ON THE PARALLEL PROJECTION OF A SPHERE

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The following problem which is closely related to POHLKE's theorem is solved: The unit sphere of o given space $Oxyz$ is mapped by parallel projection onto the plane. The map ellipse is to be determined. The result is found by using the metric properties derived from parallel projection.

The construction of the paralel projection of a sphere in a case, in which are given the maps of three orthogonal diameters of the sphere, is closely related to the proof of POHLKE's theorem [1]. The solution ellipse is obtained by twice using RYTZ's construction for finding the principal axes.

In [2] the metric properties of parallel projection were studied. The results obtained immediately give a solution of the problem mentioned.

Let us formulate the problem precisely. Let $Oxyz$ be a rectangular space coordinate system with the unit e. Denote its parallel map onto the projection plane T by $O'x'y'z'$. Denote the angles between the axes x', y', z'

$$(x', y') = \varphi_1, \ (y', z') = \varphi_2, \ (z', x') = \varphi_3,$$

and the units on the x'- , y'- , z'- axis respectively e_1, e_2, e_3. Call the coordinate system $O'x'y'z'$ determined by the numbers

$$e_\nu, \ \varphi_\nu \qquad (\nu = 1, 2, 3)$$

briefly «*Pohlke's figure*».

Suppose now that in the space $Oxyz$ there is a unit sphere $K = K_e(O)$, with radius e and centre O. The parallel projection K' of this sphere is to be constructed. As a map of a circle on K, the figure K' is an ellipse

The symmetry-plane of the figure involves the normal n of the plane T, and the major axis of the ellipse K' sought. This major axis thus lies on the orthogonal projection r' of the central projection line $r = OO'$. The minor axis of K' has the half-length e, and the half-length of the major axis is found to be $e/\cos\vartheta$, vhere $\vartheta = (n, r')$, which is the angle between n and r'. We thus arrive at the following result:

Result. *The parallel projection from the space $Oxyz$ (with the unit e) on to the plane T is defined by giving the «*POHLKE*'s figure» through the units e_ν and the angles φ_ν. The unit sphere $K = K_e(O)$ of the space is, by means of this projection, mapped on to the plane T. The map K' of K is to be determined.*

The major axis of the ellipse K' lies on the straight line r', the direction of which is determined by the angle

$$\psi = (x', r').$$

The half-lengths of the minor-and major axis of K' are

$$e, \quad e/\cos\vartheta.$$

The numbers e, ϑ, ψ used are, according to [2], *determined by the formulae*

$$\begin{cases} 2e^2 = e^2_1 + e^2_2 + e^2_3 \\ \qquad\qquad - \sqrt{(e^2_1 + e^2_2 + e^2_3)^2 - 4\,(e^2_1\, e^2_2 \sin^2\varphi_1 + e^2_2\, e^2_3 \sin^2\varphi_2 + e^2_3\, e^2_1 \sin^2\varphi_3)}, \\ \operatorname{tg}^2\vartheta = s^2_1 + s^2_2 + s^2_3 - 2, \\ \operatorname{tg}\psi = \dfrac{1}{2}\, \dfrac{s^2_3 \sin 2\varphi_3 - s^2_2 \sin 2\varphi_1}{s^2_2 \sin^2\varphi_1 + s^2_3 \sin^2\varphi_3 - (s^2_1 + s^2_2 + s^2_3 - 1)}, \\ s_\nu = \dfrac{e_\nu}{e}. \end{cases}$$

Example. Take $e_1 = 1/2$, $e_2 = e_3 = 1$; $\varphi_2 = 90°$. These numbers define the POHLKE figure belonging to a classic case. From the above result, we found immediately: $e = 1$, $\psi = 0°$, $\cos\vartheta = 2/\sqrt{5}$. The major axis of K' lies on the x'-axis having the half length $\sqrt{5}/2$. The minor axis is $= 1$.

REFERENCES

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[2] TAMMI, O. : *On parallel projection of a rectangular space coordinate system on to a plane*, Ann. Acad. Sci. Fenn. Ser. A 1, No. 254, (1958).

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ÖZET

Bu makalede, POHLKE teoremlyle yakından ilgili olan aşağıdaki problem çözülmektedir : Verilen bir $Oxyz$ uzayının birim küresi paralel izdüşümle düzlem üzerine tasvir edilmekte ve tasvir elipsinin belirlenmesi istenmektedir. Netice paralel izdüşümden elde edilen metrik özelikler kullanılarak bulunmaktadır.