NORMAL CURVATURE OF A VECTOR FIELD IN A HYPERSURFACE OF A GENERALISED FINSLER SPACE

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The idea of generalised FINSLER spaces was developed in [5]. PAN [1] had studied the normal curvature of a vector field lying in a hypersurface of a Riemannian space which was generalised to a vector field lying in a hypersurface of a FINSLER space by NAGATA [3]. In this note we have extended these results to a generalised FINSLER space. The corresponding results for generalised Riemannian spaces and FINSLER spaces [2, 4] follow as particular cases.

1. Introduction. Let F_n be an n-dhnensional generalised FINSLER space endowed with a local coordinate system

$$x^i \ (i = 1, \dots, n).$$

The distance between two neighbouring points $P(x^i)$ and $Q(x^i + dx^i)$ is given by

$$ds = F(x^i, dx^i) \tag{1}$$

where the distance function F satisfies th3 following conditions:

a) $F(x, dx)$ is continuously differentiable at least up to the fourth order in its $2n$ arguments;

b) $F(x, dx)$ is positive provided all dx^i do not vanish simultaneously;

c) $F(x, dx)$ is positively homogeneous of the first degree in the dx^i, *i. e.* $F(x^i, x\, dx^i) = x\, F(x^i, dx^i)$ for $x > 0$;

d) $g_{(ij)}(x, \dot{x})\, \xi^i \xi^j > 0$ for all real ξ^i satisfying $\sum_i (\xi^i)^2 \neq 0$, where

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(3) $$g_{(ij)}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}.$$

This space is based on a non-symmetric metric tensor $g_{ij}(x, \dot{x})$ whose symmetric part $g_{(ij)}(x, \dot{x})$ is defined by (3) and whose skew-symmetric part $g_{[ij]}(x, \dot{x})$ is a function of coordinates only.

The covariant differential of a vector field $X^i(x^k)$ of F_n is given by [6]

(4) $$DX^i = dX^i + P^i_{hk}(x, dx)\, X^h\, dx^k$$

where

(5a) $$g_{(ij)}\, h^{ik} = \delta_j^k$$

(5b) $$\Delta^i_{hk} = h^{il} \Delta_{hjk} = \frac{1}{2} h^{ij} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^h} - \frac{\partial g_{hk}}{\partial x^j} \right)$$

and

(5c) $$P^i_{hk}(x, \dot{x}) = \Delta^i_{hk} - C^i_{hm}\, \Delta^m_{pk}\, \dot{x}^p.$$

The covariant derivative of X^i with respect to x^k is given by

(6) $$\delta_k X^i = \frac{\partial X^i}{\partial X^k} + P^{*i}_{hk} X^h$$

where

(7) $$P^{*i}_{hk} = \Delta^i_{hk} - h^{im} (C_{hml} P^l_{kj} + C_{kml} P^l_{hj} - C_{hkl} P^l_{mj})\, \dot{x}^j.$$

In the case when

$$C_{ijk} \equiv \frac{1}{2} \frac{\partial g_{ij}}{\partial \dot{x}^k} = 0,$$

i. e. when the space under consideration is a generalised Riemannian space, P^i_{hk} and P^{*i}_{hk} both reduce to Δ^i_{hk}, the usual connection parameters taken for that space. P^i_{hk} and P^{*i}_{hk} reduce to the similar quantities of RUND [4], if $g_{[ij]} = 0$.

Let a hypersurface F_{n-1} of F_n be given by

(8) $$x^i = x^i(u^\alpha)$$

so that the matrix

$$\| B^i_\alpha \|$$

where

$$B^i_\alpha \equiv \frac{\partial x^i}{\partial u^\alpha}$$

is of rank $n-1$. The two kinds of normals n^i and n^{*i} to F_{n-1} are defined by [6]

(9) $$g_{(ij)}(x, n) \quad B^i_\alpha n^j = 0, \quad F(x, n) = 1$$

and

(9′) $$g_{ij}(x, \dot{x}) \; B^i_\alpha n^{*j} = 0, \quad F(x, n^*) = 1$$

respectively.

The relations regarding the projection factors are given by [5]

(10) $$\text{a)} \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha = B^i_\alpha b^\beta_i$$

and

$$\text{b)} \quad B^i_\alpha B^\alpha_j + n^i n_j = \delta^i_j = B^i_\alpha b^\alpha_j + \frac{1}{\psi} n^{*i} n^*_j .$$

Writing $X^i_{\alpha\beta}$ for the generalised covariant derivative of $B^i_{\alpha\beta}$, we have

(11) $$X^i_{\alpha\beta} = \partial_\beta B^i_\alpha - B^i_\delta P^{*\delta}_{\alpha\beta} + B^h_\alpha B^k_\beta P^{*i}_{hk}$$

the induced connection parameters $P^{*\nu}_{\alpha\beta}$ for F_{n-1} being given by

(12) $$P^{*\nu}_{\alpha\beta} = B^\nu_i (\partial_\beta B^i_\alpha + P^{*i}_{hk} B^h_\alpha B^k_\beta).$$

The coefficients of second and secondary second fundamental form are given by [5]

(13) $$\Omega_{\alpha\beta} = n_i X^i_{\alpha\beta} = n_i (\partial_\beta B^i_\alpha + P^{*i}_{hk} B^h_\alpha B^k_\beta)$$

and

(14) $$\Omega^*_{\alpha\beta} = \Omega_{\alpha\beta} \sec (n, n^*)$$

respectively.

For differentiation in the direction

(10) $$dx^i = B^i_\alpha \, du^\alpha$$

of the hypersurface the covariant differential of B^i_α is given by

(16) $$D(B^i_\alpha) = X^i_{\alpha\beta} \, du^\beta = (\partial_\beta B^i_\alpha - B^i_\delta P^{*\delta}_{\alpha\beta} + B^h_\alpha B^k_\beta P^{\;i}_{hk}) \, du^\beta .$$

Multiplying (7) by B^i_ν, we obtain

(17) $$B^i_\nu P^{*\nu}_{\alpha\beta} = B^i_\nu B^\nu_j (\partial_\beta B^j_\alpha + B^h_\alpha B^k_\beta P^{*j}_{hk}).$$

From (10), (13) and (17), we obtain

(18) $$B^i_\nu P^{*\nu}_{\alpha\beta} = \partial_\beta B^i_\alpha + P^{*i}_{hk} B^h_\alpha B^k_\beta - n^i \Omega_{\alpha\beta}$$

which implies

$$\partial_\beta B^i_\alpha = B^i_\nu P^{*\nu}_{\alpha\beta} - B^h_\alpha B^k_\beta P^{\ i}_{hk} + n^i \Omega_{\alpha\beta}. \tag{19}$$

It should be clearly understood that [5]

$$\frac{\partial g_{[\alpha\beta]}}{\partial \dot{u}^\nu} = 0.$$

Let v^i be a unit vector field defined at all points of F_{n-1}. If $C: u^\alpha = u^\alpha(s)$ be a curve ou F_{n-1}, then the vector field v^i is automatically defined for each point P of the curve C. Surely, we have

$$\begin{aligned} &v^i = B^i_\alpha v^\alpha \\ &g_{(\alpha\beta)}(u^\delta, \dot{u}^\delta) v^\alpha v^\beta = 1 \end{aligned} \tag{20}$$

v^α being the components of v^i in F_{n-1}.

Differentiating (20) with respect to s, we obtain

$$\frac{dv^i}{ds} = \left\{ (\partial_\beta B^i_\alpha) v^\alpha + \frac{\partial v^\alpha}{\partial u^\beta} B^i_\alpha \right\} \frac{du^\beta}{ds}. \tag{21}$$

Substituting the value of $\partial_\beta B^i_\alpha$ from (19), we obtain

$$\frac{Dv^i}{Ds} = \frac{\delta v^\alpha}{\delta s} B^i_\alpha + v^\nu n^i \Omega_{\nu\beta} \frac{du^\beta}{ds} \tag{22}$$

where Dv^i is the covariant differential of v^i in F_n and δv^α that of v^α in F_{n-1}, when these vectors are transported in the direction of du^α.

The vector Dv^i/Ds is defined as the *absolute curvature vector* of v^i at a point of C. Its magnitude, to be denoted by v^k is called the *absolute curvature* of v^i with respect to C at P.

2. Normal curvature. The magnitude of the normal component of the absolute curvature vector is defined to be the normal curvature of the vector field at a point of C.

3. Asymptotic directions and lines. An asymptotic direction of a vector field is that direction with respect to which the normal curvature of the vector field is zero.

A curve on F_{n-1} which is such that the direction of its tangent at each and every point coincides with an asymptotic direction of the vector field is called an asymptotic line of the vector field.

The asymptotic lines are analytically given by

$$\Omega_{(\alpha\beta)}(a, u') v^\alpha du^\beta = 0.$$

4. Principal directions. The directions with respect to which the normal curvature given by

$$v^{kn} = \frac{\Omega_{(\alpha\beta)}\, v^{\alpha}\, du^{\beta}}{\sqrt{g_{\alpha\beta}(u, v)\, v^{\alpha} v^{\beta}}\, \sqrt{g_{\gamma\delta}(u, du)\, du^{\gamma}\, du^{\delta}}} \tag{23}$$

has an extreme value are known as principal directions.

Direct calculation shows that the principal directions are determined by

$$[\Psi_{\beta\delta} - (v^{kn})^2\, g_{(\beta\delta)}]\, du^{\delta} = 0 \tag{24}$$

where

$$\Psi_{\beta\delta} = \frac{\Omega_{\alpha\beta}\, \Omega_{\gamma\delta}\, v^{\alpha}\, v^{\gamma}}{g_{(\alpha\beta)}\, v^{\alpha}\, v^{\beta}}\, . \tag{25}$$

We observe that $\Psi_{\beta\delta}$ are symmetric in their indices inspite of the fact that $\Omega_{\alpha\beta}$ are non-symmetric in their indices.

In view of the fact that $\|\psi_{\beta\delta}\|$ has rank equal to unity, the equation

$$\text{det.}\ \mid \Psi_{\beta\delta} - (v^{kn})^2\, g_{(\alpha\delta)} \mid = 0 \tag{26}$$

which determines the extreme values of the normal curvature, has only one non-zero root. This root will be denoted by $\bar{v}^{kn}$. The value of the normal curvature in this direction is called *the principal normal curvature.*

5. Conjugate directions. Two directions du^{k} and δu^{α} are said to be conjugate to each other if and only if

$$\Omega_{(\alpha\beta)}\, du^{\alpha}\, \delta u^{\beta} = 0. \tag{27}$$

We now simply quote the following theorems which are easy deductions of the above definitions:

Theorem 1. *The normal curvature of a vector field in F_{n-1} is invariant for all curves touching one another at that point.*

Theorem 2. *The normal curvature of a vector field in F_{n-1} with respect to the indicatrix $\bar{F}(u, v) = 1$ of the field is numerically equal to the absolute curvature of the field in F_n with respect to the indicatrix $F(x, v) = 1$ of the field in F_{n-1}. The absolute curvature vector of a vector field with respect to the indicatrix of the field has at each point the normal direction of F_{n-1}.*

Theorem 3. *If a vector field undergoes a parallel displacement with respect to an asymptotic line l of F_{n-1}, then it undergoes parallel displacement with respect to C considered as a curve of F_n. (The vector field is supposed to lie in F_{n-1}).*

Theorem 4. *The asymptotic line of a vector field lying in F_{n-1} is conjugate to the direction of the vector field at any point P lying on it.*

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ÖZET

Teşmil edilmiş FINSLER uzayı fikri [5] te tetkik edilmiştir. Diğer taraftan bir RIEMANN uzayındaki bir hiperyüzeyin içinde bulunan bir vektör alanının normal eğriliği PAN [1] tarafından incelenmiş ve bu fikir bir FINSLER uzayındaki bir hiperyüzeye NAGATA [3] tarafından teşmil edilmiştir. Bu yazıda bu neticeleri, teşmil edilmiş bir FINSLER uzayı için elde ettik. Teşmil edilmiş bir RIEMANN uzayı [2, 4] için bulunan neticeler bunların hususî halleridir.