad n=1,2,\ldots, \\ &c_h\neq 0, \quad h=s_n \qquad n=0,1,\ldots, \end{aligned}$$

where the coefficients c_h are algebraic numbers in a constant number field $\mathrm{K}=\mathrm{K}(\theta)$ such that $[\mathrm{K}:\mathbb{Q}]=c$, and $c_h=0$ if $r_n<h<s_n$, but $c_{r_n}\neq 0$, $c_{s_n}\neq 0$ $(n=1,2,\ldots)$, and let

$$\overline{\lim_{i\to\infty}}\sqrt[i]{\left|c_i^{\{j\}}\right|_p}<+\infty \quad (j=1,\ldots,c), \tag{3.2}$$

where $c_i^{\{j\}}$ $(j=1,\ldots,c)$ denote the conjugates of c_i over K. Furthermore, suppose that the following conditions hold:

$$\lim_{n\to\infty}\frac{s_n}{r_n}=+\infty, \tag{3.3}$$

$$\overline{\lim_{n\to\infty}}\frac{r_n}{s_{n-1}}:=\tau<+\infty, \tag{3.4}$$

$$\overline{\lim_{n\to\infty}}\frac{\log A_n}{n}:=\sigma<+\infty, \tag{3.5}$$

$$\overline{\lim_{n\to\infty}}\frac{\log h_n}{n}:=l<+\infty \quad \left(h_n=H(c_n)\right), \tag{3.6}$$

where a_v is a suitable natural number such that $a_v c_v$ is an algebraic integer and $A_v=[a_0,\ldots,a_v]$ is the least common multiple of $a_0,\ldots,a_v$. Let α be an algebraic numbers of degree m satisfying $0<\left|\overline{\alpha}\right|_p<R$ such that $\left|\overline{\alpha}\right|_p=\max\left|\alpha^{(i)}\right|_p$ and $R=\min_{j=1}^{c}\frac{1}{\overline{\lim}_{i\to\infty}\sqrt[i]{\left|c_i^{\{j\}}\right|_p}}$. Then $F(\alpha)\in U_t$ for $z=\alpha$, where t is the maximum of the

degrees of the partial sums $F_n(\alpha)=\sum_{k=0}^{n-1} P_k(\alpha)$ and $t \leq [\mathbb{Q}(\theta, \alpha):\mathbb{Q}] := g \leq c\,m$. Also, assume that $P_n(\alpha) \neq 0$ for infinitely many integers n.

Proof. 1°) The radius of convergence of (3.1) is $\geq R$ (see [6]).

2°) Let's take $F(\alpha)=\beta$. We can write

$$\beta = \beta_n + \rho_n, \tag{3.7}$$

such that

$$\beta_n = \sum_{k=0}^{n-1} P_k(\alpha) = \sum_{v=s_0}^{r_n} c_v(\theta)\alpha^v, \tag{3.8}$$

$$\rho_n = \sum_{k=n}^{\infty} P_k(\alpha) = \sum_{v=s_n}^{\infty} c_v(\theta)\alpha^v. \tag{3.9}$$

Then we obtain an upper bound for the height $H(\beta_n)$ of β_n such that

$$H(\beta_n) \leq c_0^{r_n} \quad (n > N_0), \tag{3.10}$$

where $c_0\,(>1)$ and N_0 are sufficiently large numbers (see [6]).

3°) Let the minimal polynomial of the algebraic number β_n be

$$\mathrm{P}_n(x) = f_0 + f_1 x + f_2 x^2 + \ldots + f_l x^l \;; \qquad l \leq m,\; f_i \in \mathbb{Z}\; (i = 0, 1, \ldots, l).$$

Now we shall give an upper bound for $\left|\mathrm{P}_n(\beta)\right|_p$.

From (3.7), we have

$$\begin{aligned}\mathrm{P}_n(\beta) &= f_0 + f_1(\beta_n + \rho_n) + f_2(\beta_n + \rho_n)^2 + \ldots + f_l(\beta_n + \rho_n)^l \\ &= \mathrm{P}_n(\beta_n) + \rho_n \gamma_n\;,\end{aligned}$$

and so

$$\mathrm{P}_n(\beta)=\rho_n\gamma_n \tag{3.11}$$

since $\mathrm{P}_n(\beta_n)=0$, where

$$\gamma_n=f_1+f_2(2\beta_n+\rho_n)+\ldots+f_l(\binom{l}{1}\beta_n^{l-1}+\ldots+\binom{l}{l}\rho_n^{l-1}). \tag{3.12}$$

Using similar ideas as in [6], we obtain the inequalities

$$\left|c_n^{\{j\}}\right|_p\leq\frac{M^*}{(\rho^*)^n}\quad (j=1,\ldots,c)\ (n=0,1,\ldots) \tag{3.13}$$

and so

$$|\rho_n|_p\leq M^*\left(\frac{\left|\overline{\alpha}\right|_p}{\rho^*}\right)^{s_n}\quad (n=1,2,\ldots), \tag{3.14}$$

where M^*, ρ^* $(\frac{\left|\overline{\alpha}\right|_p}{\rho^*}<1)$ are sufficiently large numbers.

Since $\lim_{n\to\infty}|\beta_n|_p=|\beta|_p$, there is exits a number $M>0$ such that

$$|\beta_n|_p\leq M. \tag{3.15}$$

Then from (3.12),(3.14) and (3.15) we have

$$\begin{aligned}|\gamma_n|_p&=\left|f_1+f_2(2\beta_n+\rho_n)+\ldots+f_l(\binom{l}{1}\beta_n^{l-1}+\ldots+\binom{l}{l}\rho_n^{l-1})\right|_p\\&\leq c_1\qquad\qquad (n=0,1,2,\ldots),\end{aligned} \tag{3.16}$$

where c_1 (>1) is a sufficiently large number. Hence from (3.11), (3.14) and (3.16), we write

$$\left|\mathrm{P}_n(\beta)\right|_p = \left|\rho_n\right|_p \left|\gamma_n\right|_p \leq \frac{c_1 M^*}{\left(\dfrac{\rho^*}{\left|\overline{\alpha}\right|_p}\right)^{s_n}} = \frac{c_1 M^*}{c_2^{s_n}} \qquad (n=0,1,\ldots), \tag{3.17}$$

where $c_2 = \dfrac{\rho^*}{\left|\overline{\alpha}\right|_p}$ $(c_2 > 1)$. Since the polynomial $\mathrm{P}_n(x)$ is the minimal polynomial of the algebraic number β_n, it is $H(\beta_n) = H(\mathrm{P}_n)$. Putting $\dfrac{\log c_2}{\log c_0} = c_3\,(>0)$, $c_1 M^* = c_4$, we have from (3. 10) and (3. 17)

$$\left|\mathrm{P}_n(\beta)\right|_p \leq \frac{c_4}{(H(\beta_n))^{\frac{s_n}{r_n}c_3}} \qquad (n > N_0). \tag{3.18}$$

From (3.3), we see that

$$\lim_{n\to\infty} \frac{s_n}{r_n} c_3 = +\infty. \tag{3.19}$$

4°) Now, we will examine the sequence of height $\{H(\mathrm{P}_n)\}$ and the sequence of degree $\{d(\mathrm{P}_n)\}$ of the polynomials P_n. These sequences provide the following conditions A, B, C.

A) $\{H(\mathrm{P}_n)\}$ is not bounded from above.

Proof. Firstly, the sequence $\{\beta_n\}$ contains infinitely many different elements (see [6]). Now let's show that the sequence $\{H(\mathrm{P}_n)\}$ is not bounded from above: If $\{H(\mathrm{P}_n)\}$ were bounded from above, then $\{\mathrm{P}_n\}$ would contain only finitely many different elements since the degrees of the polynomials P_n are bounded from above with m. Therefore the sequence $\{\beta_n\}$ corresponding to the roots of the polynomials P_n would contain finitely many different elements. But the sequence $\{\beta_n\}$ contains infinitely many different elements; hence the sequence $\{H(\mathrm{P}_n)\}$ is not bounded from above.

B) Starting from a suitable n, $\{d(\beta_n)\}$ (or $\{d(\beta_n)\}$) is a constant sequence.

Proof. As in [6], there are two different cases:

a) Let $d(\mathrm{P}_n)=1$ (or $d(\beta_n)=1$) as starting from a suitable n. Then the condition B) is satisfied.

b) Let $d(\mathrm{P}_n)>1$ (or $d(\beta_n)>1$) for infinitely many integers n. Using similar ideas as in [6], if $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ for a fixed pair (i, j) $(i \neq j)$ and for any sufficiently large n, then $\beta_{n+1}^{\{i\}} \neq \beta_{n+1}^{\{j\}}$. In this case we have $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ for all n which are larger than a suitable n. This is exactly valid, because, $\beta_n^{\{i\}} \neq \beta_n^{\{j\}}$ is satisfied for at least a pair (i, j) and for infinitely many integers n from the hypothesis b). This can also be provided for all pairs (i, j). Hence we have

$$d(\beta_{N_1}) \leq d(\beta_{N_1+1}) \leq d(\beta_{N_1+2}) \leq \ldots$$

for a sufficiently large N_1. Since $d(\beta_n) \leq g$, for a sufficiently large N_2 we can write that

$$d(\beta_{N_2}) = d(\beta_{N_2+1}) = d(\beta_{N_2+2}) = \ldots$$

such that $N_2 \geq N_1$. If the common value is shown by t, then

$$d(\beta_n) = t, \quad n \geq N_2. \tag{3.20}$$

C) We can choose a subsequence of the sequence $\{ \mathrm{P}_n \}$ such that

0) $\mathrm{P}_{n_j}(\beta) \neq 0 \quad (j=1, 2, \ldots)$.

1) $\{H(\mathrm{P}_{n_j})\}$ is the monotone increasing sequence of natural numbers, hence it is diverges to $+\infty$.

2) $\{ d(\mathrm{P}_{n_j}) \}$ is a constant sequence.

Proof. The proof is obtained from the properties A) and B); the constant value of $d(\mathrm{P}_{n_j})$ is t **.**

Also, the sequence $\{H(\beta_{n_j})\}$, which is the sequence of the heights of the algebraic numbers β_{n_j} corresponding to the polynomials P_{n_j}, is the monotone increasing sequence which is diverges to $+\infty$: Since $\{H(\mathrm{P}_{n_j})\}$ is the monotone increasing sequence of natural numbers, which is diverges to $+\infty$, $H(\mathrm{P}_{n_k}) \neq H(\mathrm{P}_{n_l})$ for $k \neq l$, and since the polynomials P_{n_j} are the irreducible polynomials, the polynomials P_{n_j} are different from each other. Since $H(\beta_{n_j}) = H(\mathrm{P}_{n_j})$, it is $H(\beta_{n_k}) \neq H(\beta_{n_l})$ (for $k \neq l$), and the algebraic numbers β_{n_j} are different from each other. Hence the sequence $\{H(\beta_{n_j})\}$ is the monotone increasing sequence which is diverges to $+\infty$.

5°) We shall show that the number β is a U-number. To show this, we will use the subsequence $\{\mathrm{P}_{n_j}\}$ defined in C).

Putting $H_{n_j} = H(\mathrm{P}_{n_j})$, from C)-0) and (3.18) we can write

$$w_t(H_{n_j}, \beta) := \min_{\substack{H(\mathrm{P}) \leq H_{n_j} \\ \deg(\mathrm{P}) \leq t \\ \mathrm{P}(\beta) \neq 0}} \left\{ \left| \mathrm{P}(\beta) \right|_p \right\}$$

$$\leq \left| \mathrm{P}_{n_j}(\beta) \right|_p < \frac{1}{H_{n_j}^{\frac{s_{n_j}}{r_{n_j}} c_3}} \quad , \quad n_j > \max(N_0, N_2) \,. \tag{3.21}$$

We have from (3.21)

$$\frac{-\log(w_t(H_{n_j}, \beta))}{\log H_{n_j}} > \frac{s_{n_j}}{r_{n_j}} c_3 \tag{3.22}$$

and from (3.19) and (3.22)

$$\lim_{n_j \to \infty} \frac{-\log(w_t(H_{n_j}, \beta))}{\log H_{n_j}} = +\infty . \tag{3.23}$$

Finally, we obtain

$$w_t(\beta) = \overline{\lim_{H_{n_j} \to \infty}} \frac{-\log(w_t(H_{n_j}, \beta))}{\log H_{n_j}} = +\infty . \tag{3.24}$$

Thus it follows from the definition of $\mu(\beta)$ that

$$\mu(\beta) \le t . \tag{3.25}$$

This shows that the number β is a U -number.

6°) Now we will show that $\mu(\beta) = t$.

a) If $t = 1$, then $\mu(\beta) = 1$ from (3.25). In this case $\beta \in U_1$.

b) If $t > 1$, then we shall show that

$$w_{t-1}(\beta) < +\infty . \tag{3.26}$$

Consider the polynomial

$$\mathrm{B}(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_\gamma x^\gamma ; \qquad 1 \le \gamma < t, \; b_i \in \mathbb{Z} \; (i = 0, 1, \ldots, \gamma) . \tag{3.27}$$

For $x = \beta$, we have from (3.7), (3.8) and (3.9)

$$\begin{aligned} \mathrm{B}(\beta) &= \mathrm{B}(\beta_l) + \rho_l r_l \\ r_l &= b_1 + b_2 (2\beta_l + \rho_l) + \ldots + b_\gamma \left(\binom{\gamma}{1} \beta_l^{\gamma-1} + \ldots + \rho_l^{\gamma-1} \right). \end{aligned} \tag{3.28}$$

By (3.4), we have the relation

$$\frac{r_n}{s_{n-1}} < \tau_1 \tag{3.4$'$}$$

for all sufficiently large n, where $\tau_1 > \tau$.

The degree of β_l for $l \geq N_2$ is exactly t. Therefore, we can use Lemma, and so we obtain

$$\left| \mathrm{B}(\beta_l) \right|_p \geq \frac{c_5}{H(\beta_l)^{t-1} H(\mathrm{B})^t} \tag{3.29}$$

for $l > N_2$, where c_5 is a positive constant independent of the polynomial B. For $l > \max(N_0, N_2)$, we obtain from (3.10) and (3.29)

$$\left| \mathrm{B}(\beta_l) \right|_p \geq \frac{c_5}{c_0^{(t-1)r_l} H(\mathrm{B})^t}$$

and from $(3.4')$

$$\left| \mathrm{B}(\beta_l) \right|_p \geq \frac{c_5}{c_0^{(t-1)s_{l-1}\tau_1} H(\mathrm{B})^t} \tag{3.30}$$

since $c_0 > 1$. We see that from (3.14)

$$\left| \rho_l \right|_p = \left| \sum_{v=s_l}^{\infty} c_v \left(\frac{b}{a}\right)^v \right|_p \leq M^* \left(\frac{\left|\overline{\alpha}\right|_p}{\rho^*} \right)^{s_l} \quad (l = 1, 2, \ldots) \tag{3.31}$$

and from (3.28)

$$\left| r_l \right|_p = \left| b_1 + b_2 (2\beta_l + \rho_l) + \ldots + b_\gamma \left(\binom{\gamma}{1} \beta_l^{\gamma-1} + \ldots + \rho_l^{\gamma-1} \right) \right|_p \leq c_6,$$

and so we have

$$\left| \rho_l r_l \right|_p \leq \frac{M^* c_6}{\left(\dfrac{\rho^*}{\left|\overline{\alpha}\right|_p} \right)^{s_l}} = \frac{c_7}{c_2^{s_l}} \qquad (l = 1, 2, \ldots), \tag{3.32}$$

where $c_7 = M^* c_6$. Hence we obtain

$$|\rho_l r_l|_p \leq \frac{c_7}{c_0^{s_l c_3}} \qquad (l=1,2,\ldots) \tag{3.33}$$

where $c_3 = \dfrac{\log c_2}{\log c_0}$ $(c_3 > 0)$.

Let's take a number λ such that

$$\lambda > 1 \tag{3.34}$$

(the value of λ will be announced later). Since $s_{n-1} \leq r_n$, we get from (3.3)

$$\lim_{n\to+\infty} \frac{s_n}{s_{n-1}} = +\infty .$$

Therefore, for the number μ which is chosen such that

$$\mu > \lambda , \tag{3.35}$$

there exists $N_3 \in \mathbb{N}$ such that

$$\frac{s_n}{s_{n-1}} > \mu \tag{3.36}$$

for $n > N_3$ (the value of μ will be announced later).

Now let's consider the inequality

$$c_0^{s_{n-1}} \leq H(\mathrm{B}) < c_0^{s_n} \tag{3.37}$$

for any polynomial B satisfying the relation

$$H(\mathrm{B}) > H_0 \tag{3.38}$$

such that

$$H_0 = \max\left(c_0^{s_{N0}}\ ,\ c_0^{s_{N2}}\ ,\ c_0^{s_{N3}}\ ,\ \left(\frac{2c_7}{c_5}\right)^{1/c_3} \right). \tag{3.39}$$

There is exactly only one n satisfying the inequality (3.37) (see [6]).

From (3.37), (3.38) and (3.39), we have

$$n > \max(N_0\ ,\ N_2\ ,\ N_3)\,. \tag{3.40}$$

We see that from (3.35), (3.36) and (3.40)

$$s_{n-1} < \frac{s_n}{\lambda} \tag{3.41}$$

and from (3.34)

$$\frac{s_n}{\lambda} < s_n\,. \tag{3.42}$$

In this case, the interval $\left[\, c_0^{s_{n-1}}\ ,\ c_0^{s_n} \right)$ can be divided into two subintervals such that these subintervals are $\left[\, c_0^{s_{n-1}}\ ,\ c_0^{s_n/\lambda} \right)$ and $\left[\, c_0^{s_n/\lambda}\ ,\ c_0^{s_n} \right)$. Then $H(\mathrm{B})$ satisfying the relation (3.37) belong to one of the following two subintervals:

I) $c_0^{s_{n-1}} \le H(\mathrm{B}) < c_0^{s_n/\lambda}$,

II) $c_0^{s_n/\lambda} \le H(\mathrm{B}) < c_0^{s_n}$.

Case I) If we write the relation (3.30) with l replaced by n, then we get

$$\left|\mathrm{B}(\beta_n)\right|_p \ge \frac{c_5}{H(\mathrm{B})^{t+\tau_1(t-1)}} \tag{3.43}$$

and we have from (3.33)

$$\left|\rho_n r_n\right|_p < \frac{c_7}{H(\mathrm{B})^{\lambda c_3}}\ .$$

If we choose

$$\lambda := \frac{t+\tau_1(t-1)}{c_3} + 1, \tag{3.44}$$

then

$$|\rho_n r_n|_p < \frac{c_7}{H(\mathrm{B})^{t+\tau_1(t-1)+c_3}}. \tag{3.45}$$

From (3.38) and (3.39), we have

$$H(\mathrm{B}) > \left(\frac{2c_7}{c_5}\right)^{1/c_3} \Rightarrow c_5 - \frac{c_7}{H(\mathrm{B})^{c_3}} > \frac{c_5}{2} > 0. \tag{3.46}$$

Therefore we obtain

$$\frac{c_5}{H(\mathrm{B})^{t+\tau_1(t-1)}} > \frac{c_7}{H(\mathrm{B})^{t+\tau_1(t-1)+c_3}}. \tag{3.47}$$

Then

$$|\mathrm{B}(\beta_n)|_p > |\rho_n r_n|_p . \tag{3.48}$$

Hence we have from (3.28) and (3.48)

$$|\mathrm{B}(\beta)|_p = |\mathrm{B}(\beta_n)|_p \geq \frac{c_5}{H(\mathrm{B})^{t+\tau_1(t-1)}}. \tag{3.49}$$

Case II) If we write the relation (3.30) with l replaced by $n+1$, then we get

$$|\mathrm{B}(\beta_{n+1})|_p \geq \frac{c_5}{H(\mathrm{B})^{t+\lambda\tau_1(t-1)}}, \tag{3.50}$$

and from (3.33) and (3.36), we obtain

$$|\rho_{n+1} r_{n+1}|_p \leq \frac{c_7}{c_0^{s_n \mu c_3}} < \frac{c_7}{H(\mathrm{B})^{\mu c_3}} \tag{3.51}$$

If we choose

$$\mu := \frac{t+\lambda\tau_1(t-1)}{c_3}+1, \tag{3.52}$$

then from (3.51)

$$\left|\rho_{n+1} r_{n+1}\right|_p < \frac{c_7}{H(\mathrm{B})^{t+\lambda\tau_1(t-1)+c_3}}. \tag{3.53}$$

Therefore we obtain from (3.46)

$$\frac{c_5}{H(\mathrm{B})^{t+\lambda\tau_1(t-1)}} > \frac{c_7}{H(\mathrm{B})^{t+\lambda\tau_1(t-1)+c_3}}. \tag{3.54}$$

Then

$$\left|\mathrm{B}(\beta_{n+1})\right|_p > \left|\rho_{n+1} r_{n+1}\right|_p . \tag{3.55}$$

Hence we have from (3.28) and (3.55)

$$\left|\mathrm{B}(\beta)\right|_p = \left|\mathrm{B}(\beta_{n+1})\right|_p \geq \frac{c_5}{H(\mathrm{B})^{t+\lambda\tau_1(t-1)}}. \tag{3.56}$$

The inequality (3.56) is also satisfied in case I), since $t+\lambda\tau_1(t-1)>t+\tau_1(t-1)$ from (3.34). Putting $x=t+\lambda\tau_1(t-1)$ in both cases, we have

$$\left|\mathrm{B}(\beta)\right|_p \geq \frac{c_5}{H(\mathrm{B})^x} \;\; ; \; H(\mathrm{B})>H_0. \tag{3.57}$$

From 5°), we have

$$w_{t-1}(H_0,\beta) = \min_{\substack{\gamma\leq t-1\\ H(\mathrm{B})\leq H_0\\ \mathrm{B}(\beta)\neq 0}} \left\{ \left|\mathrm{B}(\beta)\right|_p \right\} \leq \left|\mathrm{B}(\beta)\right|_p \tag{3.58}$$

for all polynomials B with integer coefficients which have $\gamma\leq t-1$ and $H(\mathrm{B})\leq H_0$. Hence we can write

$$\left|\mathrm{B}(\beta)\right|_p \geq \frac{w_{t-1}(H_0,\beta)}{H(\mathrm{B})^x}; \quad \gamma\leq t-1, \quad H(\mathrm{B})\leq H_0. \tag{3.59}$$

Putting $c_8 = \min\left(c_5,\, w_{t-1}(H_0,\beta)\right)$, we obtain from (3.57), (3.59)

$$\left|B(\beta)\right|_p \geq \frac{c_8}{H(B)^x} \tag{3.60}$$

for $\gamma \leq t-1$ and $H(B)=1, 2, \ldots$. We see from (3.60) that

$$\left|B(\beta)\right|_p \geq \frac{c_8}{H(B)^x} \geq \frac{c_8}{H^x} \tag{3.61}$$

for all polynomials B which have $\gamma \leq t-1$ and $H(B) \leq H$ (where H is any positive integer). Thus

$$w_{t-1}(H,\beta) \geq \frac{c_8}{H^x} \qquad \text{for all } H. \tag{3.62}$$

From (3.62), we obtain

$$\frac{-\log(w_{t-1}(H,\beta))}{\log H} \leq x - \frac{\log c_8}{\log H} \tag{3.63}$$

and so

$$w_{t-1}(\beta) = \overline{\lim_{H\to\infty}} \frac{-\log(w_{t-1}(H,\beta))}{\log H} \leq x. \tag{3.64}$$

Therefore it follows from definition of $\mu(\beta)$ that

$$\mu(\beta) > t-1, \quad \text{that is} \quad \mu(\beta) \geq t. \tag{3.65}$$

Finally, from (3.25) and (3.65), we have

$$\mu(\beta) = t, \quad t > 1. \tag{3.66}$$

In other words, $\beta \in U_t$. Hence we obtain $\beta \in U_t$ in both of the cases 6) a) and b).

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